

## Maths Circle India: online math challenge

1. Suppose we have 1000 doors in a row marked  $1, 2, 3, \dots, 999, 1000$ . All doors are closed. Now, we pick a number  $n$  lying between 1 and 1000, and we change the condition of the doors (opening it if it is closed, and closing it if it is open), whose numbers are divisible by  $n$ . To elaborate, initially all the doors were closed. We pick 1, and as every number is divisible by 1, we open all the doors. Then we pick 2, and as only even numbers are divisible by 2, we close the even numbered doors, leaving the odd numbered doors open. We continue this process for  $3, 4, 5, \dots, 1000$ .

Now the question is, How many doors are open and how many are closed at the end of this process ?

If we change the procedure a little bit, instead of running the procedure for every  $n$  lying between 1 and 1000, we run this procedure only for primes lying between 1 and 1000. What will be the outcome then ?

What if for integer  $n$  lying between 1 and 1000, instead of changing the condition of the doors whose numbers are divisible by  $n$ , we change the condition of the doors whose numbers are co-prime to  $n$ . What would be the outcome then ?

Note: If  $a, n$  are positive integers,  $a, n$  are called co-prime to each other, if  $\gcd(a, n) = 1$ .

2. (a) Choose  $n + 1$  integers from the set  $\{ 1, 2, \dots, 2n \}$ . Then there will be two which are co-prime. Prove or disprove.  
(b) Choose  $n + 1$  integers from the set  $\{ 1, 2, \dots, 2n \}$ . Then there will be two numbers such that one divides the other. Is it true ?  
(c) Suppose every point in  $\mathbb{R}^2$  is colored with one of the 3 colors Red, Green and Blue. Then in at least one of the colors (depends on the coloring) for any given distance  $d$  there are two points  $x$  and  $y$  with distance  $d$  between them. Is it always true ?

3. Amit and Smita play a series of games where in each game they can either win or lose one rupee. In each round, Amit earns Re. 1 with probability  $p \in (0, 1)$ , and in this case Smita loses Re. 1. Conversely, Amit loses Re. 1 with probability  $q = 1 - p$  and Smita gains Re. 1. Suppose Amit starts with Rs.  $x > 0$  and Smita starts with Rs.  $y > 0$ . What is the probability that Amit loses all his money *eventually*?
  
4. (a) There's a stick of a fixed length  $L$ . A person breaks the stick into some pieces. How do you arrange those pieces so that area enclosed by the arrangement is maximum?  
 (b) If you are given the option of breaking the stick into ' $n$ ' number of pieces how would you do that so that the area enclosed by them is largest among all possible ways.
  
5. A new test has been developed for a rare disease. The test is 99% accurate. This means that 99 out of 100 times, the test correctly identifies whether or not a person has the disease. Studies have estimated that, at any given time, this disease affects about 1 in 10000 people in the general population. Uma decides to get tested for the disease.
  - (a) Suppose that the test result is positive (i.e., the test says that Uma has the disease). What is the probability that Uma actually has the disease? If the test result is negative, then what is the probability that Uma really does not have the disease?
  - (b) How do your answers change if the disease is less rare, and is found in 1% of the general population?
  
6. The object shown on the left in Figure 1 is made of a material that can be stretched, compressed, bent, distorted and molded at will. That is, one can change its shape in any way one pleases. However, one cannot tear the material, or stick two parts of it together. Convince yourself that it is possible to deform the object on the left in Figure 1 so that it is transformed into the object shown on the right in the same figure. Draw a "movie" (a sequence of stills) that shows this deformation process unfold.

Make linked rings by joining your index fingers with your thumbs. Observe that what you have demonstrated above is the following: if

the human body were elastic enough, it would be possible to move your hands apart without ever separating the joined fingertips.

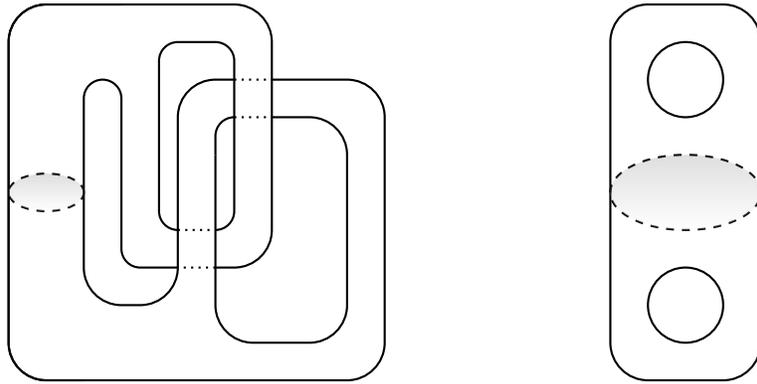


Figure 1: Can the rings be unlinked?