Existence and homogenization results for a class of singular elliptic problems in perforated domains

Federica RAIMONDI [⋆]

Joint works with P. DONATO [⋆] *and S. MONSURRÒ* [†]

[⋆] *Université de Rouen Normandie (France)*

[†] *Università degli Studi di Salerno (Italy)*

Multi-Scale Analysis and Theory of Homogenization

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EXISTENCE AND UNIQUENESS RESULTS FOR A CLASS OF SINGULAR ELLIPTIC PROBLEMS IN PERFORATED DOMAINS

P. Donato, S. Monsurrò and F. R., *Existence and uniqueness results for a class of singular elliptic problems in perforated domains*, Ric. Mat., **66**, pp. 333–360, 2017.

The perforated domain

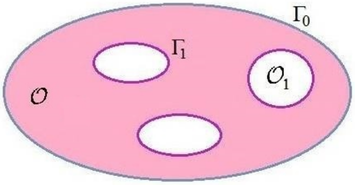

Figure: The perforated domain $\mathcal{O}$

- $\mathcal{O}_0$ a connected bounded open set in $\mathbb{R}^N$, $N \geq 2$, and Γ_0 its boundary,
- $\mathcal{O}_1$ a bounded open set such that $\overline{\mathcal{O}_1} \subset \mathcal{O}_0$ and its boundary Γ_1 is Lipschitz continuous,
- $\mathcal{O} := \mathcal{O}_0 \setminus \overline{\mathcal{O}_1}$ the perforated domain,
- $\partial\mathcal{O} = \Gamma_0 \cup \Gamma_1$.

The singular elliptic problem

$$\begin{cases} -\operatorname{div}(B(x,u)\nabla u) + \lambda u = f\zeta(u) & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \Gamma_0, \\ (B(x,u)\nabla u)\nu + \rho h(u) = g & \text{on } \Gamma_1, \end{cases}$$

where ν denotes the unit external normal vector to $\mathcal{O}$.

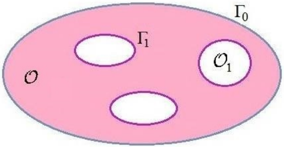

Figure: The perforated domain $\mathcal{O}$

Assumptions on the data

H$_1$) The real $N \times N$ matrix field $B : (x,t) \in \mathcal{O} \times \mathbb{R} \mapsto B(x,t) = (b_{i,j}(x,t))_{i,j=1,\dots,N} \in \mathbb{R}^{N^2}$ satisfies the following conditions:

$$\begin{cases} \text{i) } B \text{ is a Carathéodory function,} \\ \text{i.e. } B(x,\cdot) \text{ is continuous for a.e. } x \in \mathcal{O} \\ \text{and } B(\cdot,t) \text{ is measurable for every } t \in \mathbb{R}; \\ \\ \text{ii) } B(\cdot,t) \in M(\alpha,\beta,\mathcal{O}), \text{ for every } t \in \mathbb{R}. \end{cases}$$

where $M(\alpha,\beta,\mathcal{O})$ is the set of matrix fields $B \in (L^\infty(\mathcal{O}))^{N\times N}$ such that $(B(x)\lambda,\lambda) \geq \alpha|\lambda|^2$ and $|B(x)\lambda| \leq \beta|\lambda|$, $\forall \lambda \in \mathbb{R}^N$ and almost everywhere on $\mathcal{O}$.

H$_2$)
$$\lambda \geq 0.$$

H$_3$)
$$\begin{cases} \text{i) } \zeta : [0,+\infty[\to [0,+\infty] \text{ is a function such that} \\ \quad \zeta \in \mathcal{C}^0([0,+\infty[), \ 0 \le \zeta(s) \le \dfrac{1}{s^k} \\ \quad \text{for every } s \in]0,+\infty[, \ \text{with } 0 < k \le 1; \\ \text{ii) } f \ge 0 \text{ a.e. in } \mathcal{O}, \ \text{with } f \in L^l(\mathcal{O}), \ \text{for } l \ge \dfrac{2}{1+k} (\ge 1). \end{cases}$$

H$_4$) The functions g and ρ are such that

$$\begin{cases} \text{i) } 0 \le g \in L^s(\Gamma_1), \ \text{with } \begin{cases} s \ge \dfrac{2(N-1)}{N} & \text{if } N > 2, \\ s > 1 & \text{if } N = 2; \end{cases} \\ \text{ii) } 0 \le \rho \in L^\infty(\Gamma_1). \end{cases}$$

Assumptions on the data

H$_5$) Either

$$f \not\equiv 0$$

or

$$\partial H_{N-1}(\Gamma_1) > 0 \quad \text{and} \quad g \not\equiv 0,$$

where ∂H_{N-1} is the $(N-1)$-Hausdorff measure in $\mathbb{R}^N$.

H$_6$) The function h is an increasing and continuously differentiable function such that for some positive constant C and an exponent q one has

$$\begin{cases} h(0) = 0, \\ |h'(s)| \leq C(1 + |s|^{q-1}), \forall s \in \mathbb{R}, \\ \text{with } 1 \leq q < \infty \text{ if } N = 2, \text{ and } 1 \leq q \leq \dfrac{N}{N-2} \text{ if } N > 2. \end{cases}$$

Difficulties

- Quasilinear matrix field
- Singular datum
- Nonlinear Robin boundary condition

Physical meaning of the problem

- Quasilinear matrix field: heat diffusion depends on the temperature.
- Singular datum: electrical conductivity.
- Nonlinear Robin boundary condition: chemical reactions at the boundaries of the perforations.

The functional framework

$$V = \{v \in H^1(\mathcal{O}) : v = 0 \text{ on } \Gamma_0\}$$

Due to the Poincaré inequality, there exists a positive constant $C_{\mathcal{O}}$ such that

$$\|v\|_{L^2(\mathcal{O})} \leq C_{\mathcal{O}} \|\nabla v\|_{L^2(\mathcal{O})}, \ \forall v \in V.$$

Consequently, V can be equipped by the norm

$$\|v\|_V := \|\nabla v\|_{L^2(\mathcal{O})}, \ \forall v \in V$$

which is equivalent to the H^1-norm.

The variational formulation

$$
\begin{cases}
\text{Find } u \in V \text{ such that } u > 0 \;\; \text{a.e. in } \mathcal{O}, \\
\displaystyle\int_{\mathcal{O}} f\zeta(u)\varphi dx < +\infty \text{ and} \\
\displaystyle\int_{\mathcal{O}} B(x,u)\nabla u \nabla \varphi dx + \int_{\Gamma_1} \rho h(u)\varphi d\sigma + \int_{\mathcal{O}} \lambda u \varphi dx \\
\displaystyle= \int_{\mathcal{O}} f\zeta(u)\varphi dx + \int_{\Gamma_1} g\varphi d\sigma, \;\; \forall \varphi \in V.
\end{cases}
$$

Proposition

Under assumptions H_1)-H_6), let $u \in V$ be a solution of the problem. The following a priori estimate holds:

$$\|u\|_V \leq c,$$

where c depends on N, α, s, $C_{\mathcal{O}}$, $\|f\|_{L^l(\mathcal{O})}$ and $\|g\|_{L^s(\Gamma_1)}$.

Proposition

Under assumptions H_1)-H_6), let $u \in V$ be a solution of the problem. Then

$$\|f\zeta(u)\varphi\|_{L^1(\mathcal{O})} \leq c,$$

for every positive $\varphi \in V$ with c depending on N, α, β, s, q, $C_{\mathcal{O}}$, λ, $\|\varphi\|_V$, $\|f\|_{L^l(\mathcal{O})}$, $\|\rho\|_{L^\infty(\Gamma_1)}$ and $\|g\|_{L^s(\Gamma_1)}$.

A third bound

Proposition

Under assumptions H_1)-H_6), let $u \in V$ be a solution of the problem and δ a fixed positive real number. Then,

$$\int_{\{0\leq u\leq\delta\}} f\zeta(u)\varphi dx \leq \int_{\mathcal{O}} B(x,u)\nabla u\nabla\varphi Z_\delta(u)dx$$

$$+ h(2\delta)\|\rho\|_{L^\infty(\Gamma_1)}\|\varphi\|_{L^1(\Gamma_1)} + 2\delta\lambda\|\varphi\|_{L^1(\mathcal{O})},$$

for every $\varphi \in V$, $\varphi \geq 0$, ...

...where Z_δ is defined by

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \delta, \\ -\dfrac{s}{\delta} + 2, & \text{if } \delta \leq s \leq 2\delta, \\ 0, & \text{if } s \geq 2\delta. \end{cases}$$

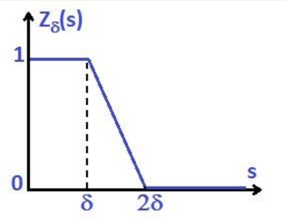

Figure: The auxiliary function

The approximate problem

As usual in the literature, in order to prove the existence of a solution for our problem, we approximate it with a sequence of nonsingular problems with a bounded nonlinearity in the equation:

$$\begin{cases} -\operatorname{div}(B(x,u_m)\nabla u_m) + \lambda u_m = T_m(f\zeta(|u_m|)) & \text{in } \mathcal{O}, \\ u_m = 0 & \text{on } \Gamma_0, \\ (B(x,u_m)\nabla u_m)\nu + \rho h(u_m) = g & \text{on } \Gamma_1, \end{cases}$$

where, for every $m \in \mathbb{N}$, the function T_m is the usual truncation function at level m.

▷ Thanks to this approximation, we avoid the singularity at zero!

The existence result for the approximate problem

Fix $w \in V$ and consider the following problem

$$\begin{cases} \text{Find } u \in V \text{ such that } \forall v \in V, \\ \displaystyle\int_{\mathcal{O}} B(x,w)\nabla u \nabla v dx + \int_{\Gamma_1} \rho h(u) v d\sigma + \int_{\mathcal{O}} \lambda u v dx \\ \displaystyle = \int_{\mathcal{O}} T_m(f\zeta(|w|)) v dx + \int_{\Gamma_1} g v d\sigma, \end{cases}$$

that admits a unique solution.

D. Cioranescu, P. Donato and R. Zaki, *Asymptotic behaviour of elliptic problems in perforated domains with nonlinear boundary conditions*, Asymptotic Analysis, 53 (2007), 209-235.

▷ $T : w \in V \mapsto T(w) = u \in V$

Theorem (Schauder fixed-point theorem)

Let K be a convex closed set of a Banach space B. Let $T : K \subset B \to B$ be a continuous map such that $T(K) \subset K$ and $T(K)$ is relatively compact in K. Then T has a fixed point in K.

$$\Downarrow$$

Theorem

Under assumptions H_1)-H_6), for every fixed $m \in \mathbb{N}$, the approximate problem admits at least a solution.
Moreover, $u_m \geq 0$ almost everywhere in $\mathcal{O}$.

How to find our solution

Fix $m \in \mathbb{N}$, let $u_m \in V$ be a solution of the approximate problem.

▷ The three a priori estimates apply to u_m, uniformly with respect to m.

▷ There exists $u \in V$ such that, up to a subsequence,

$$\begin{cases} \text{i) } u_m \rightharpoonup u \ \text{ weakly in } V, \\ \text{ii) } u_m \to u \ \text{ strongly in } L^2(\mathcal{O}), \\ \text{iii) } u_m \to u \ \text{ a.e. in } \mathcal{O}. \end{cases}$$

▷ By passing to the limit in the approximate problem as $m \to +\infty$, one has that the limit function u is a solution of our problem.

The uniqueness result

$H_{1'}$) There exists a real function $\omega : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

$$\begin{cases} \text{i) } \omega \text{ is continuous and non decreasing, with } \omega(t) > 0 \; \forall \, t > 0; \\ \text{ii) } |B(x,t_1) - B(x,t_2)| \le \omega(|t_1 - t_2|) \text{ for a.e. } x \in \mathcal{O}, \forall \, t_1 \ne t_2; \\ \text{iii) } \forall y > 0, \; \lim_{x \to 0^+} \int_x^y \frac{dt}{\omega(t)} = +\infty. \end{cases}$$

$H_{3'}$) The function ζ is non increasing in $[0, +\infty[$.

Theorem

Under assumptions H_1)-H_6) and the additional ones $H_{1'}$) and $H_{3'}$), the problem admits a unique solution.

HOMOGENIZATION AND CORRECTOR RESULTS FOR SINGULAR ELLIPTIC PROBLEMS IN PERIODICALLY PERFORATED DOMAINS

P. Donato, S. Monsurrò and F. R., *Homogenization of a class of singular elliptic problems in perforated domains*, Nonlinear Analysis, **173**, pp. 180–208 (2018).

The domain

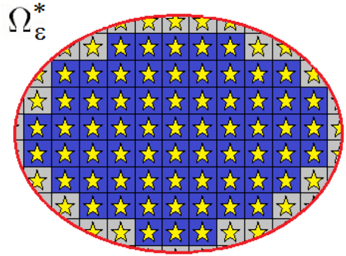

Figure: The periodically perforated domain Ω^*_ε

- Ω a connected bounded open set in $\mathbb{R}^N$, $N \geq 2$, and $\partial\Omega$ its Lipschitz-continuous boundary.

The perforated cell

Figure: The perforated cell Y^*

- Y the reference cell,
- T a nonempty open subset such that $\overline{T} \subset Y$ and its boundary ∂T is Lipschitz-continuous with a finite number of connected components,
- $Y^* := Y \setminus \overline{T}$ the perforated cell,
- $\theta \doteq \frac{|Y^*|}{|Y|}$.

The domain

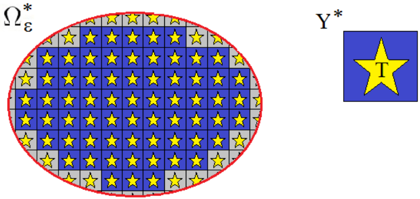

Figure: The periodically perforated domain Ω_ε^* and the perforated cell Y^*

- $\{\varepsilon\}_{\varepsilon>0}$ a positive parameter taking values in a sequence converging to zero,
- T_ε the union of the disjoint traslated sets of T rescaled by ε,
- $\Omega_\varepsilon^* := \Omega \setminus \overline{T}_\varepsilon$ the periodically perforated domain,
- $\partial\Omega_\varepsilon^* = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon$.

To study the asymptotic behavior, as ε goes to zero, of

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon) = f\zeta(u_\varepsilon) & \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ (A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon)\nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \end{cases}$$

where ν denotes the unit external normal vector to the domain.

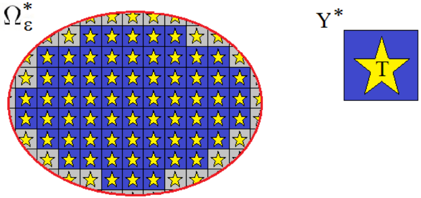

Figure: The perforated domain Ω_ε^*

Assumptions on the data

- $\hookrightarrow$ A, f, ζ, ρ, h and g as before.
- $\hookrightarrow$ $A(\cdot, t)$ is Y-periodic for every t.
- $\hookrightarrow$ ρ and g are Y-periodic.
- $\hookrightarrow$ $\gamma \geq 1$.

For almost every $x \in \Omega$ and every $t \in \mathbb{R}$, we set

$$A^\varepsilon(x, t) \doteq A\left(\frac{x}{\varepsilon}, t\right).$$

For almost every $x \in \Gamma_1^\varepsilon$, we set

$$g^\varepsilon(x) \doteq \varepsilon g\left(\frac{x}{\varepsilon}\right) \qquad \text{and} \qquad \rho^\varepsilon(x) \doteq \varepsilon \rho\left(\frac{x}{\varepsilon}\right).$$

The functional framework

$$V_\varepsilon = \{v \in H^1(\Omega_\varepsilon^*) : v = 0 \text{ on } \Gamma_0^\varepsilon\}$$

The Poincaré inequality in V_ε holds with a constant c_p independent on ε, that is

$$\|v\|_{L^2(\Omega_\varepsilon^*)} \leq c_P \|\nabla v\|_{L^2(\Omega_\varepsilon^*)}, \ \forall v \in V_\varepsilon.$$

Consequently, V_ε can be equipped by the norm

$$\|v\|_{V_\varepsilon} := \|\nabla v\|_{L^2(\Omega_\varepsilon^*)}, \ \forall v \in V_\varepsilon$$

which is equivalent to the H^1-norm via a constant independent on ε.

D. Cioranescu and J. Saint Jean Paulin, *Homogenization in open sets with holes*, J. Math. Anal. Appl., **71**, pp. 590–607 (1979).

The ε-probem

$$
\begin{cases}
\text{Find } u_\varepsilon \in V_\varepsilon \text{ such that } u_\varepsilon > 0 \ \text{ a.e. in } \Omega_\varepsilon^*, \\[2ex]
\displaystyle\int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx < +\infty \text{ and} \\[2ex]
\displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon \nabla\varphi dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma \\[2ex]
\displaystyle= \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon \varphi d\sigma, \quad \forall \varphi \in V_\varepsilon.
\end{cases}
$$

⋆ This problem admits a unique solution, for every fixed ε.

Under our assumptions, let $u_\varepsilon \in V_\varepsilon$ be the solution of the problem. The following a priori estimates hold:

$$\star \qquad \|u_\varepsilon\|_{V_\varepsilon} \leq c,$$

where c depends on α, c_P, $\|f\|_{L^l(\Omega)}$ and $\mathcal{M}_{\partial T}(g)$.
Also, up to a subsequence,

$$\star \qquad \|f\zeta(u_\varepsilon)\varphi\|_{L^1(\Omega_\varepsilon^*)} \leq c,$$

for every nonnegative $\varphi \in H_0^1(\Omega) \cup L^\infty(\Omega)$ with c depending on α, β, c_P, $|Y|$, $\|\nabla\varphi\|_{L^2(\Omega_\varepsilon^*)}$, $\|f\|_{L^l(\Omega)}$, $\|\rho\|_{L^\infty(\partial T)}$ and $\mathcal{M}_{\partial T}(g)$.

Under our assumptions, let $u_\varepsilon \in V_\varepsilon$ be the solution of the problem and δ a fixed positive real number. Then,

$$\star \quad \int_{\{0<u_\varepsilon\leq\delta\}} f\zeta(u_\varepsilon)\varphi dx \leq \int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla\varphi Z_\delta(u_\varepsilon)dx + c\,\varepsilon^{\gamma-\frac{1}{2}} h(2\delta)\|\rho\|_{L^\infty(\partial T)}\|\varphi\|_{V_\varepsilon},$$

for every $\varphi \in V_\varepsilon$, $\varphi \geq 0$ and c independent on ε.

For any Lebesgue-measurable function ϕ on Ω_ε^*,

$$\mathcal{T}_\varepsilon^*(\phi)(x,y) \doteq \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. for } (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

For any Lebesgue-measurable function ϕ on $\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon$, the boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined as follows:

$$\mathcal{T}_\varepsilon^b(\phi)(x,y) \doteq \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x,y) \in \widehat{\Omega}_\varepsilon \times \partial T, \\ 0 & \text{a.e. for } (x,y) \in \Lambda_\varepsilon \times \partial T. \end{cases}$$

Some properties

For all $\phi \in L^1(\Omega_\varepsilon^*)$, the following **integration formula** holds:

$$\int_{\Omega_\varepsilon^*} \phi(x)dx = \int_{\Lambda_\varepsilon^*} \phi(x)dx + \frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x,y)dxdy.$$

Unfolding criterion for integrals

Let ϕ_ε be in $L^1(\Omega_\varepsilon^*)$. If

$$\int_{\Lambda_\varepsilon^*} |\phi_\varepsilon| dx \to 0 \ \text{ as } \varepsilon \to 0,$$

then

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} \phi_\varepsilon dx = \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon)dxdy.$$

The homogenization process

Proposition

Let $u_\varepsilon \in V_\varepsilon$ be the unique solution of the problem.
Then, there exist a subsequence and two functions $u_0 \in H_0^1(\Omega)$ and $\widehat{u} \in L^2(\Omega; H^1_{per}(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{u}) = 0$, such that

$$(\clubsuit)\begin{cases} i)\ \mathcal{T}^*_\varepsilon(u_\varepsilon) \to u_0 & \text{strongly in } L^2(\Omega; H^1(Y^*)) \\ & \text{and a.e. in } \Omega \times Y^*, \\ ii)\ \mathcal{T}^*_\varepsilon(\zeta(u_\varepsilon)) \to \zeta(u_0) & \text{a.e. in } \Omega \times Y^*, \\ iii)\ \widetilde{u_\varepsilon} \rightharpoonup \theta u_0 & \text{weakly in } L^2(\Omega), \\ iv)\ \mathcal{T}^*_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u} & \text{weakly in } L^2(\Omega \times Y^*), \\ v)\ \mathcal{T}^b_\varepsilon(h(u_\varepsilon)) \to h(u_0) & \text{strongly in } L^t(\Omega; W^{1-\frac{1}{t},t}(\partial T)) \end{cases}$$

where

$$t \doteq \begin{cases} \in (1;2) & \text{if } N = 2 \text{ and } q > 1, \\ \dfrac{2N}{q(N-2)+2} & \text{otherwise.} \end{cases}$$

Moreover,

$u_0 \geq 0$ almost everywhere in Ω and

$$\int_\Omega f\zeta(u_0)\varphi dx < +\infty, \quad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

D. Cioranescu, A. Damlamian, P. Donato, G. Griso and R. Zaki, *The periodic unfolding method in domains with holes*, SIAM J. Math. Anal., **44**, No. 2, pp. 718–760 (2012).

Identification of $\hat{u}$

Let $(u_0, \widehat{u})$ be given by the previous proposition. Then

$$\widehat{u}(y, x) = -\sum_{i=1}^{N} \widehat{\chi}_{e_i}(y, u_0(x)) \frac{\partial u_0}{\partial x_i}(x) \in L^2(\Omega; H^1_{per}(Y^*)),$$

where $\{e_1, ..., e_N\}$ is the canonical basis of $\mathbb{R}^N$ and $\widehat{\chi}_\lambda(y, t)$ is solution of the following cell problem:

$$\begin{cases} -div(A(\cdot, t)\nabla_y \widehat{\chi}_\lambda(\cdot, t)) = -div(A(\cdot, t)\lambda) & \text{in } Y^*, \\ A(\cdot, t)(\lambda - \nabla_y \widehat{\chi}_\lambda(\cdot, t))\nu = 0 & \text{on } \partial T, \\ \widehat{\chi}_\lambda(\cdot, t) \text{ Y-periodic,} & \\ \dfrac{1}{|Y^*|} \displaystyle\int_{Y^*} \widehat{\chi}_\lambda(y, t) dy = 0, & \end{cases}$$

for every $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}^N$.

The homogenized matrix

For every fixed $t \in \mathbb{R}$, the homogenized matrix $A^0(t)$ is defined by (see Artola-Duvaut)

$$A^0(t)\lambda \doteq \frac{1}{|Y|}\int_{Y^*} A(y,t)\nabla_y\widehat{\omega}_\lambda(y,t)dy \quad \forall\lambda \in \mathbb{R}^N,$$

where, for every $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}^N$,

$$\widehat{\omega}_\lambda(y,t) = \lambda \cdot y - \widehat{\chi}_\lambda(y,t)$$

and $\widehat{\chi}_\lambda(y,t)$ previously given.

Moreover, it verifies the following properties:

- $A^0(u_0)\nabla u_0 = \frac{1}{|Y|}\int_{Y^*} A(y, u_0)(\nabla u_0 + \nabla_y \widehat{u})dy$;
- A^0 is continuous and $A^0(t) \in M\left(\alpha, \frac{\beta^2}{\alpha}, \Omega\right)$ for every $t \in \mathbb{R}$;
- there exists a positive constant C, depending on α, β, Y and T, s.t. $|A^0(t_1) - A^0(t_2)| \leq C\omega(|t_1 - t_2|)$

  for every $t_1, t_2 \in \mathbb{R}$, with $t_1 \neq t_2$,

  where ω is the function given for the uniqueness of the solution.

B. Cabarrubias and P. Donato, *Homogenization of a quasilinear elliptic problem with nonlinear Robin boundary comditions*, Appl. Anal. (6), **91**, No. 6, 1111–1127 (2012).

The homogenization theorem

Let $u_\varepsilon \in V_\varepsilon$ be the unique solution of the problem and $(u_0, \widehat{u})$ be given by the previous propositions.
Then $(u_0, \widehat{u})$ is the unique solution of the unfolded limit equation

$$\begin{cases} \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega) \text{ and } \forall \psi \in L^2(\Omega; H^1_{per}(Y^*)) \\ \displaystyle\int_{\Omega \times Y^*} A(y, u_0)(\nabla u_0 + \nabla_y \widehat{u})(\nabla \varphi + \nabla_y \psi) dxdy \\ \displaystyle + |Y| c_\gamma \int_\Omega h(u_0) \varphi dx = |Y^*| \int_\Omega f \zeta(u_0) \varphi dx \\ \displaystyle + |\partial T| \mathcal{M}_{\partial T}(g) \int_\Omega \varphi dx, \end{cases}$$

$$\text{where} \quad c_\gamma \doteq \begin{cases} \dfrac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(\rho) & \text{if } \gamma = 1, \\ 0 & \text{if } \gamma > 1. \end{cases}$$

Furthermore, $u_0 > 0$ almost everywhere in Ω and u_0 is the unique solution of the following singular limit problem:

$$\begin{cases} -div(A^0(u_0)\nabla u_0) + c_\gamma h(u_0) = \theta f \zeta(u_0) + \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

This implies the uniqueness of $\widehat{u}$ under the condition $\mathcal{M}_{Y^*}(\widehat{u}) = 0$, in view of the expression of $\widehat{u}$ in terms of u_0.

Consequently, convergences (♣) hold for the whole sequence.

Sketch of the proof

How to pass to the limit in

$$\int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon \nabla\varphi dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma$$

$$= \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon \varphi d\sigma, \qquad \varphi \in V_\varepsilon.$$

▷ Two main tools

Proposition

There exists a linear operator $\mathcal{L}_\varepsilon : H^{-1}(\Omega) \to V'_\varepsilon$ *satisfying the following condition:*

If $\{\varphi_\varepsilon\}$ *is a sequence such that*

$$\|\varphi_\varepsilon\|_{V_\varepsilon} \leq c \quad \text{and} \quad \widetilde{\varphi_\varepsilon} \rightharpoonup \theta\varphi_0 \quad \text{weakly in } L^2(\Omega),$$

then

$$\lim_{\varepsilon \to 0} \langle \mathcal{L}_\varepsilon(Z), \varphi_\varepsilon \rangle_{V'_\varepsilon, V_\varepsilon} = \langle Z, \varphi_0 \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}.$$

I. Chourabi and P. Donato, *Homogenization of elliptic problems with quadratic growth and nonhomogenous Robin conditions in perforated domains*, Chin. Ann. Math. Ser. B, **37**, No. 6, pp. 833–852 (2016).

Let u_0 be a weak cluster point of the sequence $\{u_\varepsilon\}$.
Let us introduce the following linear problem associated with u_0:

$$\begin{cases} -div(A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon) & \text{in } \Omega_\varepsilon^*, \\ \qquad = \mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla u_0) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)), & \\ v_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ (A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon)\nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \end{cases}$$

whose homogenization has been studied in

I. Chourabi and P. Donato, *Homogenization and correctors of a class of elliptic problems in perforated domains*, Asymptot. Anal., **92**, No. 1-2, pp. 1–43 (2015).

$\rightsquigarrow \quad \|v_\varepsilon\|_{V_\varepsilon} \leq c$ and $\widetilde{v_\varepsilon} \rightharpoonup \theta u_0$ weakly in $L^2(\Omega)$.

Tool 1: A crucial auxiliary result

Theorem

Under our assumptions, let u_ε and v_ε be solutions of the problems. Then, for the subsequence verifying (♣),

$$\lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - \nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon^*)} = 0.$$

A. Bensoussan, L. Boccardo and F. Murat, *H-convergence for quasi-linear elliptic equations with quadratic growth*, Appl. Math. Optim., **26**, No. 3, pp. 253–272 (1992).

P. Donato and D. Giachetti, *Existence and homogenization for a singular problem through rough surfaces*, SIAM J. Math. Anal., **48**, pp. 4047–4086 (2016).

If we take as test function

$$\psi_\varepsilon(x) = \varphi(x) + \varepsilon\phi(x)\xi\left(\frac{x}{\varepsilon}\right) \in V_\varepsilon,$$

with $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi \in \mathcal{D}(\Omega)$ and $\xi \in \mathcal{C}^1_{per}(Y^*)$, then

- $\displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon \nabla\psi_\varepsilon dx$

$$= \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla(u_\varepsilon - v_\varepsilon)\nabla\psi_\varepsilon dx + \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla v_\varepsilon \nabla\psi_\varepsilon dx$$

$$\longrightarrow \frac{1}{|Y|}\int_{\Omega\times Y^*} A(y, u_0)(\nabla u_0 + \nabla_y \widehat{u})(\nabla\varphi + \phi\nabla_y\xi)dxdy.$$

Tool 2: Decomposition of the singular term

- Split the integral of the singular term into two terms:

$$\int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\psi_\varepsilon dx = \int_{\{0<u_\varepsilon\leq\delta\}} f\zeta(u_\varepsilon)\psi_\varepsilon dx + \int_{\{u_\varepsilon>\delta\}} f\zeta(u_\varepsilon)\psi_\varepsilon dx.$$

- We assume that ψ_ε is nonnegative without loss of generality.

$$\int_{\{0<u_\varepsilon\leq\delta\}} f\zeta(u_\varepsilon)\psi_\varepsilon dx \leq \int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla\psi_\varepsilon Z_\delta(u_\varepsilon)dx$$

$$+c\varepsilon^{\gamma-\frac{1}{2}} h(2\delta)\|\rho\|_{L^\infty(\partial T)}\|\psi_\varepsilon\|_{V_\varepsilon}$$

$$\longrightarrow \; 0$$

and

$$\int_{\{u_\varepsilon>\delta\}} f\zeta(u_\varepsilon)\psi_\varepsilon dx \;\longrightarrow\; \theta\int_\Omega f\zeta(u_0)\varphi dx.$$

□

The corrector result

Let us introduce the usual corrector matrix field for perforated domains defined by

$$C^\varepsilon(\cdot, t) = \left(C^\varepsilon_{i,j}(\cdot, t)\right)_{i,j=1,\dots,N} \in \mathbb{R}^{N^2}, \quad \text{for every } t \in \mathbb{R},$$

where

$$\begin{cases} C^\varepsilon(x,t) = C\left(\dfrac{x}{\varepsilon}, t\right) & \text{a.e. in } \Omega^*_\varepsilon, \\ C_{i,j}(y,t) = \dfrac{\partial \widehat{\omega}_j}{\partial y_i}(y,t), \ \ i,j = 1, \dots, N & \text{a.e. on } Y^*. \end{cases}$$

▷ Thanks to Tool 1, we get

Theorem

Under the above assumptions, one has

$$\lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - C^\varepsilon(\cdot, u_\varepsilon)\nabla u_0\|_{L^1(\Omega^*_\varepsilon)} = 0.$$

Hoping to see you again in two years... in Houston!