Asymptotic analysis and correctors for elliptic problems in cylinders

Adrien Ceccaldi, Université de Rouen Normandie

Multi-Scale Analysis and Theory of Homogenization, ICTS 26 august - 6 september 2019

6 september 2019

Based on a joint work with S. Mardare

Outline

Outline

Outline

The framework

Let p be an integer such that $1 \leq p \leq n-1$ and ω_1 be a bounded domain of $\mathbb{R}^p$, verifying

$$\omega_1 \text{ is star-shaped with respect to an open ball of } \mathbb{R}^p \text{ centered at } 0.$$

Note that in particular, any bounded open convex set containing 0 satisfies the previous property.
Let ω_2 be a bounded Lipschitz domain of $\mathbb{R}^{n-p}$. We set

$$\Omega_\ell = \ell\omega_1 \times \omega_2 \ , \ \Omega_\infty = \mathbb{R}^p \times \omega_2 \, .$$

We note $X_1 = (x_1, \cdots, x_p) \in \ell\omega_1$ and $X_2 = (x_{p+1}, \cdots, x_n) \in \omega_2$.
For $\beta \in \mathbb{R}$, we define

$$V_\beta(\Omega_\infty) = \{ f \in L^2_{loc}(\bar{\Omega}_\infty) \mid \exists\, C_0 \geq 0 \text{ such that } \|f\|_{L^2(\Omega_\ell)} \leq C_0 e^{\beta\ell} \ \ \forall\, \ell > 0 \},$$

$$W_\beta(\Omega_\infty) = \{ f \in H^{-1}_{loc}(\bar{\Omega}_\infty) \mid \exists\, \tilde{C}_0 \geq 0 \text{ such that } \|f\|_{H^{-1}(\Omega_\ell)} \leq \tilde{C}_0 e^{\beta\ell} \ \ \forall\, \ell > 0 \}.$$

The problem in Ω_ℓ

$$\begin{cases} -div(A\nabla u_\ell) = f \quad \text{in} \quad \Omega_\ell \\ u_\ell = g \quad \text{on} \quad \partial\Omega_\ell \end{cases}$$

We consider a matrix field $A = (a_{ij})_{1\leq i,j\leq n} \in L^\infty(\Omega_\infty; \mathcal{M}_n(\mathbb{R}))$, i.e. $a_{ij} \in L^\infty(\Omega_\infty)$ for all $i,\ j \in \{1, \dots, n\}$, satisfying the following properties:

$$\lambda|\xi|^2 \leq A(x)\xi\cdot\xi \quad \forall\,\xi \in \mathbb{R}^n,\ \text{a.e. } x \in \Omega_\infty,$$
$$|A(x)\xi| \leq \Lambda|\xi| \quad \forall\,\xi \in \mathbb{R}^n,\ \text{a.e. } x \in \Omega_\infty,$$

for some constants $\lambda > 0$ and $\Lambda > 0$.
We take $f \in H^{-1}_{loc}(\bar{\Omega}_\infty)$ and $g \in H^1_{loc}(\bar{\Omega}_\infty)$, so that $f \in H^{-1}(\Omega_\ell)$ and $g \in H^1(\Omega_\ell)$ for any $\ell > 0$.

Thanks to the Lax-Milgram theorem, there exists a unique weak solution to the previous problem, i.e. there exists a unique solution $u_\ell \in H^1(\Omega_\ell)$ to the variational problem:

$$\begin{cases} \displaystyle\int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla v \, dx = \langle f, v\rangle \quad \text{for all } \; v \in H^1_0(\Omega_\ell)\,, \\ u_\ell = g \quad \text{on } \partial\Omega_\ell\,. \end{cases}$$

Notice that the last equality is to be taken is the sense of the trace theory in $H^1(\Omega_\ell)$, i.e. u_ℓ satisfies $\gamma(u_\ell) = g$ on $\partial\Omega_\ell$.

The type of result are interested in

Does u_ℓ converges as ℓ goes to infinity, and if this is the case, what is its limit ?

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_?)} \to 0 \text{ as } \ell \to +\infty,$$

if possible with a good rate of convergence. Then, if we want $\Omega_? = \Omega_\ell$, we can add a corrector. Hence we look for a w_ℓ such that

$$\|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_\ell)} \to 0 \text{ as } \ell \to +\infty.$$

Outline

Chipot, M. and Rougirel, A. (2002)

- *On the asymptotic behaviour of the solution of elliptic problems in cylindrical domains becoming unbounded*

The authors consider the case where $A = A(X_2)$, $f = f(X_2)$. Then, for all fixed $\ell_0 > 0$ and all $\gamma > 0$, there exist a constant C not depending on ℓ such that

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell_0})} \leq C\ell^{-\gamma} \text{ for all } \ell > \ell_0.$$

Chipot, M. and Yeressian, K. (2008)

- *Exponential rates of convergence by an iteration technique*

Theorem

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-\alpha\ell} \quad \textit{for all } \ell > 0,$$

where $\alpha > 0$ is a constant independant of ℓ.

With the following assumptions: $f = f(X_2)$, $f \in H^{-1}(\omega_2)$ and $A(x) = \begin{pmatrix} A_{11}(X_1, X_2) & A_{12}(X_2) \\ A21(X_1, X_2) & A_{22}(X_2) \end{pmatrix}$ where A_{11} is a $p \times p$ and A_{22} is a $(n-p) \times (n-p)$ matrix.

Chipot, M. and Mardare, S. (2008)

- *Asymptotic behaviour of the Stokes problem in cylinders becoming unbounded in one direction*

The introduction of an important technique : proving that u_ℓ is a "Cauchy sequence". This allows to consider more general hypotheses on the data (the matrix field A and f) and to construct the limit u_∞ as a solution in the infinite cylinder.

Chipot, M. (2014, 2016)

This general result is proved by Chipot in his book *Asymptotic Issues for Some Partial Differential Equations* of 2016:

Theorem

Under the general assumptions from the first part and for β *small enough, we have*

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-\alpha\ell} \quad \textit{for all } \ell > 0.$$

The proof uses the two techniques introduced in the previously mentioned works.

Remark

If $\|f\|_{H^{-1}(\Omega_\ell)} = O(\ell^\gamma)$ *(*$\gamma > 0$ *constant), then using the same iteration technique we obtain*

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-\eta \ln(\ell)})} \underset{\ell \to +\infty}{\longrightarrow} 0,$$

for some constant $\eta > 0$ *large enough.*

Outline

A Poincaré inequality to start with

Lemma

Let $v \in H^1(\Omega_\ell)$ such that $v = 0$ on $\ell\omega_1 \times \partial\omega_2$ and $\tilde{\omega}_1 \subset \ell\omega_1$ a measurable set. Then there exists a constant c_{ω_2} depending only on ω_2 such that:

$$\|v\|_{L^2(\tilde{\omega}_1\times\omega_2)} \leq c_{\omega_2}\|\nabla_{X_2} v\|_{L^2(\tilde{\omega}_1\times\omega_2)} .$$

Theorem

Let $f \in W_\beta(\Omega_\infty)$ *and* $g \in H^1_{loc}(\bar{\Omega}_\infty)$ *such that* $\nabla g \in (V_\beta(\Omega_\infty))^n$ *for some small enough* $\beta > 0$ *and let* $u_\ell \in H^1(\Omega_\ell)$ *be the solution of the variational problem in* Ω_ℓ*. Then for all* $\ell_0 > 0$,

$$u_\ell \to u_\infty \text{ strongly in } H^1(\Omega_{\ell_0})$$

as $\ell \to \infty$*, where* $u_\infty \in H^1_{loc}(\bar{\Omega}_\infty)$ *is the weak solution to the following non-homogenous Dirichlet problem in the cylinder* Ω_∞*:*

$$\begin{cases} -\operatorname{div}(A\nabla u_\infty) = f \text{ in } \Omega_\infty \\ u_\infty = g \text{ on } \partial\Omega_\infty \\ \|\nabla u_\infty\|_{L^2(\Omega_\ell)} \leq C_\infty e^{2\beta\ell} \quad \forall\, \ell > 0, \end{cases}$$

with $C_\infty \geq 0$ *not depending on* ℓ*.*
Furthermore, we have the estimate

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-\alpha\ell} \quad \text{for all } \ell > 0\,.$$

Remarks

- The variational formulation corresponding to the first equation of the previous problem is

$$\int_{\Omega_\infty} A\nabla u_\infty \cdot \nabla v \, dx = \langle f, v\rangle \quad \text{for all } \; v \in \bigcup_{\ell>0} H_0^1(\Omega_\ell)$$

- If the boundary condition in the problem verified by u_ℓ is replaced by $u_\ell = g_\ell$ on $\partial\Omega_\ell$ (with $g_\ell \in H^{\frac{1}{2}}(\partial\Omega_\ell)$ satisfying $g_\ell = g$ on $\partial\Omega_\ell \cap \partial\Omega_\infty$), the condition $\nabla g \in (V_\beta(\Omega_\infty))^n$ should be replaced by

$$\|g_\ell\|_{H^{\frac{1}{2}}(\partial\Omega_\ell)} \leq \; C_0 e^{\beta\ell} \; \text{ for all } \; \ell > 0\,,$$

  where the norm on $H^{\frac{1}{2}}(\partial\Omega_\ell)$ is given by

$$\|w\|_{H^{\frac{1}{2}}(\partial\Omega_\ell)} = \; \inf\{\|h\|_{H^1(\Omega_\ell)} \; ; \; h \in H^1(\Omega_\ell) \text{ and } \gamma(h) = w \text{ on } \partial\Omega_\ell\}.$$

- The growth condition is mandatory in order to have the uniqueness of the solution.

Proof, Step I

There exists a constant $a \in (0,1)$ only depending on ω_1, ω_2, λ and Λ such that

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 \, dx \leq a \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 \, dx$$

for all $\ell > 1$, for all $\ell_1 \leq \ell - 1$ and for all $r \geq 0$.

- We define a function $\rho \in W^{1,\infty}(\mathbb{R}^p \times \omega_2)$ only depending on X_1 such that $0 \leq \rho \leq 1$, $\rho = 1$ on Ω_{ℓ_1}, $\rho = 0$ on $(\mathbb{R}^p \times \omega_2) \setminus \Omega_{\ell_1+1}$, and $|\nabla_{X_1}\rho| \leq c_0$ in Ω_ℓ, with c_0 a constant depending only on ω_1, and therefore independent of ℓ_1 or ℓ.
- The function $v = \rho(u_\ell - u_{\ell+r})$ is in $H_0^1(\Omega_\ell)$ and therefore can be used as test function is the variationnal equation verified by $u_\ell - u_{\ell+r}$.

Proof, Step I (2)

- Observing that $\nabla_{X_1}\rho = 0$ in $\Omega_{\ell_1} \cup (\Omega_\ell \setminus \Omega_{\ell_1+1})$, we have

$$
\begin{aligned}
\lambda \int_{\Omega_\ell} \rho |\nabla(u_\ell - u_{\ell+r})|^2 \, dx &\leq \int_{\Omega_\ell} A\nabla(u_\ell - u_{\ell+r}) \cdot \rho \nabla(u_\ell - u_{\ell+r}) \, dx \\
&\leq \int_{\Omega_{\ell_1+1}\setminus\Omega_{\ell_1}} |A\nabla(u_\ell - u_{\ell+r})||\nabla_{X_1}\rho||u_\ell - u_{\ell+r}| \, dx \\
&\leq c_0\Lambda \int_{\Omega_{\ell_1+1}\setminus\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})||u_\ell - u_{\ell+r}| \, dx \\
&\leq c_0\Lambda \|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\ell_1+1}\setminus\Omega_{\ell_1})} \|(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\ell_1+1}\setminus\Omega_{\ell_1})} \\
&\leq \; c_0\Lambda c_{\omega_2} \int_{\Omega_{\ell_1+1}\setminus\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 \, dx
\end{aligned}
$$

which is

$$
\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 \, dx \leq \frac{C}{1+C} \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 \, dx \, .
$$

Proof, Step II

There exists constants $C \geq 0$ and $\alpha > 0$, depending only on ω_1, ω_2, λ, Λ, C_0, $\tilde{C}_0$ and β, such that

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-\alpha\ell} \quad \textit{for all } \ell > 0 \textit{ and all } r \in [0,1].$$

- We start by iterating the inequality of Step I $[\frac{\ell}{2}]$ times, which leads to

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 \, dx \leq a^{\frac{\ell}{2}-1} \int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 \, dx$$

  and therefore

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq ce^{-\tilde{\alpha}\ell} \|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_\ell)} .$$

Proof, Step II (2)

- The function $z_\ell = u_\ell - g$ is in H_0^1 and can be used as test function, leading (after a good use of the properties verified by f, g and A) to

$$\|\nabla z_\ell\|_{L^2(\Omega_\ell)} \leq Ce^{\beta\ell}.$$

- Since $u_\ell - u_{\ell+r} = z_\ell - z_{\ell+r}$ on Ω_ℓ, we have

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_\ell)} = \|\nabla(z_\ell - z_{\ell+r})\|_{L^2(\Omega_\ell)} \leq Ce^{\beta\ell}.$$

- Combining our inequalities gives

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-(\tilde{\alpha}-\beta)\ell}.$$

Proof, Step III

There exists two constants $C \geq 0$ and $\alpha > 0$ depending only on ω_1, ω_2, λ, Λ, C_0, $\tilde{C}_0$ and β such that

$$\|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-\alpha\ell},$$

for all positive ℓ and all non-negative t.

$$\begin{aligned}\|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell}{2}})} &\leq \sum_{i=0}^{[t]-1} \|\nabla(u_{\ell+i} - u_{\ell+i+1})\|_{L^2(\Omega_{\frac{\ell}{2}})} \\ &\qquad + \|\nabla(u_{\ell+[t]} - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell}{2}})} \\ &\leq C\frac{1}{1-e^{-\alpha}}e^{-\alpha\ell}.\end{aligned}$$

Proof, Step IV

There exists $u_\infty \in H^1_{loc}(\bar{\Omega}_\infty)$ such that for all $\ell_0 > 0$, $u_\ell \to u_\infty$ in $H^1(\Omega_{\ell_0})$, and $u_\ell - u_\infty$ verifies

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-\alpha\ell} \quad \textit{for all } \ell > 0\,,$$

for some constants $C \geq 0$ and $\alpha > 0$, depending only on $\omega_1,\ \omega_2,\ \lambda$, Λ, C_0, $\tilde{C}_0$ and β.

- By Poincaré inequality,

$$\|u_\ell - u_{\ell+t}\|_{H^1(\Omega_{\ell_0})} \leq C\|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\ell_0})} \leq Ce^{-\alpha\ell}.$$

- It implies that $(u_\ell)_{\ell>0}$ is a Cauchy “sequence” for the norm of in the Banach space $H^1(\Omega_{\ell_0})$.
- We finally set

$$u_\infty = \begin{cases} u^1_\infty \text{ in } \Omega_1 \\ u^k_\infty \text{ in } \Omega_k \setminus \Omega_{k-1} \quad \text{for all } k \geq 2. \end{cases}$$

Proof, Step V

The limit u_∞ from the previous step is a solution to the variationnal problem given in the theorem.

- Fixing ℓ_0 and taking v in H_0^1, we have

$$\int_{\Omega_{\ell_0}} A\nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla v \, dx = \langle f, v\rangle \, .$$

- Since $\nabla u_\ell \to \nabla u_\infty$ strongly in $(L^2(\Omega_{\ell_0}))^n$, letting ℓ go to infinity leads to

$$\int_{\Omega_\infty} A\nabla u_\infty \cdot \nabla v \, dx = \int_{\Omega_{\ell_0}} A\nabla u_\infty \cdot \nabla v \, dx = \langle f, v\rangle \, .$$

- We then have that u_∞ satisfies the variational equation given in the remarks, since ℓ_0 is taken arbitrarily.

Proof, Step V (2)

- Keeping in mind that $u_\ell \to u_\infty$ in $H^1(\Omega_{\ell_0})$ and using the continuity of the trace operator,$\gamma(u_\ell) \to \gamma(u_\infty)$ in $L^2(\partial\Omega_{\ell_0})$ and particularly in $L^2(\ell_0\omega_1 \times \partial\omega_2)$. Thus, $\gamma(u_\infty) = g$ on $\ell_0\omega_1 \times \partial\omega_2$.
- Since ℓ_0 is arbitrarily taken, we derive $\gamma(u_\infty) = g$ on $\partial\Omega_\infty = \mathbb{R}^p \times \partial\omega_2 = \bigcup_{\ell_0 > 0} (\ell_0\omega_1 \times \partial\omega_2)$.
- Finally, the last inequality given in the problem from the theorem comes this way:

$$
\begin{aligned}
\|\nabla u_\infty\|_{L^2(\Omega_\ell)} &\leq \|\nabla(u_\infty - u_{2\ell})\|_{L^2(\Omega_\ell)} + \|\nabla u_{2\ell}\|_{L^2(\Omega_\ell)} \\
&\leq \|\nabla(u_\infty - u_{2\ell})\|_{L^2(\Omega_\ell)} + \|\nabla u_{2\ell}\|_{L^2(\Omega_{2\ell})} \\
&\leq C(e^{-2\alpha\ell} + e^{2\beta\ell}) \\
&\leq C_\infty e^{2\beta\ell}.
\end{aligned}
$$

Proof, Step VI

There exists a unique solution to the variationnal problem given in the theorem.

- Let $u_\infty, \tilde{u}_\infty$ be two solutions of the problem. Using the computations made for u_ℓ an $u_{\ell+r}$, we have

$$\int_{\Omega_{\ell_1}} |\nabla(u_\infty - \tilde{u}_\infty)|^2 \, dx \leq a \int_{\Omega_{\ell_1+1}} |\nabla(u_\infty - \tilde{u}_\infty)|^2 \, dx.$$

- It follows that

$$\|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} \leq a^{\frac{1}{2}} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1+1})}.$$

- Iterating this inequality k times gives

$$\begin{aligned} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} &\leq a^{\frac{k}{2}} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1+k})} \\ &= e^{-2\tilde{\alpha}k} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1+k})}. \end{aligned}$$

Proof, Step VI (2)

- Combining the previous inequality with the one verified by the gradients of u_∞ and $\tilde{u}_\infty$, we have

$$\begin{aligned}\|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} &\leq 2C_\infty e^{2\beta(\ell_1+k)} e^{-2\tilde{\alpha}k} \\ &= 2C_\infty e^{2\beta\ell_1} e^{-2(\tilde{\alpha}-\beta)k}.\end{aligned}$$

- Fixing ℓ_1 and making k go to infinity, we have that $\|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} = 0$, since $\beta < \tilde{\alpha}$.
- On the boundary, $u_\infty = \tilde{u}_\infty = g$ on $\partial\Omega_\infty$, implying $u_\infty - \tilde{u}_\infty = 0$ on $\partial\Omega_\infty$.
- It follows that $u_\infty - \tilde{u}_\infty = 0$ a.e. in Ω_{ℓ_1}, and since ℓ_1 was arbitrarily chosen, this leads to $u_\infty - \tilde{u}_\infty = 0$ a.e. in Ω_∞.

□

Remark

If we consider that A is divided into four blocks

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is a $p \times p$ matrix and A_{22} a $(n-p) \times (n-p)$ matrix, then if the data of the problem satisfy the conditions

$$A_{12} = A_{12}(X_2), \quad A_{22} = A_{22}(X_2), \quad f = f(X_2), \quad g = g(X_2),$$

with $f \in H^{-1}(\omega_2)$ and $g \in H^1(\omega_2)$, we retrieve the result by M. Chipot and K. Yeressian.

Outline

Outline

Some previous results

- Chipot, M. and Mardare, S. (2008)
  On correctors for the Stokes problem in cylinders
- Chipot, M. and Guesmia, S. (2010)
  Correctors for some asymptotic problems

In the second work, the authors are constructing correctors for the case of the laplacian (i.e. $A = I_n$) with $f = f(X_2)$. Then, one can construct a convinient corrector $w_\ell \in H^1(\Omega_\ell)$ such that

$$\|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_\ell)} \leq Ce^{-\alpha\ell}.$$

The framework

For this work, we only consider cylinders Ω_ℓ of the form

$$\Omega_\ell = (-\ell, \ell) \times \omega, \text{ i.e. } \omega_1 = (-1, 1) \text{ and } \omega_2 = \omega \in \mathbb{R}^{n-1}$$

where ω is a bounded Lipschitz domain of $\mathbb{R}^{n-1}$, and $f \in H^{-1}_{loc}(\bar{\Omega}_\infty)$ satisfying

$$\|f\|_{H^{-1}(\Omega_\ell)} \leq Ce^{\beta\ell} \qquad \forall\ \ell > 0.$$

A first remark

Observe first that for a given ℓ, the perfect corrector would be $w_\ell = u_\ell - u_\infty$, verifying

$$\begin{cases} w_\ell \in H^1(\Omega_\ell) \\ \displaystyle\int_{\Omega_\ell} A\nabla w_\ell \cdot \nabla v \, dx = 0 \text{ for all } v \in H^1_0(\Omega_\ell) \\ w_\ell = -u_\infty \text{ on } \{-\ell\} \times \omega \text{ and on } \{\ell\} \times \omega \\ w_\ell = 0 \quad \text{on } (-\ell, \ell) \times \partial\omega. \end{cases}$$

The construction

We build our corrector separately on $\Omega_\ell^+ = (0,\ell) \times \omega$ and on $\Omega_\ell^- = (-\ell, 0) \times \omega$. Thus, w_ℓ will be defined as

$$w_\ell = \begin{cases} w_\ell^- \text{ on } \Omega_\ell^- \\ w_\ell^+ \text{ on } \Omega_\ell^+ \end{cases}$$

where w_ℓ^+ is the solution to the problem

$$\begin{cases} w_\ell^+ \in H^1((-\infty,\ell)\times\omega) \\ \displaystyle\int_{(-\infty,\ell)\times\omega} A\nabla w_\ell^+ \cdot \nabla v \, dx = 0 \quad \forall \ v \in H_0^1((-\infty,\ell)\times\omega) \\ w_\ell^+ = -u_\infty(\ell,\cdot) \text{ on } \{\ell\}\times\omega \\ w_\ell^+ = 0 \quad \text{on } (-\infty,\ell)\times\partial\omega. \end{cases}$$

The function $w_\ell^- \in ((-\ell,+\infty)\times\omega)$ is contructed in a similar way.

As one could notice, this corrector w_ℓ belongs to $H^1(\Omega_\ell \backslash (\{0\} \times \omega))$ but does not necessarily belong to $H^1(\Omega_\ell)$.
To get a $H^1(\Omega_\ell)$-corrector, it is enough to consider the corrector $\tilde{w}_\ell = \psi(x_1) w_\ell$, with $\psi : \mathbb{R} \to \mathbb{R}$ a Lipschitz-continuous function such that

$$\psi(x_1) = \begin{cases} 1 & \text{if } |x_1| > 1 \\ 0 & \text{if } |x_1| < \frac{1}{2}. \end{cases}$$

First step

To start with, we have the following result:

Theorem

There exists a unique solution w_ℓ^+ to the problem

$$\begin{cases} w_\ell^+ \in H^1((-\infty,\ell)\times\omega) \\ \displaystyle\int_{(-\infty,\ell)\times\omega} A\nabla w_\ell^+ \cdot \nabla v \, dx = 0 \quad \forall\, v \in H_0^1((-\infty,\ell)\times\omega) \\ w_\ell^+ = -u_\infty(\ell,\cdot) \text{ on } \{\ell\}\times\omega \\ w_\ell^+ = 0 \quad \text{on } (-\infty,\ell)\times\partial\omega, \end{cases}$$

Some intermediate results

Lemma

We have

$$\|\nabla w_\ell^+\|_{L^2((-\infty,\ell)\times\omega)} \leq Ce^{2\beta\ell},$$

where $C > 0$ is a constant depending only on λ, Λ, ω and C_0.

Theorem

For the functions u_ℓ, u_∞ and w_ℓ^+, there exists a constant C such that

$$\|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)} \leq C(\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_1^+)} + \|\nabla w_\ell^+\|_{L^2(\Omega_1)})$$

Some intermediate results (2)

Theorem

Let $k \in (-\infty, \ell - 1]$ *be fixed. Then there exist* $C \geq 0$ *depending only on* Λ, λ, k *and* ω, *and* $\alpha > 0$ *depending only on* Λ, λ *and* ω *such that*

$$\|\nabla w_\ell^+\|_{L^2((-\infty,k)\times\omega)} \leq Ce^{-\alpha\ell}\|\nabla w_\ell^+\|_{L^2((-\infty,\ell)\times\omega)}.$$

Proof

Let us first denote by $h = h(x_1)$ the function defined by

$$h(x_1) = \begin{cases} 1 & \text{for } x_1 \leq k \\ 0 & \text{for } x_1 \geq k+1 \\ -x_1 + k + 1 & \text{for } k < x_1 < k+1. \end{cases}$$

Then, for $k + 1 < \ell$,

$$hw_\ell^+ \in H_0^1((-\infty, \ell) \times \omega)$$

so that we have

$$\int_{(-\infty,\ell)\times\omega} A\nabla w_\ell^+ \nabla(hw_\ell^+)\, dx = 0.$$

This implies

$$\lambda \int_{(-\infty,k)\times\omega} |\nabla w_\ell^+|^2\, dx \leq \Lambda C_\omega \|\nabla w_\ell^+\|^2_{L^2((k,k+1)\times\omega)},$$

using the Cauchy-Schwarz and the Poincaré inequalities.

Proof (2)

Consequently

$$\int_{(-\infty,k)\times\omega} |\nabla w_\ell^+|^2\, dx \leq \frac{\Lambda C_\omega}{\lambda} \int_{(k,k+1)\times\omega} |\nabla w_\ell^+|^2\, dx$$
$$= \frac{\Lambda C_\omega}{\lambda} \int_{(-\infty,k+1)\times\omega} |\nabla w_\ell^+|^2\, dx -$$
$$\frac{\Lambda C_\omega}{\lambda} \int_{(-\infty,k)\times\omega} |\nabla w_\ell^+|^2\, dx$$

which, for $C = \frac{\Lambda C_\omega}{\lambda}$, reads

$$\int_{(-\infty,k)\times\omega} |\nabla w_\ell^+|^2\, dx \leq \left(\frac{C}{C+1}\right) \int_{(-\infty,k+1)\times\omega} |\nabla w_\ell^+|^2\, dx.$$

Some intermediate results (3)

Theorem

Assuming that the constant β *is small enough, there exists* C *depending only on* Λ, λ, C_0 *and* ω *and* α *depending only on* Λ, λ, ω *and* β *two positive constants such that*

$$\|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)} \leq Ce^{-\alpha\ell}$$

for all $\ell > 0$.

Proof.

We know that

$$\|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)} \leq C(\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_1^+)} + \|\nabla w_\ell^+\|_{L^2(\Omega_1^+)}).$$

Combining this inequality with some previous ones, we directly have, since $\Omega_1^+ \subset (-\infty, 1) \times \omega$,

$$\begin{aligned}\|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)} &\leq C(Ce^{-\alpha\ell} + Ce^{-\alpha\ell}\|\nabla w_\ell^+\|_{L^2((-\infty,\ell)\times\omega)}) \\ &\leq C(Ce^{-\alpha\ell} + Ce^{-\alpha\ell}e^{2\beta\ell}) \leq Ce^{-\alpha\ell}\end{aligned}$$

where the constant α is positive. □

The result we were looking for

Theorem

There exists some positive constants C and α such that

$$\|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_\ell)} \leq Ce^{-\alpha\ell}$$

for all $\ell > 0$, where w_ℓ is our corrector.

Here, ∇ is taken in the $\mathcal{D}'(\Omega_\ell \setminus \{0\} \times \omega)$-sense, since $w_\ell \notin H^1(\Omega_\ell)$.

Outline

In this part of the talk, we take as hypotheses that $A = A(X_2)$ and $f = f(X_2)$. Note that in this case, $\|f\|_{H^{-1}(\Omega_\ell)} = O(\ell^{\frac{1}{2}})$.
The functions w_ℓ^+ and w_ℓ^- are solutions of the variational problems

$$\begin{cases} w_\ell^+ \in H^1((-\infty,\ell)\times\omega) \\ \displaystyle\int_{(-\infty,\ell)\times\omega} A\nabla w^+ \cdot \nabla v \, dx = 0 \quad \forall\ v \in H_0^1((-\infty,\ell)\times\omega) \\ w_\ell^+ = -u_\infty \quad \text{on } \{\ell\}\times\omega \\ w_\ell^+ = 0 \quad \text{on } (-\infty,\ell)\times\partial\omega \end{cases}$$

and

$$\begin{cases} w_\ell^- \in H^1((-\ell,+\infty)\times\omega) \\ \displaystyle\int_{(-\ell,+\infty)\times\omega} A\nabla w^- \cdot \nabla v \, dx = 0 \quad \forall\ v \in H_0^1((0,+\infty)\times\omega) \\ w_\ell^- = -u_\infty \quad \text{on } \{-\ell\}\times\omega \\ w_\ell^- = 0 \quad \text{on } (-\ell,+\infty)\times\partial\omega. \end{cases}$$

In this setting, the functions w_ℓ^+ and w_ℓ^- can be obtained by translation from a fixed function:

$$w_\ell^+(x_1, X_2) = w^+(x_1 - \ell, X_2),$$

where w^+ is defined by

$$\begin{cases} w^+ \in H^1((-\infty,0)\times\omega) \\ \displaystyle\int_{(-\infty,0)\times\omega} A\nabla w^+ \nabla v\, dx = 0 \quad \forall\ v \in H_0^1((-\infty,0)\times\omega) \\ w^+ = -u_\infty \quad \text{on } \{0\}\times\omega \\ w^+ = 0 \quad \text{on } (-\infty,0)\times\partial\omega \end{cases}$$

and

$$w_\ell^- = w^-(x_1 + \ell, X_2)$$

with w^- defined in a similar way as w^+.

The optimality result

We remind that with the iteration method introduced in Chipot-Yeressian (2008), one can prove that

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-\eta\ln(\ell)})} \underset{\ell\to+\infty}{\longrightarrow} 0,$$

for some constant $\eta > 0$ large enough.
The result above is almost optimal in the following sense: we are looking for the largest domain that can be considered in the L^2-norm appearing in the previous slide such that the convergence of u_ℓ to u_∞ described remains valid.
Using the properties of w^+ and w^-, we can justify that in general the following negative result takes place:
For a constant in $a > 0$, we have that in general

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-a})} \not\to 0.$$

Justification

Let us suppose that, on the contrary,

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-a})} \to 0.$$

Then, since

$$\|\nabla w_\ell\|_{L^2(\Omega_{\ell-a})} \leq \|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_{\ell-a})} + \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-a})}$$

and both $\|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_{\ell-a})} \to 0$ and $\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-a})} \to 0$ as ℓ goes to $+\infty$, we have

$$\|\nabla w_\ell\|_{L^2(\Omega_{\ell-a})} \to 0.$$

Obviously $\|\nabla w_\ell^+\|_{L^2(\Omega^+_{\ell-a})} = \|\nabla w^+\|_{L^2((\ell-a)\times\omega)}$. Since the function $\ell \mapsto \|\nabla w^+\|_{L^2((-\ell,-a)\times\omega)}$ is positive and increasing, it follows that $\|\nabla w^+\|_{L^2((-\infty,-a)\times\omega)} = 0$.
Using the Poincaré inequality, we deduce that

$$w^+ = 0 \text{ in } (-\infty, -a) \times \omega.$$

Let us now give an example where this is impossible:

- If the coefficients of A are analytic, then it follows from its definition that w^+ is analytic on $(-\infty,\ 0) \times \omega$. Using the previous equality we therefore derive $w^+ = 0$ in $(-\infty,\ 0) \times \omega$, which contradicts $w^+ = -u_\infty$ on $\{0\} \times \omega$ in the case where $u_\infty \neq 0$, i.e. if $f \neq 0$.

Thank you very much for having paid attention !