Asymptotic analysis of a Boundary Optimal Control Problem

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Over view of my talk

- Domain description

Over view of my talk

- Domain description
- Problem description

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- Problem description
- Limit Problem

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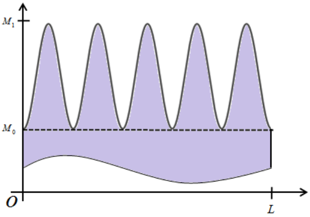
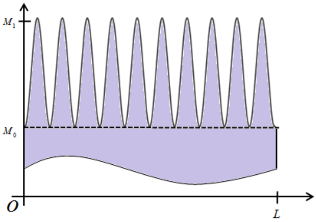
- Main results

Domain description

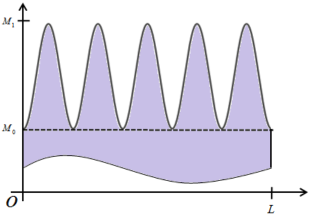


$\varepsilon = \frac{1}{5}$

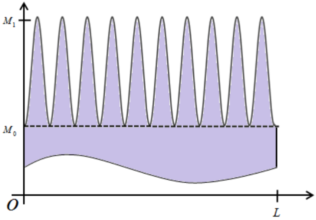


$\varepsilon = \frac{1}{10}$

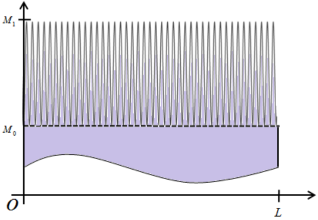
Domain description



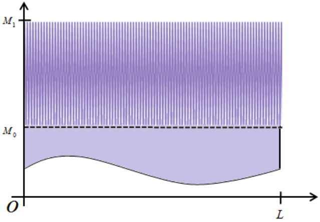
$\varepsilon = \frac{1}{5}$



$\varepsilon = \frac{1}{10}$



$\varepsilon = \frac{1}{50}$



$\varepsilon = \frac{1}{100}$

Domain description

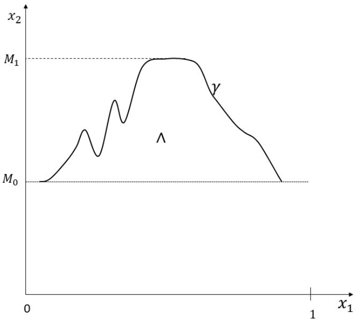


Figure: Reference cell (Graph of η)

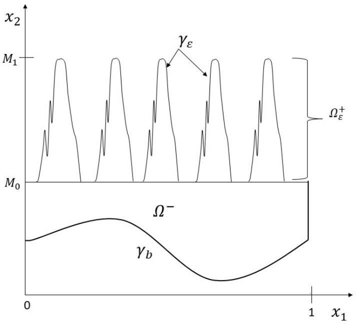


Figure: Oscillating Domain $\varepsilon = \frac{1}{5}$

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Optimal control problem

- For $\theta \in L^2_{per}(\gamma)$ define

$$\theta^\varepsilon(y', \eta(y')) = \tau_\varepsilon(y', \eta(y'))\theta(y', \eta(y')),$$

where,

$$\tau_\varepsilon(y', \eta(y')) = \left(\chi_{F_{M_0}} + \chi_{F_{M_1}} + \varepsilon^\alpha \frac{\sqrt{|\nabla\eta|}\sqrt{1+|\nabla\eta|^2}}{\sqrt{\varepsilon^2+|\nabla\eta|^2}}\chi_S\right)(y', \eta(y')),$$

for $\alpha \geq 1$

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  for $\alpha \geq 1$
- $\hat{\theta}^\varepsilon(y', \eta(y')) = \theta^\varepsilon\left(\frac{y'}{\varepsilon}, \eta\left(\frac{y'}{\varepsilon}\right)\right).$
- State equation

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f \quad in \quad \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \hat{\theta}^\varepsilon \quad on \quad \gamma_\varepsilon, \\ u_\varepsilon = 0 \quad on \ \gamma_b, \\ u_\varepsilon \quad is \quad \gamma_s \quad periodic. \end{cases} \tag{1}$$

- For $\theta \in L^2_{per}(\gamma)$, we consider the following L^2-cost functional

$$J_\varepsilon(u_\varepsilon, \theta) = \frac{1}{2} \int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2} \int_\gamma |\theta|^2.$$

- Now consider the following optimal control problem: find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1_\varepsilon \times L^2_{per}(\gamma)$, satisfies PDE (1) such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = inf\{J_\varepsilon(u_\varepsilon, \theta)| \ \ \theta \in L^2_{per}(\gamma)\}. \tag{2}$$

Theorem 1

For each $\varepsilon > 0$, the minimization problem (2) admits a unique solution $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1_\varepsilon \times L^2_{per}(\gamma)$.

Characterization

- Adjoint state equation

$$\begin{cases} -\Delta \bar{v}_\varepsilon + \bar{v}_\varepsilon = \bar{u}_\varepsilon - u_d \quad \text{in} \quad \Omega_\varepsilon, \\ \dfrac{\partial \bar{v}_\varepsilon}{\partial \nu_\varepsilon} = 0 \quad \text{on} \quad \gamma_\varepsilon, \\ \bar{v}_\varepsilon = 0 \quad \text{on } \gamma_b, \\ \bar{v}_\varepsilon \quad \text{is} \quad \gamma_s \quad \text{periodic.} \end{cases} \tag{3}$$

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Theorem 2

Given $f \in L^2(\Omega)$, and let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem (2) and $\bar{v}_\varepsilon$ satisfies (3). Then, the optimal control $\bar{\theta}_\varepsilon \in L^2_{per}(\gamma)$ is given by

$$\begin{cases} \bar{\theta}_\varepsilon\big|_S = -\dfrac{\varepsilon^{\alpha-1}\sqrt{|\bigtriangledown_{y'}\eta|(y')}}{\beta}\displaystyle\int_{(0,1)^2} T^\varepsilon \bar{v}_\varepsilon(x', \eta(y'), y')dx' \\ \bar{\theta}_\varepsilon\big|_{F_{M_i}} = -\dfrac{1}{\beta}\displaystyle\int_{(0,1)^2} T_i^\varepsilon \bar{v}_\varepsilon(x', M_i, y')dx', \ i = 0, 1. \end{cases}$$

Converse is also true,

Limit domain

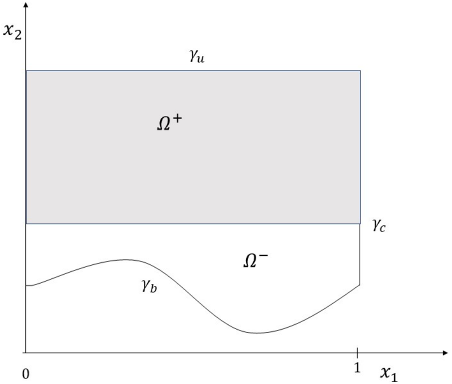

Figure: Limit Domain

Notations

- For $x_3 \in (M_0, M_1)$, $h(x_3) = |Y(x_3) = \{x' \in (0,1)^2 : \ \eta(x') > x_3\}$, and $h(M_i) = |\{x \in (0,1)^2 : \ \eta(x') = M_i\}|$ for $i = 0, 1$.

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- For each $x_3 \in [M_0, M_1]$, we denote $d\gamma(x_3)$ is the surface measure on the level curve

$$\gamma_{x_3}(x') = \{(x', x_3)\big| \ x' \in (0,1)^2, \ \eta(x') = x_3\}.$$

We define ω on (M_0, M_1) by

$$\omega(x_3) = \int_{\gamma_{x_3}} \sqrt{1 + |\nabla_{x'}\eta(x')|^2} d\gamma(x_3).$$

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$$\omega(x_3) = \int_{\gamma_{x_3}} \sqrt{1 + |\nabla_{x'}\eta(x')|^2} d\gamma(x_3).$$
- The Limit solution space
$$W(\Omega) = \left\{ \psi \in L^2(\Omega) | \ \ \frac{\partial \psi}{\partial x_3} \in L^2(\Omega), \quad \psi \in H^1(\Omega^-) \right\}$$
with the following inner product
$$\langle u, v \rangle_{W(\Omega)} = \langle hu, v \rangle_{L^2(\Omega^+)} + \left\langle h \frac{\partial u}{\partial x_3}, \frac{\partial v}{\partial x_3} \right\rangle_{L^2(\Omega^+)} + \langle u, v \rangle_{H^1(\Omega^-)}.$$
Note that $W(\Omega)$ is a Hilbert space with respect to the given inner product.

Limit optimal control problem $\alpha = 1$

- For $f \in L^2(\Omega), \varrho_1, \varrho_2 \in \mathbb{R},\ \theta \in L^2(M_0, M_1)$, consider the following optimal control problem: find $(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2) \in W(\Omega) \times L^2(M_0, M_1) \times \mathbb{R} \times \mathbb{R}$ such that

$$J(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2) = inf \left\{ J(u, \theta, \varrho_1, \varrho_2) \middle|\ \ \varrho_1, \varrho_2 \in \mathbb{R},\ \theta \in L^2(M_0, M_1) \right\} \tag{4}$$

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- where J is an L^2-cost functional defined as

$$J(u, \theta, \varrho_1, \varrho_2) = \frac{1}{2} \int_{\Omega^+} (h(x_3)\chi_{\Omega+} + \chi_{\Omega-}) |u - u_d|^2 + \frac{\beta}{2} \int_{M_0}^{M_1} \frac{1}{|\omega(x_3)|} |\theta(x_3)|^2 + \frac{\beta h(M_1)}{2|Y_{M_1}|} \varrho_2^2 + \frac{\beta}{2|Y_{M_0}|} \varrho_1^2.$$

For the controls $\theta \in L^2(M_0, M_1)$, $\varrho_1, \varrho_2 \in \mathbb{R}$, u solves the following PDE

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(h(x_3)\dfrac{\partial u^+}{\partial x_3}\right) + h(x_3)u^+ = h(x_3)f^+ + \theta \ \text{ in } \Omega^+, \\ -\Delta u^- + u^- = f^- \text{ in } \Omega^- \\ \dfrac{\partial u^+}{\partial x_3} = \varrho_2, \\ u^+ = u^-, \ \dfrac{\partial u^-}{\partial x_3} - h(M_0)\dfrac{\partial u^+}{\partial x_3} = \varrho_1 \text{ on } \gamma_c, \\ u^- = 0 \ \text{ on } \ \gamma_b, \text{ and } u \text{ is } \ \gamma_s \text{ periodic.} \end{cases} \tag{5}$$

Theorem 3

The optimal control problem (4) admits a unique solution.

Convergence

Theorem 4

Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ *and* $(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2)$ *be the solution of the problem (2) and (4) respectively, then,*

$$\begin{aligned}
&\widetilde{\bar{u}_\varepsilon^+} \rightharpoonup h(x_3)\bar{u}^+ \ \textit{weakly in } L^2((0,1)^2; H^1(M_0, M_1)),\\
&\widetilde{\bar{v}_\varepsilon^+} \rightharpoonup h(x_3)\bar{v}^+ \ \textit{weakly in } L^2((0,1)^2; H^1(M_0, M_1)),\\
&\bar{u}_\varepsilon^- \rightharpoonup \bar{u}^- \textit{in weakly } H^1(\Omega^-),\\
&\bar{v}_\varepsilon^- \rightharpoonup \bar{v}^- \ \textit{weakly in } H^1(\Omega^-),\\
&\langle \hat{\bar{\theta}}_\varepsilon, \phi\rangle_{L^2(\gamma_\varepsilon)} \rightarrow \langle \Theta, \phi\rangle \ \ \textit{for all} \ \ \phi \in H^1(\Omega^+),
\end{aligned} \tag{6}$$

where,

$$\langle \Theta, \phi\rangle = \int_{(0,1)^2} \bar{\varrho}_1 \phi(x', M_0) dx' + \int_{(0,1)^2} \bar{\varrho}_2 \phi(x', M_1) dx' + \int_{\Omega^+} \bar{\theta}(x_3)\phi(x) dx.$$

Limit optimal control problem $\alpha > 1$

- Given $f \in L^2(\Omega)$, $\varrho_1, \varrho_2 \in \mathbb{R}$, consider the following optimal control problem: find $(\bar{u}, \bar{\varrho}_1, \bar{\varrho}_2)$ such that

$$J(\bar{u}, \bar{\varrho}_1, \bar{\varrho}_2) = \inf\{J(u, \varrho_1, \varrho_2,)\big| \; \varrho_1, \varrho_2 \in \mathbb{R}\} \tag{7}$$

- where J is an L^2-cost functional defined as

$$J(u, \varrho_1, \varrho_2) = \int_{\Omega^+} (h(x_3)\chi_{\Omega^+} + \chi_{\Omega^-})|u - u_d|^2 + \frac{\beta h(M_1)}{2|Y_{M_1}|}\varrho_2^2 + \frac{\beta}{2|Y_{M_0}|}\varrho_1^2.$$

- u satisfies the following PDE

$$\begin{cases} -\dfrac{\partial}{\partial x_3}\left(h(x_3)\dfrac{\partial u^+}{\partial x_3}\right) + h(x_3)u^+ = h(x_3)f^+ \text{ in } \Omega^+, \\ -\Delta u^- + u^- = f^- \text{ in } \Omega^- \\ \dfrac{\partial u^+}{\partial x_3} = \varrho_2 \;\text{ on }\; \gamma_u, \quad u^+ = u^- \text{ on } \gamma_c, \\ \dfrac{\partial u^-}{\partial x_3} - h(M_0)\dfrac{\partial u^+}{\partial x_3} = \varrho_1 \text{ on } \gamma_c \\ u^- = 0, \quad \text{on } \;\gamma_b \text{ and } u \;\gamma_s \;\text{ periodic.} \end{cases} \tag{8}$$

Convergence

Theorem 5

Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ *and* $(\bar{u}, \bar{\theta}, \bar{\varrho}_1, \bar{\varrho}_2)$ *be the solution of the problem (2) and (4) respectively, then,*

$$\begin{aligned}
&\widetilde{\bar{u}_\varepsilon^+} \rightharpoonup h(x_3)\bar{u}^+ \text{ weakly in } L^2((0,1)^2; H^1(M_0, M_1)),\\
&\widetilde{\bar{v}_\varepsilon^+} \rightharpoonup h(x_3)\bar{v}^+ \text{ weakly in } L^2((0,1)^2; H^1(M_0, M_1)),\\
&\bar{u}_\varepsilon^- \rightharpoonup \bar{u}^- \text{in weakly } H^1(\Omega^-),\\
&\bar{v}_\varepsilon^- \rightharpoonup \bar{v}^- \text{ weakly in } H^1(\Omega^-),\\
&\langle \hat{\bar{\theta}}_\varepsilon, \phi \rangle_{L^2(\gamma_\varepsilon)} \to \langle \Theta, \phi \rangle \text{ for all } \phi \in H^1(\Omega^+),
\end{aligned} \tag{9}$$

where,

$$\langle \Theta, \phi \rangle = \int_{(0,1)^2} \bar{\varrho}_1 \phi(x', M_0) dx' + \int_{(0,1)^2} \bar{\varrho}_2 \phi(x', M_1) dx'.$$

Another point of view

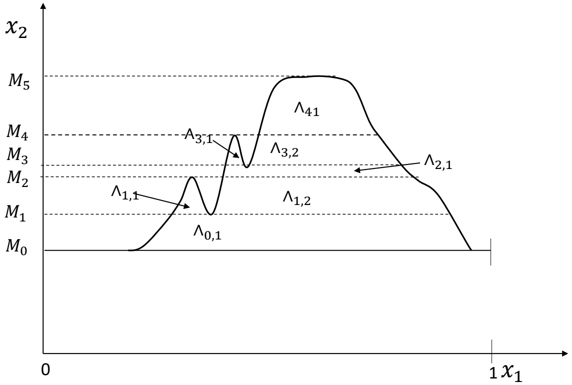

Figure: Reference cell

- Let $\{M_r | \; r \in \{0, 1, \cdots, k_0 + 1\}\}$ be the set of all local extremal values of η with $M_0 < M_1 < \cdots < M_{k_0} < M_{k_{0+1}}$.

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- Let Λ_r has n_r number of connected components and are denoted as $\Lambda_{k,r}$ for $k \in \{1, 2, \cdots, n_r\}$, one can write $\Lambda_r = \bigcup_{k=1}^{n_r} \Lambda_{k,r}$.

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- $S_r = \{(x', \eta(x'))| \; x' \in (0,1)^2, \; M_r < \eta(x') < M_{r+1}\}$,
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  $S_{r,k} = \partial\Lambda_{r,k} \cap S_r$,
- $Y_{r,k}(z) = \{x' \in (0,1)^2| \ \ (x', z) \in \Lambda_{r,k}\}$,

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- Let Λ_r has n_r number of connected components and are denoted as $\Lambda_{k,r}$ for $k \in \{1, 2, \cdots, n_r\}$, one can write $\Lambda_r = \bigcup_{k=1}^{n_r} \Lambda_{k,r}$.
- $S_r = \{(x', \eta(x'))| \; x' \in (0,1)^2, \;\; M_r < \eta(x') < M_{r+1}\}$,
  $S_{r,k} = \partial\Lambda_{r,k} \cap S_r$,
- $Y_{r,k}(z) = \{x' \in (0,1)^2| \;\; (x', z) \in \Lambda_{r,k}\}$,
- $h_{r,k}(z) = |Y_{r,k}(z)|$.

Characterization

- Adjoint state equation

$$\begin{cases} -\Delta \bar{v}_\varepsilon + \bar{v}_\varepsilon = \bar{u}_\varepsilon - u_d \quad \text{in} \quad \Omega_\varepsilon, \\ \bar{v}_\varepsilon = 0 \quad \text{on} \ \gamma_b, \\ \dfrac{\partial \bar{v}_\varepsilon}{\partial \nu_\varepsilon} = 0 \quad \text{on} \quad \gamma_\varepsilon, \\ \bar{v}_\varepsilon \quad \text{is} \quad \gamma_s \quad \text{periodic.} \end{cases} \tag{10}$$

Theorem 6

Given $f \in L^2(\Omega)$, and let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem (2) and $\bar{v}_\varepsilon$ satisfies (10). Then, the optimal control $\bar{\theta}_\varepsilon \in L^2_{per}(\gamma)$ is given by

$$\begin{cases} \bar{\theta}_\varepsilon\big|_{S_{r,k}} = -\dfrac{\sqrt{|\nabla_{y'}\,\eta|(y')}}{\beta}\displaystyle\int_{(0,1)^2} T^\varepsilon_{r,k}\bar{v}_\varepsilon(x',\eta(y'),y')dx' \\ \text{for } 0 \le r \le k_0 \text{ and } 1 \le k \le n_r, \\ \bar{\theta}_\varepsilon\big|_{F_0} = -\dfrac{1}{\beta}\displaystyle\int_{(0,1)^2} T^\varepsilon_0\bar{v}_\varepsilon(x',M_0,y')dx', \\ \bar{\theta}_\varepsilon\big|_{F_{k_0+1,k}} = -\dfrac{1}{\beta}\displaystyle\int_{(0,1)^2} T^\varepsilon_{k_0,k}\bar{v}_\varepsilon(x',M_{k_0+1},y')dx. \end{cases}$$

The converse is also true.

- Our aim is to analyze the asymptotic behavior of $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon, \bar{v}_\varepsilon)$ as $\varepsilon \to 0$.

Limit optimal control problem

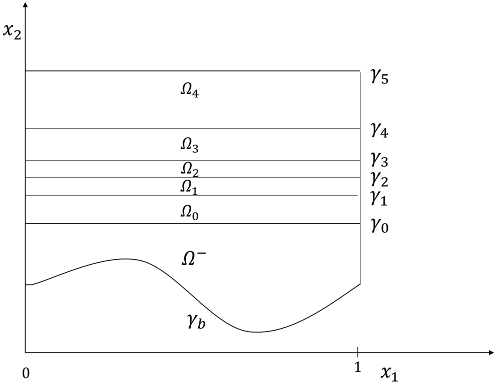

Figure: Limit domain

- $f \in L^2(\Omega)$, for $r \in \{0, 1, \cdots, k_0\}$, $k \in \{1, 2, \cdots, n_r\}$, $\theta_{r,k} \in L^2(M_r, M_{r+1})$, $\varrho_{k_0+1,k}, \varrho_0 \in \mathbb{R}$

$$
\begin{cases}
-\dfrac{\partial}{\partial x_3}\left(h_{r,k}(x_3)\dfrac{\partial u_{r,k}}{\partial x_3}\right) + u_{r,k} = f + \theta_{r,k} \quad \text{in } \Omega_r \\
-\Delta u^- + u^- = f \quad \text{in } \Omega^- \\
-\displaystyle\sum_{k=1}^{n_{k_0}} h_{r,k}(x_3)\dfrac{\partial u_{k_0,k}}{\partial x_3}(x', M_{k_0+1}) = \sum_{k=1}^{n_{k_0}} \varrho_{k_0+1,k} \quad \text{on } \gamma_{k_0+1} \\
\displaystyle\sum_{k=1}^{n_r} h_{r,k}(x_3)\dfrac{\partial u_{r,k}}{\partial x_3} - \sum_{k=1}^{n_{r+1}} h_{r,k}(x_3)\dfrac{\partial u_{r+1,k}}{\partial x_3} = 0 \quad \text{on } \gamma_{r+1} \\
\displaystyle\sum_{k=1}^{n_0} h_{0,k}(x_3)\dfrac{\partial u_{0,k}}{\partial x_3} - \dfrac{\partial u^-}{\partial x_3} = \varrho_0 \quad \text{on } \gamma_0 \\
u^- = u_{0,k} \text{ on } \gamma_0 \text{ and } u_{r,k'} = u_{r+1,k''} \quad \text{on } \gamma_r \text{ if } \Lambda_{r,k'} \text{ and } \Lambda_{r+1,k''} \\
\text{shared interface boundary, where } r \in \{0, 1, 2, \cdots, k_0 - 1\}, \\
k' \in \{1, 2, \cdots, n_r\},\ k'' \in \{1, 2, \cdots, n_{r+1}\}. \\
u = 0 \text{ on } \gamma_b \text{ and } 1-\text{periodic in } x'.
\end{cases}
\tag{11}
$$

- Consider the following optimal control problem: find $(\bar{u}, \bar{\theta}_{r,k}, \bar{\varrho}_{k_0+1,k}, \bar{\varrho}_0) \in \mathcal{H} \times L^2(M_r, M_{r+1}) \times \mathbb{R} \times \mathbb{R}$ such that

$$J(\bar{u}, \bar{\theta}_{r,k}, \bar{\varrho}_{k_0+1,k}, \bar{\varrho}_0) = \inf\{J(u, \theta_{r,k}, \varrho_{k_0+1,k}, \varrho_0)\} \qquad (12)$$

- Consider the following optimal control problem: find $(\bar{u}, \bar{\theta}_{r,k}, \bar{\varrho}_{k_0+1,k}, \bar{\varrho}_0) \in \mathcal{H} \times L^2(M_r, M_{r+1}) \times \mathbb{R} \times \mathbb{R}$ such that

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- where J is an L^2-cost functional given by

$$J(u,\theta)$$
$$= \int_{\Omega^-} |u^- - u_d|^2 + \sum_{r=0}^{k_0} \sum_{k=1}^{n_r} \int_{\Omega_r} |h_{r,k}(x_3) u_{r,k} - u_d|^2 +$$
$$\sum_{r=0}^{k_0} \sum_{k=1}^{n_r} \int_{\Omega_r} \frac{\beta}{2\omega_{r,k}(x_3)} |\theta_{r,k}|^2 + \sum_{k=1}^{n_{k_0}} \frac{\beta}{2|Y_{k_0+1}, k|} \varrho^2_{k_0+1,k} + \frac{\beta}{2|Y_{M_0}|} \varrho_0^2.$$

Main Result

Theorem 7

Let $\bar{u}_\varepsilon$, $\bar{v}_\varepsilon$ defined as earlier, then the following convergence result holds. For $r \in \{0, 1, 2, \cdots, k_0\}$ and corresponding $k \in \{1, 2, \cdots, n_r\}$,

$$\begin{aligned}
&\widetilde{\bar{u}_\varepsilon|_{\Omega^\varepsilon_{r,k}}} \rightharpoonup h_{r,k}(x_3)\bar{u}_{r,k} \text{ in } L^2((0,1)^2, H^1(M_r, M_{r+1})),\\
&\bar{u}^-_\varepsilon \rightharpoonup \bar{u}^- \text{ in } H^1(\Omega^-),\\
&\widetilde{\bar{v}_\varepsilon|_{\Omega^\varepsilon_{r,k}}} \rightharpoonup h_{r,k}(x_3)\bar{v}_{r,k} \text{ in } L^2((0,1)^2, H^1(M_r, M_{r+1})),\\
&\bar{v}^-_\varepsilon \rightharpoonup \bar{v}^- \text{ in } H^1(\Omega^-).
\end{aligned}$$

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Thank you!