ライリー切片

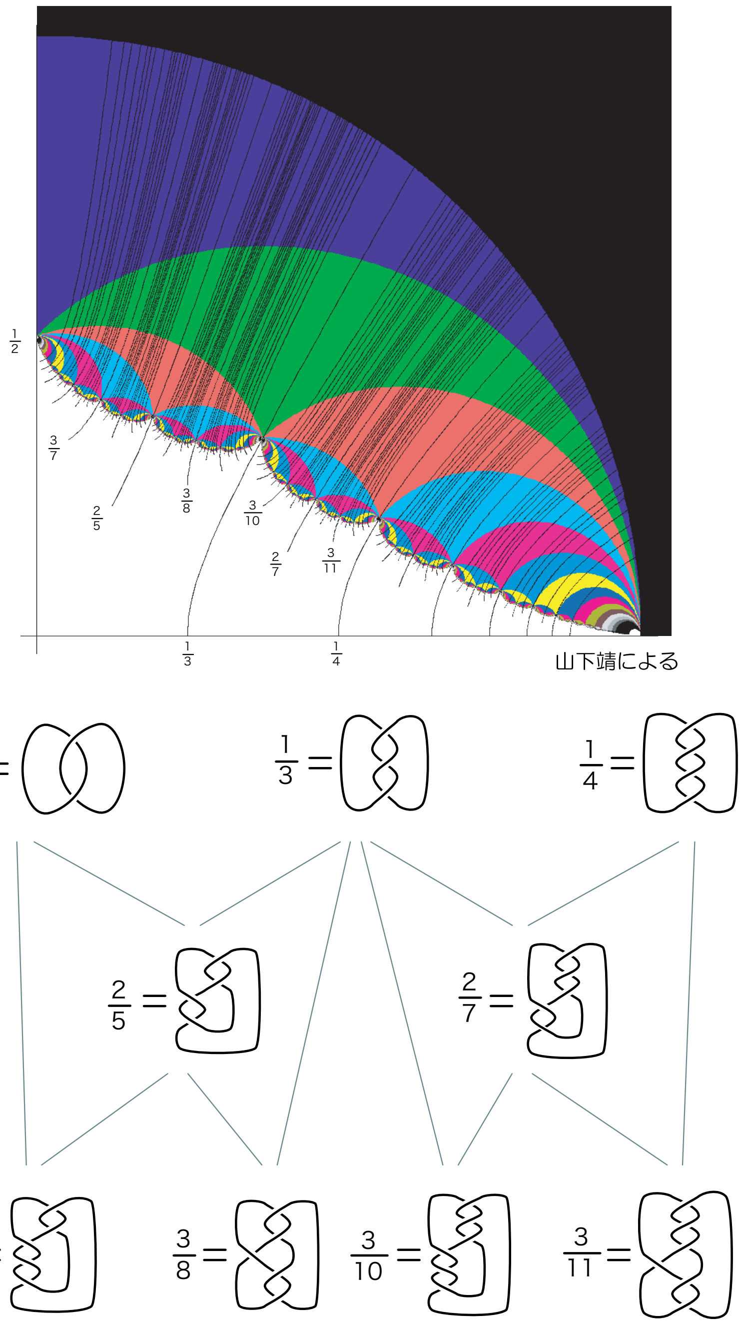
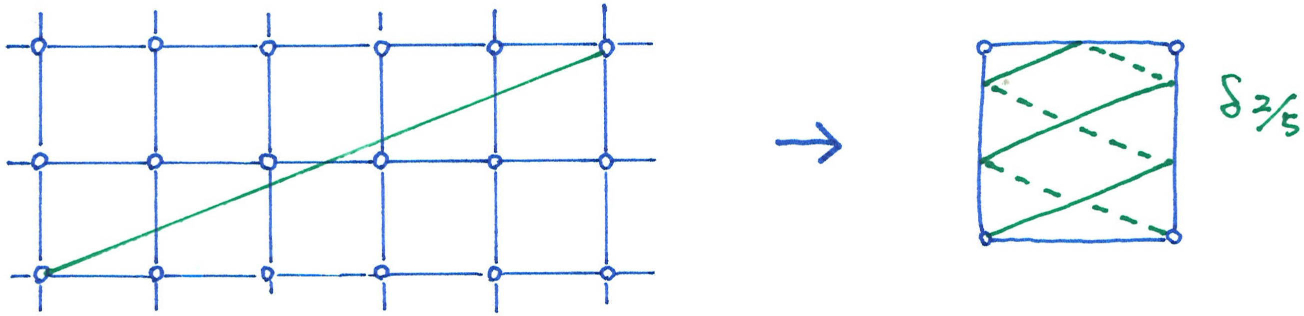


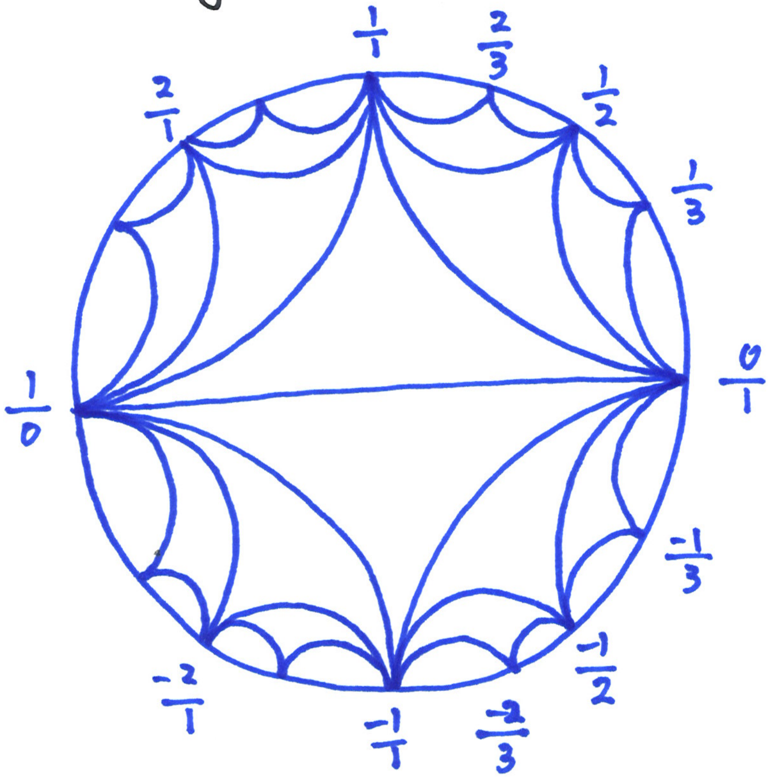
図 2: 有理結び目の双曲構造

$S := \mathbb{R}^2 - \mathbb{Z}^2 \big/ \langle \pi\text{-rotations around punctures} \rangle$: 4-punctured sphere

(Conway sphere)



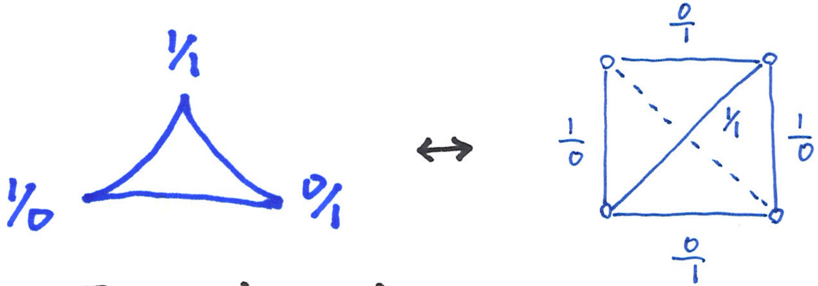
D : Farey tessellation



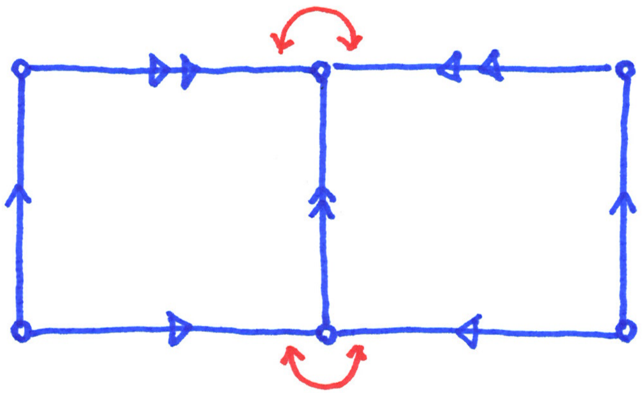
Vertex set of $D = \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\frac{1}{0}\} \ni r$

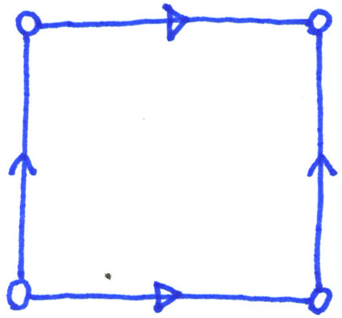
$\underset{1-1}{\leftrightarrow} \{\text{essential simple loops on } S\} \ni \alpha_r$

$\underset{1-2}{\leftrightarrow} \{\text{essential simple arcs on } S\} \ni \delta_r$

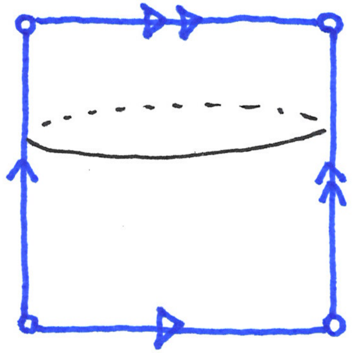


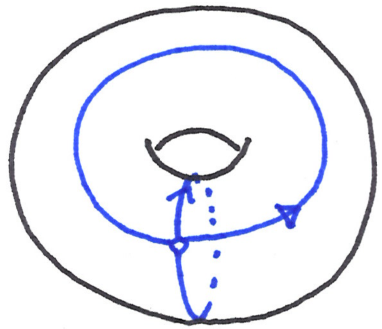
Farey triangle ideal triangulation of S





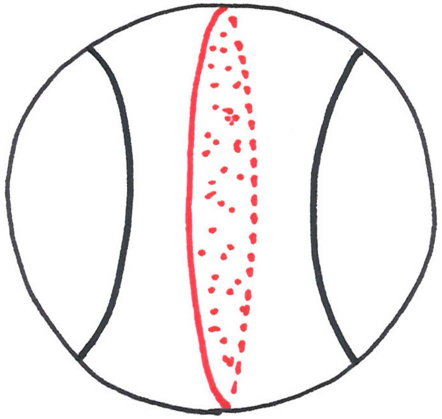
$\downarrow$ $\downarrow$

$S :=$  $\cong S^2 - 4$ points

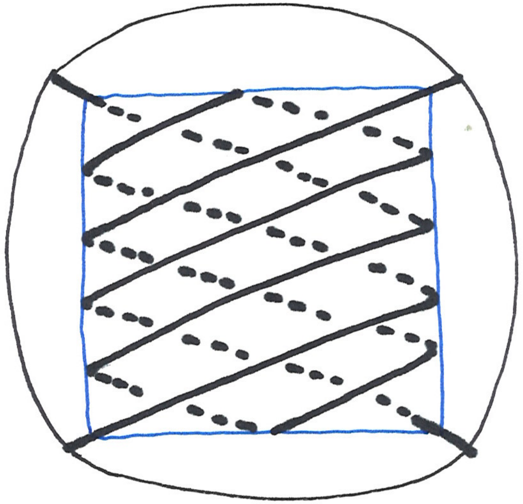
 $=: T$

S and T are "commensurable"

Rational tangle $(B^3, t(r))$ of slope r :



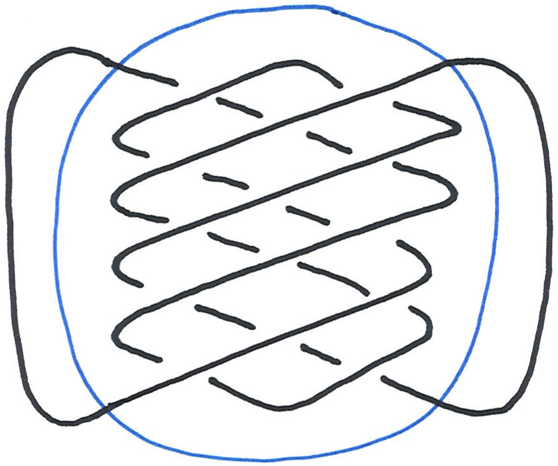
$(B^3, t(1/0))$



$(B^3, t(2/5))$

$$\pi_1(B^3 - t(r)) \cong \pi_1(S) / \langle\langle \alpha_r \rangle\rangle$$

$(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$: 2-bridge link of slope r

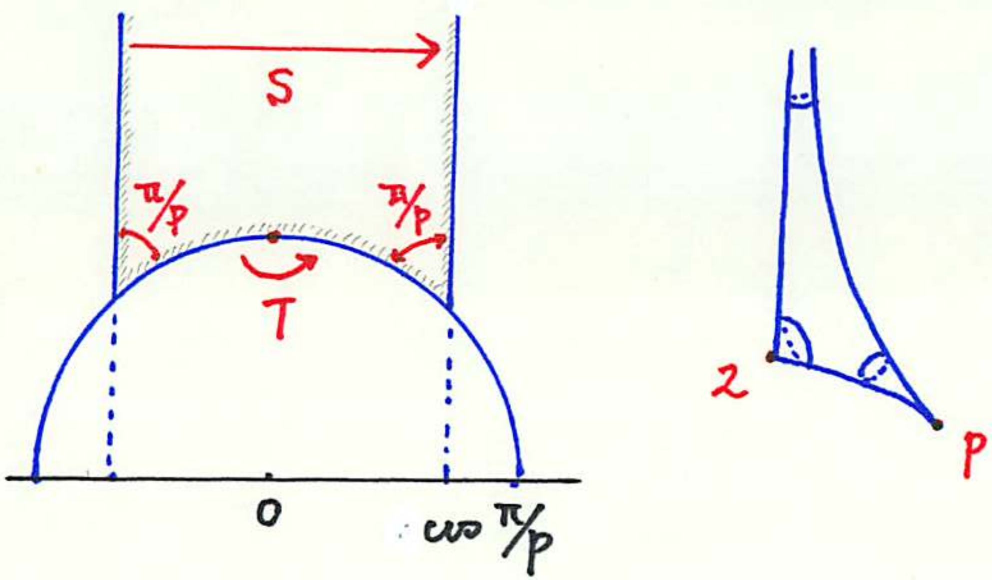


$$G(K(r)) := \pi_1(S^3 - K(r)) \cong \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$$

Hecke group

$$H(p) := \left\langle T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ S = \begin{pmatrix} 1 & 2\cos\pi/p \\ 0 & 1 \end{pmatrix} \right\rangle < PSL(2,\mathbb{R})$$

$$\mathbb{H}^2 / H(p) = (2, p, \infty)\text{-orbifold} \qquad (p \geqq 3)$$



Note : $H(3) = PSL(2, \mathbb{Z})$

- Two-parabolic subgroup of $H(p)$

$$G(p) := \langle S, TST^{-1} \rangle = \begin{cases} H(p) & \text{if } p:\text{odd} \\ \text{Index 2 subgroup} & \text{if } p:\text{even} \end{cases}$$

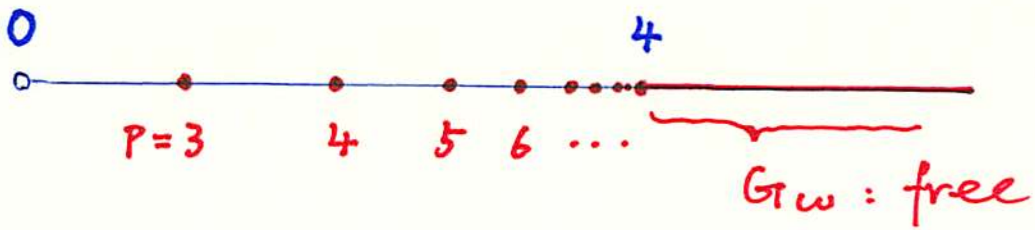
$$= \left\langle \begin{pmatrix} 1 & 2\cos\pi/p \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2\cos\pi/p & 1 \end{pmatrix} \right\rangle$$

$$\cong \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right\rangle =: G_\omega \qquad \omega = 4\cos^2\pi/p$$

[Knapp 1966, Knopp-Newman 1993]

For $\omega \in \mathbb{R} - \{0\}$, G_ω is discrete

$$\Leftrightarrow \quad \pm\omega \in \{ 4\cos^2\pi/p \mid p \geq 3 \} \cup [4, \infty)$$



Riley slice of the Schottky space

$$\mathcal{R} := \left\{ \omega \in \mathbb{C} \;\middle|\; \mathbb{H}^3 \cup \Omega(G_\omega) \big/ G_\omega \cong \right.$$

$$\left. \right\}$$

$$\overline{\mathcal{R}} = \{ \omega \in \mathbb{C} \mid G_\omega : \text{free Kleinian group} \}$$

↑ Ohshika-Miyachi

Riley's pioneering exploration of two-parabolic groups

$$\mathcal{D} := \{ \omega \in \mathbb{C} - \{0\} \mid G_\omega \text{ is discrete and non-free} \}$$

" The calculation for the diagram were pushed far enough to suggest that all essential phenomena were depicted."

(1) Each $\omega \in \mathcal{D}$ such that G_ω is torsion-free corresponds to the hyperbolic structure on a 2-bridge link complement.

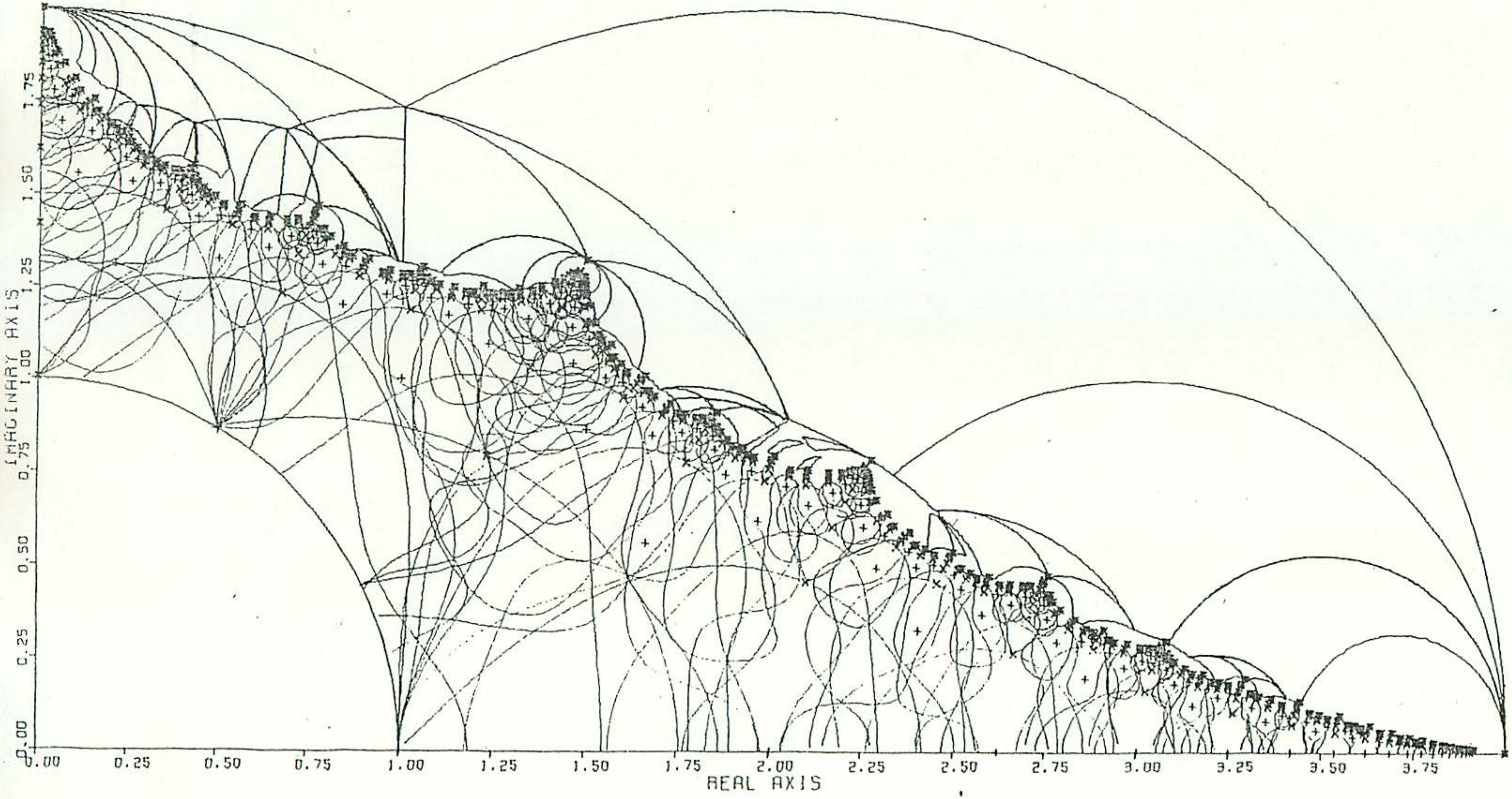
(2) Each $\omega \in \mathcal{D}$ such that G_ω is not torsion-free gives a Heckoid group for some 2-bridge link.

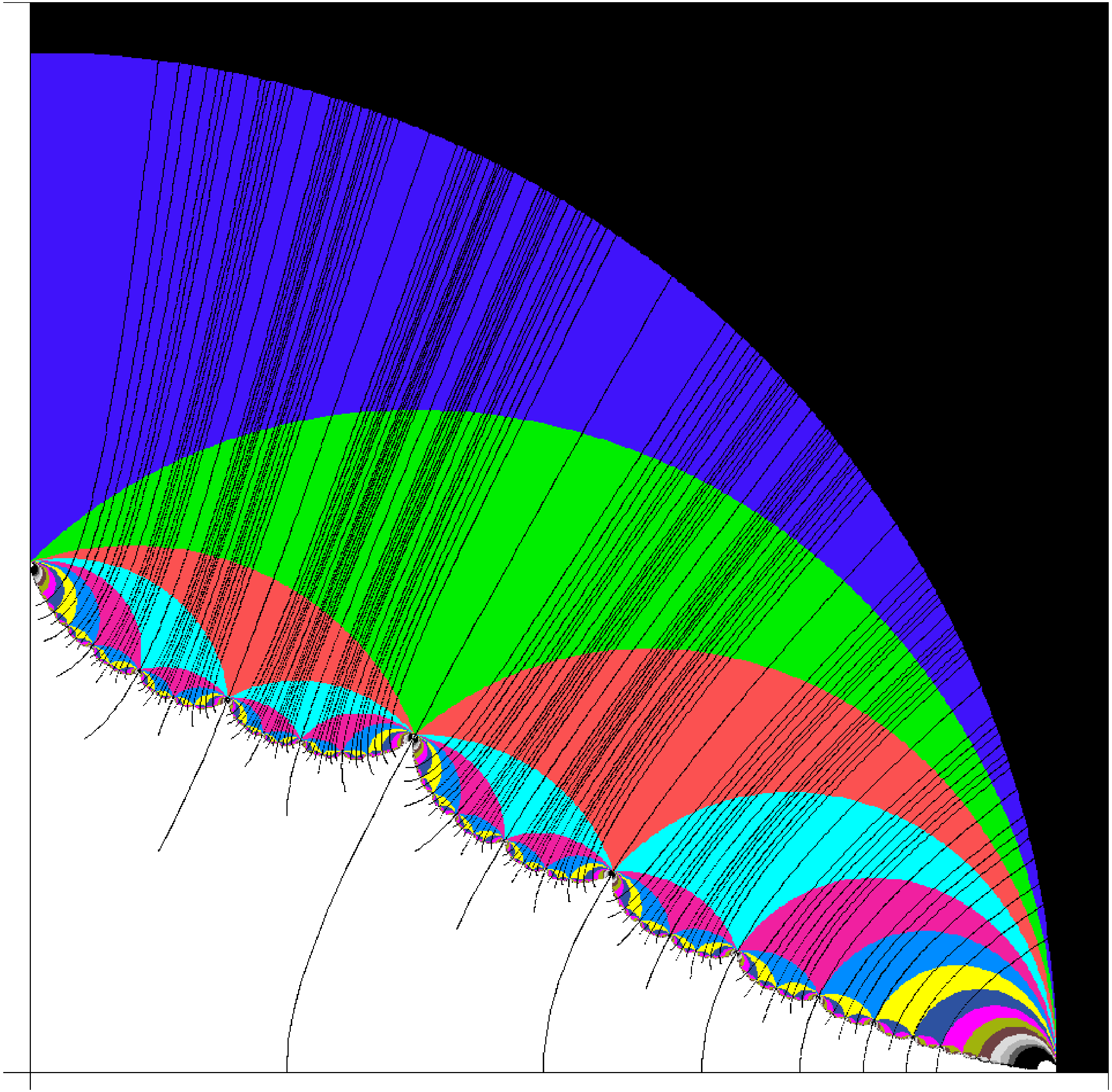
(3) The points $\omega_q(K)$, $q = 3, 4, \cdots$, determining the Heckoid groups for 2-bridge links K converge to a cusp $\omega_\infty(K)$ of the boundary of the region $\mathcal{R}$, the Riley slice of the Schottky space.

Riley's pioneering exploration of two parabolic groups

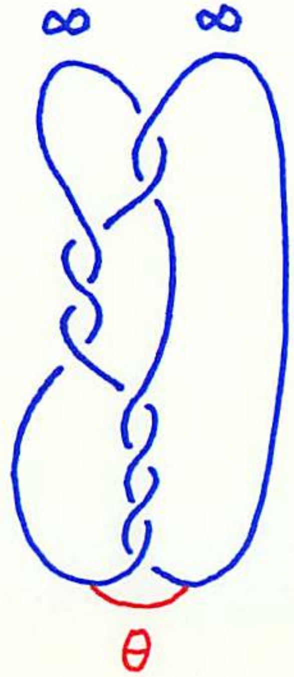
GROUPS GENERATED BY TWO PARABOLICS A AND B(W) FOR W IN THE FIRST QUADRANT

W IS MARKED BY +, CROSS, OR × ACCORDING AS G(W) IS A PELL OR REAL HECKOID GROUP, A NON-REAL HECKOID GROUP, OR A CUSP GROUP.
EACH CONTOUR IS A LEVEL CURVE ABS C21(T)(W)=1 FOR SOME WORD T IN A, B, AND IS TERMINATED AT THE AXES OR UNIT CIRCLE.
INSIDE EACH CONTOUR G(W) IS INDISCRETE WHEN C21(T).NE.0 .
OUTSIDE THE ×'S G(W) IS FREE, DISCRETE, NON-RIGID.
THESE GROUPS LIE IN CELLS WHERE THE GROUPS OF EACH CELL HAVE SIMILAR FORD DOMAINS.





[Akiyoshi - S - Wada - Yamashita 2007] announced :



- $^{\exists}\{C(r;\theta)\}_{0 \le \theta \le 2\pi}$

  continuous family of hyperbolic cone manifolds,

  except when "$r \not\equiv \pm 1/p$ in $\mathbb{Q}/\mathbb{Z}$ and $\theta = 2\pi$".

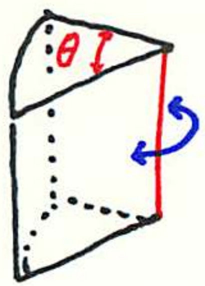
- The holonomy representation

  $\rho_\theta : \pi_1(|C(r:\theta)| - \text{cone axis}) \to PSL(2.\mathbb{C})$

  has a discrete image iff $\theta = 2\pi/d$ $\left(d \in \frac{1}{2}\mathbb{N}_{\ge 2}\right)$

- The Heckoid group $H(r;d) = \mathrm{Im}\,\rho_\theta$

  where $\theta = 2\pi/d$



cone singularity

[Agol 2002]

For $\omega \in \mathbb{C}^*$, G_ω is discrete iff

(0) G_ω : free Kleinian group

(1) $\pm\omega \in \{4\cos^2 \pi/p \mid p \geq 3\}$

(2) $G_\omega \cong$ 2-bridge knot/link group

$\cong \pi_1(S^3 - K(r))$ $r = q/p$ $(q \not\equiv \pm 1 \mod p)$

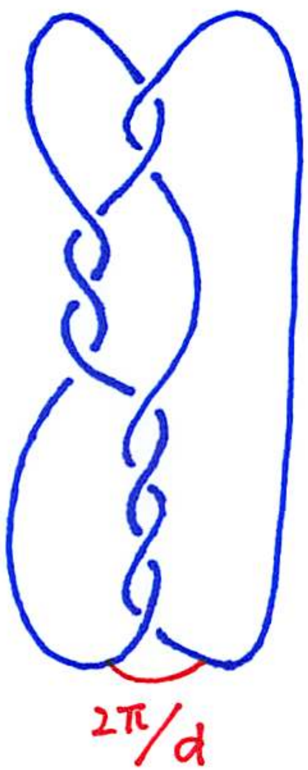
(3) $G_\omega \cong$ Heckoid group $H(r;d)$ for a 2-bridge link $K(r)$

= Orbifold fundamental group

of the Heckoid orbifold $\mathcal{O}(r;d)$

Remark The terminology "Heckoid group" was introduced by [Riley, 1992]

- Even Heckoid orbifold $\mathcal{O}(r;d)$ for 2-bridge link $K(r)$



$$r = [2,3,4] = \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}} = \frac{13}{30} \in \mathbb{Q}$$

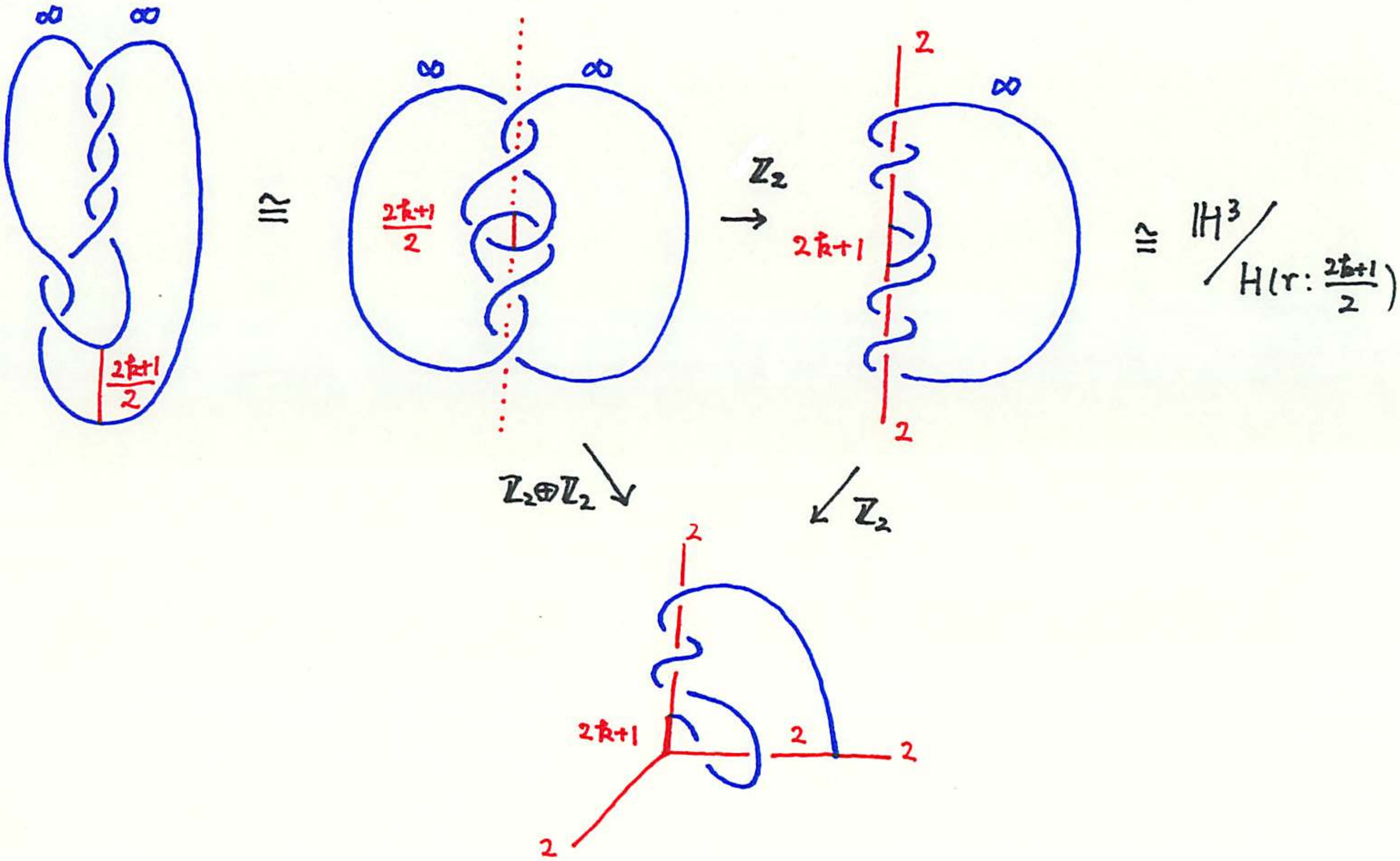
$$d \in \mathbb{N}_{\geqq 2}$$

Even Heckoid group $H(r:d) = \pi_1(\mathcal{O}(r;d))$.

- Even when $d \in \frac{1}{2}\mathbb{N}_{\geq 3}$,

  the odd Heckoid group $H(r:d) =$ "$\pi_1(\mathcal{O}(r:d))$" is defined.

Quotient orbifold of odd Heckoid group $H(r; \frac{2k+1}{2})$

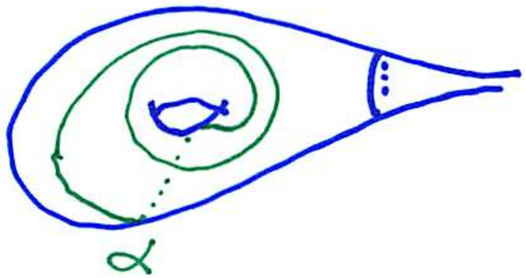


McShane's identity

For any hyperbolic once-punctured torus T

$$\sum_{\alpha} \frac{1}{1+\exp(l(\alpha))} = \frac{1}{2}$$

where α runs over essential simple closed geodesics



- Generalization + Application : Mirzakhani, Tan-Wong-Zhang
- 3-dim variation : Bowditch, Akiyoshi-Miyachi-S.

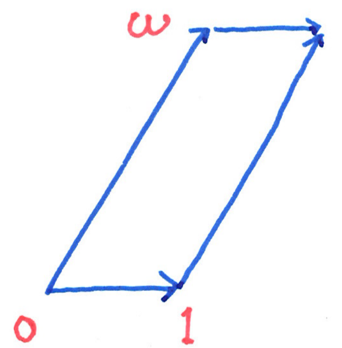
Theorem (A variation of McShane's identity for 2-bridge knots)

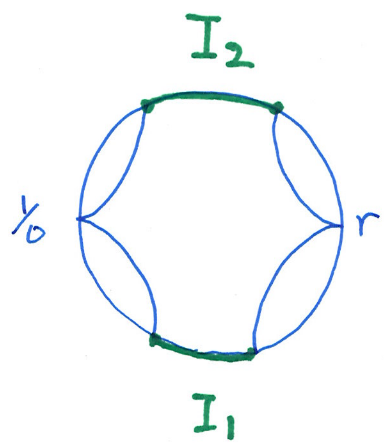
For $\rho_r : \pi_1(T) \to PSL(2,\mathbb{C})$ corresponding to the complete hyperbolic structure of $S^3 - K(r)$;

$$2\sum_{s\in \mathring{I}_1} \frac{1}{1+\exp(\ell(\rho_r(\alpha_s)))} + \sum_{s\in\partial I_1} \frac{1}{1+\exp(\ell(\rho_r(\alpha_s)))}$$

$$= -1 - 2\sum_{s\in \mathring{I}_2} \frac{1}{1+\exp(\ell(\rho_r(\alpha_s)))} - \sum_{s\in\partial I_2} \frac{1}{1+\exp(\ell(\rho_r(\alpha_s)))}$$

= Modulus of the cusp torus with a suitable choice of the longitude.





$\square$

[Ohshika - Miyachi]

The boundary of the Riley slice in $\mathbb{C}$ is a Jordan curve.

(Outline)

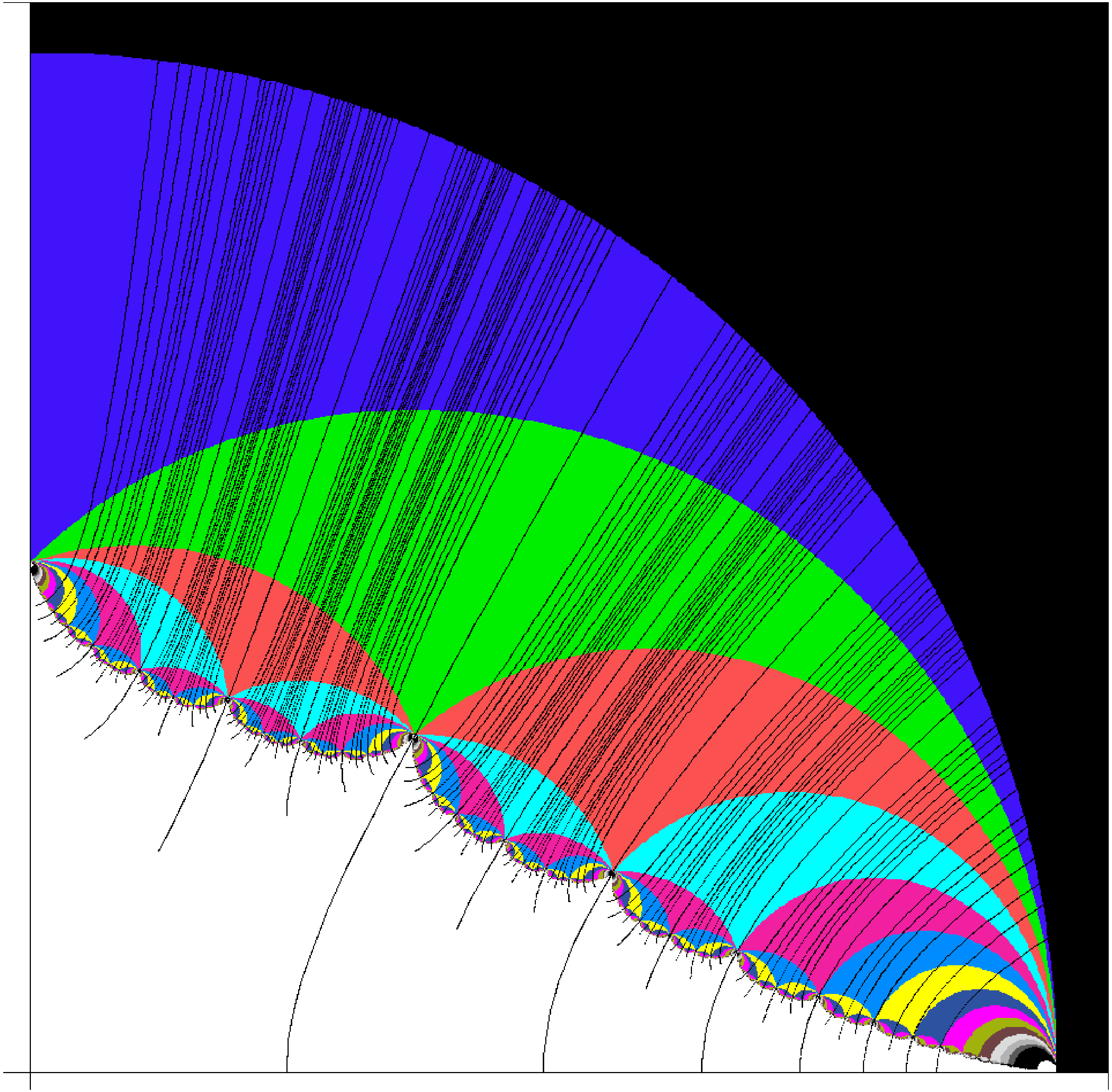
- Construct uniform combinatorial models of $\mathbb{H}^3/G_\omega$ for $\omega \in \overline{\mathcal{R}}$
- $\mathbb{H}^3/G_\omega \cong \mathbb{H}^3/G_{\omega'}$ iff their end invariants coincide
- End invariant map $\overline{\mathcal{R}} \longrightarrow \{z \in \mathbb{C} \mid \operatorname{Im} z \geqq 0\}/_{z \sim z+2}$ is a homeomorphism.

[Martin - S]

The boundary of the Riley slice is equal to the set of accumulation points of the set of 2-bridge links in $\mathbb{C}$.

(proof)

- Cusps are dense in ∂R by Canary - Sa'ar.
- Every cusp is an accumulation point of the set of 2-bridge links.



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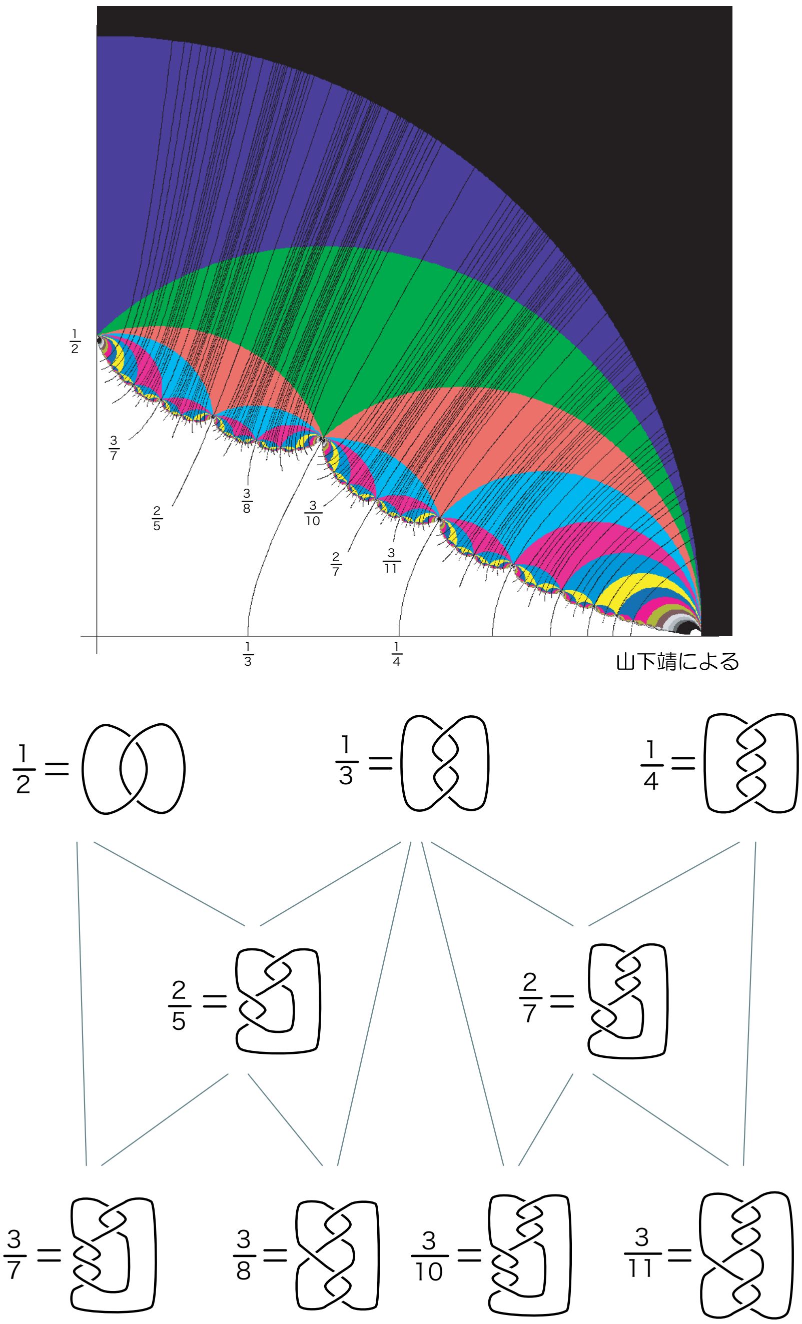
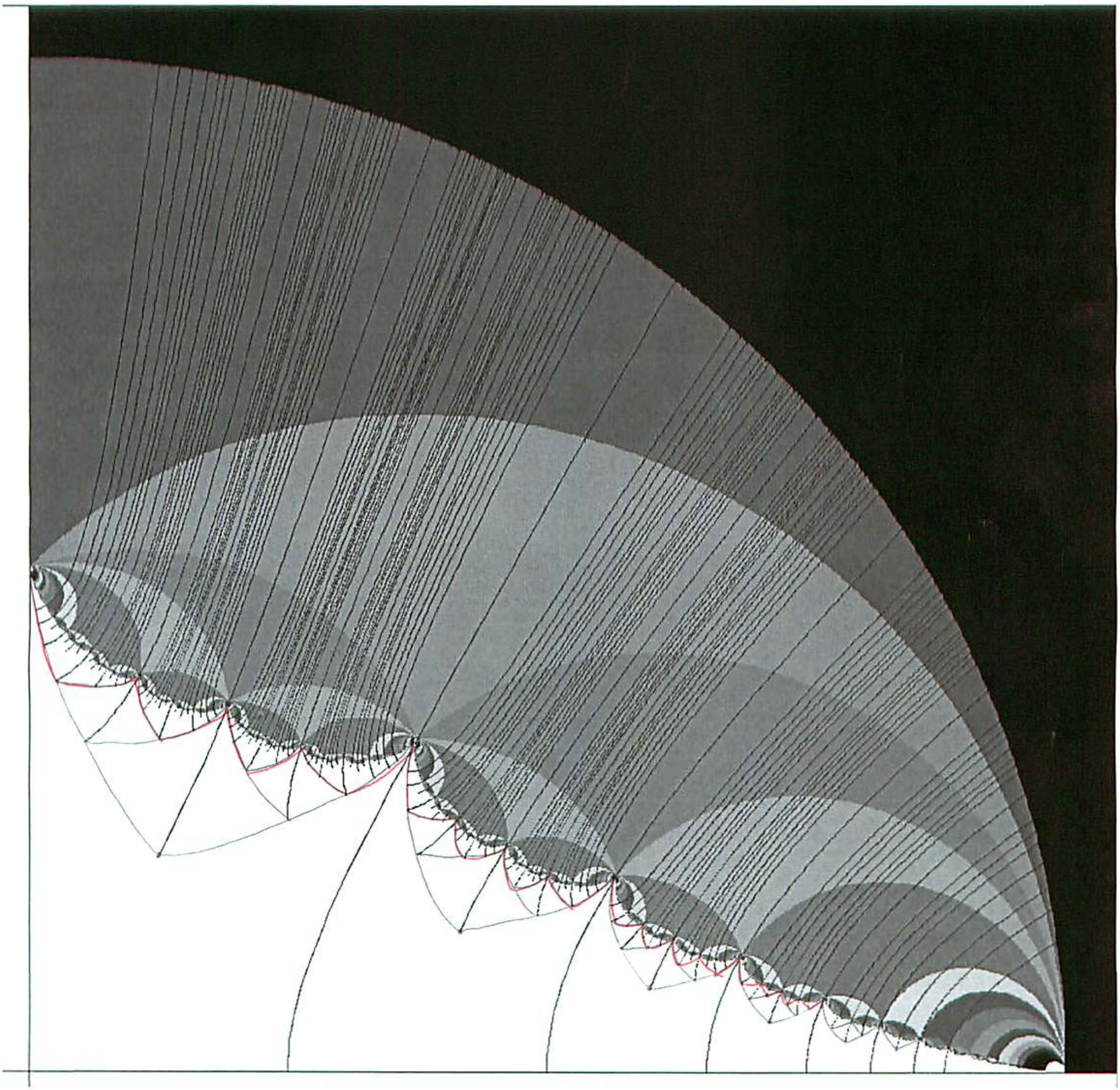


図 2: 有理結び目の双曲構造



Thank you very much!