Emmy Noether in Erlangen and Göttingen

Ravi A. Rao

School of Mathematics, Tata Institute of Fundamental Research

Abstract

We give an overview of Emmy Noether's seminal works; and her contribution to the birth of modern algebra. We also give an indication of its influence in present day research.

Her name

In 1809, the Tolerance Edict of Baden stated that the male head of every Jewish family who did not have a distinguishing heriditary surname was to assume one for himself and his children. Elias Samuel, the grandfather of Max Noether, was given the name Nöther. Max and his children used the spelling Noether, though his marriage documents still showed the official spelling as Nöther.

Erlangen

Felix Klein (1849-1925) brought world fame to Erlangen with his inaugural lecture of 1872 explaining the significance of the concept of a group in geometry, now known as the "Erlangen Program". One of Felix Klein's friends and colleagues, Paul Gordan (1837-1912) was a colleague of Max Noether (1844-1921), whose daughter Emmy was to become Gordan's only doctoral candidate. It was only in 1904 that the University of Erlangen decided that women should be provided full access; and Emmy was accepted by Gordan (the King of Invariant Theory) to be his first student (at the age of 61). She finished her doctorate (no BA or MA in Math!) in the subject of Computational classical Invariant Theory in 1907. Her thesis ended with 331 formulas on invariants of ternary biquadratic forms! Her degree was awarded "summa cum laude" the highest distinction possible.

Noether's thesis

Her thesis was on the construction of the system of forms for the ternary biquadratic form: Über der Bildung des Formensystems der ternären biquadratischen Form (registered in Erlanger Universitätsschriften 1907/08 number 202, dated June 2, 1908). This later appeared in Journal für die reine und angewandte Mathematik, 134, 1908, pg. 23-90. Emmy Noether herself later referred to her thesis, as well as to several consecutive paper on the theory of invariants, as "crap". She said it was a jungle of formulas – routine calculations.

As a lecturer (unpaid) in Erlangen she began to study the work of Hilbert. Two students of Hilbert: E. Fischer (1875-1954) and Erhard Schmidt, were the successor of Gordan (who retired in 1910) at Erlangen, It was Fischer's influence that Emmy Noether made the definitive change from the purely computational method distinctly algorithmic approach represented by Gordan to the mode of thinking characteristic of Hilbert.

Noether invited to Göttingen

In 1913 and 1914 Noether exchanged letters with David Hilbert and his Göttingen senior colleague Felix Klein discussing Einstein's Relativity Theory. Hilbert invited her in 1915 to come to Göttingen as a Lecturer; but this was opposed by the Humanities department. (What will the soldiers think if they come back from the war and be taught at the feet of a woman? Hilbert argued that: "Gentleman, the Senate is not a bathhouse; so I do not see why a woman cannot enter it. However, he did not succeed in convincing his colleagues.

Privatdozent in Göttingen

Noether was so eager to join Hilbert's department in Göttingen that, to overcome Hilbert's opponents, she agreed not to be formally appointed as a lecturer and to receive no pay. Her father continued supporting her financially (sadly her mother died in 1915) and the lectures she gave were advertised as lectures by Professor Hilbert, with assistance from Dr. E. Noether. It was only in 1922 that she was given a small salary, as a lecturer in algebra. She received the appointment as honorary Professor, as a result of Courant's efforts she received a so-called Lehraufirag, i.e. a small salary (200-400 marks per month) for her lectures; which required confirmation every year by the Ministry.
Thus, after the death of Gordan, and with her brother Fritz, and E. Fischer drafted in the military, and the sudden death of her mother, Emmy moved to Göttingen at the end of April, 1915, when she received an invitation from Klein and HIlbert to substitute for the *Privatdozenten*- someone who had the right to teach without being on the paid staff of the university.

Noether's work

Noether's Theorem had revolutionized physics. In 1919 the full force of her powerful mind turned towards pure mathematics. In this discipline, she was one of the principle architects of abstract algebra. Her name is remembered in many of its concepts, structures and objects, such as: Noetherian, Noetherian group, Noetherian induction, Noether normalization, Noether problem, Noetherian ring, Noetherian module, Noetherian scheme, Noetherian space, Albert–Brauer–Hasse–Noether theorem, Lasker–Noether theorem, and Skolem–Noether theorem.

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Her work was pivotal in the fields of:
mathematical rings she established the modern axiomatic definition of a commutative ring and developed the basis of commutative ring theory commutative number fields, linear transformations, noncommutative algebras – Hermann Weyl credited Noether with representations of noncommutative algebras by linear transformations, and their application to the study of commutative number fields and their arithmetics.

Steinitz paper on field theory

In 1910 in Vol. 137 of Crelle, 167–309, there appeared a 134 page paper with the title Algebraische Theorie der Körper [Algebraic Theory of Fields] by Ernst Steinitz.

Steinitz paper was **the first time** that an abstract algebraic structure was studied **on the basis of its system of axioms**. On the one hand the formal algebraic thinking, on the other hand the connection to set theory–these are the characteristics of the modern algebra of structures.

Noether carried this program further in her works, Idealtheorie in Ringbereischen (1921) and Abstrakter Aufbau der Idealtheorie in algebraischen Zahl und Funktionenkorpern (1926).

Invariant Theory

Three of Hilbert's fundamental contributions to modern algebra, Nullstellensatz, Basis Theorem, Syzygy Theorem, were first proved as lemmas in his Invariant Theory papers (1890, 1893).
A polynomial $f \in \mathbb{C}[X_1, \ldots, X_n]$ is said to be symmetric if it is invariant under any permutation of the variables $X_1, \ldots, X_n$ i.e.
$f(X_1, \ldots, X_n) = f(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$, $\forall \sigma \in S_n$, the group of permutations on n-symbols. For instance,

$$
\begin{aligned}
\varepsilon_1 &= X_1 + \ldots + X_n = \sum_i X_i \\
\varepsilon_2 &= X_1X_2 + X_1X_3 + \ldots + X_2X_3 + \ldots + X_{n-1}X_n \\
&= \prod_{i<j} X_iX_j \\
\varepsilon_3 &= \prod_{i<j<k} X_iX_jX_k \\
\ldots &= \ldots \\
\varepsilon_n &= X_1X_2 \ldots X_n
\end{aligned}
$$

Contd.

are invariant under the action of S_n. These are called the elementary symmetric polynomials.
Every symmetric polynomial f can be written uniquely as a polynomial $f(\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{C}[X_1, \ldots, X_n]$. These fundamental invariants are algebraically independent. There is no non-zero syzygy.

Let G be a subgroup of $GL_n(\mathbb{C})$, the general linear group. $GL_n(\mathbb{C})$ acts linearly on the variables $X_1, \ldots, X_n$:
If $\alpha \in GL_n(\mathbb{C})$, send $(X_1, \ldots, X_n)^t \mapsto \alpha(X_1, \ldots, X_n)^t$, and $f \mapsto f \circ \alpha$.
Example: Let $f = X_1^2 + 2X_1X_2$, $\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so $X_1 \mapsto X_2$, and $X_2 \mapsto -X_1$, and $f \mapsto X_2^2 - 2X_1X_2$.

The ring of invariants

$$\mathbb{C}[X_1, \dots, X_n]^G = \{f \in \mathbb{C}[X_1, \dots, X_n] \mid f = f \circ \gamma, \forall \gamma \in G\}$$

is the subring of $\mathbb{C}[X_1, \dots, X_n]$ consisting of polynomials which are invariant under every element of G.

The following are some of the relevant questions about these rings:

• **Finite Generation**: Is $\mathbb{C}[X_1, \dots, X_n]^G$ generated by a finite set of generators?

• **Syzygies**: What are (all) the relations between these generators?

• **Computational Algorithm**: Can you give an algorithm to write down an arbitrary invariant as a polynomial in some fundamental invariants?

Noether's degree bound

Hilbert's Finiteness Theorem showed that for finite groups $G \subset GL_n(\mathbb{C})$, the ring of invariants $\mathbb{C}[X_1, \ldots, X_n]^G$ has transcendence degree n over $\mathbb{C}$ and is finitely generated.

Nagata (1959) gave examples to show that certain "exotic" linear groups do not have finitely generated invariant rings. (This meant that Hilbert's sixth problem had a negative solution in general.)

In her last theorem proved in Erlangen, Noether (1916) gave an efficient version of Hilbert's theorem:

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Theorem (Noether's degree bound)
The invariant ring $\mathbb{C}[X_1, \ldots, X_n]^G$ if a finite natrix group G has a basis of at most $\binom{n+|G|}{n}$ invariants whose degrees are bounded above by $|G|$.

This theorem has great impact on **computational commutative algebra**. More generally, Noether's theorem works if $|G|$ is invertible in the base field; and so this has also led to active research in **modular invariant theory**.

Noether's thesis work

The work on algebraic invariants was started in 1841 by George Boole (1815–64), whose results were limited. Cayley was attracted to the subject, and he interested Sylvester in the subject. They were joined by George Salmon; and the three did so much work on invariants that they were dubbed **the invariant trinity** by Hermite.

Many particular invariants were sought and found –like the discriminants, Hessian, Jacobian, resultant, catalecticant, etc. This led to the major problem of invariant theory; which was to find a complete system of invariants. Cayley showed that the ones found by Eisenstein for the binary cubic form and the ones he obtained for the binary quartic form are a complete system for each case.

Gordan's work

The existence of such a basis for binary forms of any given degree was given by Paul Gordan, using results of Clebsch. He also showed how to compute the invariants. He later worked it out for the ternary quadratic form, the ternary cubic form, and for a system of two and three ternary quadratics. Kung and Rota attempted, in 1984, in their paper The 'Invariant Theory of Binary Forms', to verify the completeness of the basis of Gordan; instead Kung and Rota found it easier to begin a more systematic method, the symbolic or umbral calculus to study these questions! They record that Gordan's method (in 1885, 1887) remains the most effective one. Further insight into the explicit generation of covariants will require a systematization of Gordan's brackets of brackets (plethysms) and a concomitant deepening of the straightening algorithm.

Hilbert transforms Invariant Theory

Hilbert wrote his doctoral thesis on invariant theory in 1885, and later in 1888 he reproved Gordan's binary forms theorem, modifying an argument of Mertens. This proof did not give a process to find the invariants though. Later in 1888 he announced a totally new existential approach to the problem showing that any form of given degree and given number of variables, and any given system of forms in any given number of variables have a basis –the Hilbert Basis Theorem. Gordan exclaimed that this is not mathematics, its theology!

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Hilbert was sensitive to the criticism of Gordan, and within three years he responded by giving a constructive Nullcone method, using ideas of Cayley's Ω-process. He gave an approach which could be made algorithmic. And Gordan used this method, and said that he was convinced that theology also has its advantages. Gordan's method later led Gröbner to his idea of how a basis can be got given a set of generators; which permits an accurate division algorithm in a polynomial ring in several variables. This was completed by Gr obner's student Buchberger; who named it Gröbner basis after his guide. This is the basic reason why one can do computational algebra via a computer algebra today.

Noether's thesis

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From 1911 to 1919 Emmy Noether produced many papers on finite basis for various cases using Hilbert's techniques and her own.

Today even the invariants of the ternary quartic have not been fully classified; which attempt persuaded Emmy Noether to quit invariant theory!

The Inverse Galois Problem

Does every finite group G appear as a Galois group of some finite extension L of $\mathbb{Q}$?

Milestones:

• The **Kronecker–Weber** theorem (1853, 1886): The result is true if G is a finite abelian group. One may even take L to be a suitable subfield of $\mathbb{Q}(\omega)$, where ω is a primitive n-th root of unity.

• **Hilbert's Irreducibility Theorem (1892):** Let $f(X) \in \mathbb{Q}(T)[X]$ be an irreducible polynomial. There are infinitely many points $t_0 \in \mathbb{Q}$ such that $f_{t_0}(X)$ is irreducible in $\mathbb{Q}[X]$.

Using the above result, Hilbert showed that S_n, A_n can be realized as Galois groups over $\mathbb{Q}$.

Fischer's work on Inv Gal Problem

• Emmy Noether's work in Galois theory originated in conversations with E. Fischer, who had shown

Theorem (E. Fischer, 1915): Let G be a finite abelian group of exponent e. Let k be a field of characteristic prime to p, and containing the e-th roots of unity. Let V be a finite dimensional representation of G over k and let $K(V)$ be quotient field of $k[X_1, \ldots, X_n]$. Then $K(V)^G$ is rational over k.

• Let G be a finite group and let $X = \{X_g \mid g \in G\}$. G acts on X by left multiplication: $X_h \mapsto X_{gh}$, for all $h \in G$. This extends to an automorphism on $\mathbb{Q}(X) := \mathbb{Q}(X_e, X_g, \ldots, X_h)$.
If $\mathbb{Q}(X)^G$ is the fixed field of $\mathbb{Q}(X)$ under this action of G then $\mathbb{Q}(X)/\mathbb{Q}(X)^G$ is Galois with group G.

Theorem (E. Noether, 1918): If $\mathbb{Q}(X)^G$ is rational, i.e. purely transcendental extension of $\mathbb{Q}$ then there exists an extension of $\mathbb{Q}$ with Galois group G.

Noether's proof

Proof: Emmy Noether's philosophy was 'Given the group find the equation; given the equation find the group.'
Let $\mathbb{Q}(X)^G = \mathbb{Q}(W_1, \ldots, W_n)$. By the Primitive element theorem, $\mathbb{Q}(X) = \mathbb{Q}(X)^G(\theta)$, for some $\theta \in \mathbb{Q}(X)$. Let $f(T) \in \mathbb{Q}(X)^G[T] = \mathbb{Q}(W_1, \ldots, W_n)[T]$ be the minimal polynomial of θ.
For $t_0 = (t_{01}, \ldots, t_{0n}) \in \mathbb{Q}^n$ the $f_{t_0}(T) \in \mathbb{Q}[T]$ is got by evaluating $W_i \mapsto t_{0i} \in \mathbb{Q}$.
By Hilbert's Irreducibility Theorem $f_{t_0}(T) \in \mathbb{Q}[T]$ is irreducible for some $t_0 \in \mathbb{Q}$. If L is the splitting field of $f_{t_0}(T)$ then $L|\mathbb{Q}$ is Galois with group G.
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Thus Noether gave the idea of parametrizing all Galois extensions with a given group G: If $k(X_1, \dots, X_n)^G$ is rational over k then there is a generic Galois extension for k and G; equivalently all Galois extensions of field $L \supset K$ with group G can be parametrized by suitable mappings sending X_i to elements of L.

These ideas have been explored and enhanced in the 1987 work of David Saltman – the so-called generic freeness method.

Noether's problem

Noether's problem: Is $\mathbb{Q}[X_1, \ldots X_n)^G$ always rational over $\mathbb{Q}$? (If not, find all groups G for which it is so.)

Richard Swan (1969) showed that $\mathbb{Q}(X_1, \ldots, X_n)^G$ is not purely transcendental when G is a cyclic group of order $n = 47, 113, 233$. Hendrik Lenstra (1974) showed it when $n = 8$ or for any abelian group containing an element of order 8.

Inverse Galois Problem:

• Shafarevich showed that any solvable group can be realized as a Galois group over $\mathbb{Q}$.
• Most of the finite simple groups have been realized as Galois group over $\mathbb{Q}$. The largest one is the Monster group, and this was done by Thompson in 1984.

Present day status

Noether's method has proved decisive in attempts to prove the Inverse Galois problem. The rationality method of Noether has evolved in the work of Saltman (1987).

Since the group can always be realized over the fields $\mathbb{C}(X)$, $\mathbb{R}(X)$, or $\overline{\mathbb{Q}}(X)$ the Rigidity method, via Hilbert's Irreducibility Theorem, attempts to descend to $\mathbb{Q}$ (see Voskrenskinskii, Endo–Miyata, Colliot-Thélenè–Sansuc).

Finally we have the Galois representation methods being applied to the Inverse Galois problem; see Khare–Larsen–Savin (2008).

Noether Normalization

The paper "Der Endlichketsatz der Invarianten endlicher linearer Gruppen der Characteristik p" illustrates the power of Noether's abstract methods. She was able to complete the solution, which she had begun ten years earlier, to a special case of Hilbert's Fourteenth Problem: Let K be a subfield of the field of rational functions $k(X_1, \ldots, X_n)$ over a field k. Is $R := K \cap k[X_1, \ldots, X_n]$ a finitely generated k-algebra.

(In 1959 Masayoshi Nagata found a counterexample to Hilbert's conjecture, for a suitable ring of invariants for the action of a linear algebraic group.)

Noether Normalization

In 1926, Emmy Noether obtained a positive answer in the case of rings of invariants of a finite groups. Noether used what has become known as the Noether Normalization Lemma: which makes an affine ring a finitely generated module over a "better" ring; thereby exploiting not only finiteness conditions but all the good properties of the better ring. Normalization has today become a standard technique in Commutative Algebra.

Noether Normalization: (1926) Let R be an affine ring of dimension d over a field k. Let $I_1 \subset I_2 \subset \ldots I_m$ be a chain of ideals of R with $codim(I_j) = d_j$ and $d_1 > d_2 > \ldots > d_m \geq 0$. Then R contains a polynomial ring $S := k[X_1, \ldots, X_d]$ such that R is a finitely generated S-module and $I_j \cap S = (X_{d_{j+1}}, \ldots, X_d)$, for $j = 1, \ldots, m$.

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Noether did not do the "ideal" version. She also assumed k is an infinite field. The case when k is finite was done by O. Zariski in 1943 and later also by Nagata who gave the non-linear change of variables method. Nagata also gave the ideal case, when $m = 1$, in 1966.

Geometric intuition: For each d-dimensional affine variety $X \subset \mathbb{A}^n_k$, and chains of subvarieties, there exists a finite map taking X to an affine space $\mathbb{A}^d_k$ which carries the chain of subvarieties onto a chain of coordinate planes. If k is infinite then any sufficiently general linear projection $\mathbb{A}^m_k \longrightarrow \mathbb{A}^d_k$ induces such a finite map $X \longrightarrow \mathbb{A}^d_k$.

Remark: This geometric version was known and simply taken for granted by the Italian geometers.

Consequences

Corollary: If R is an affine domain over a field k then $dim(R)$ equals the transcendence degree of the quotient field of R over k; and this number equals the length of very maximal chain of prime ideals in R.

As a consequence of Noether normalization, one can recover
Hilbert's Nullstellensatz: Every maximal ideal of $\overline{k}[X_1, \ldots, X_n]$ is a "point" $(X_1 - \lambda_1, \ldots, X_n - \lambda_n)$ for some $\lambda_1, \ldots, \lambda_n \in \overline{k}$.

Strong form of HN: Let $\mathfrak{p}$ be a prime ideal of an affine ring R over a field k. If $\mathfrak{p}$ is a maximal ideal then $R/\mathfrak{p}$ is a finite field extension of k. In general, $\mathfrak{p}$ is the intersection of the maximal ideals of R containing $\mathfrak{p}$.

Krull Dimension

A precise topological definition of dimension of a topological space was first given by L.E.J. Brouwer (1913), working from the ideas of Poincaré.

In 1937 Krull proposed the definition of Krull dimension of a commutative ring R. Here he justifies it by quoting the geometric evidence accumulated by Emmy Noether (1923) for affine rings, and for factor rings of power series rings by W. Rickert (1932), and the analogy with the algebraic work done on Riemann surfaces.

Applications of Noether Normalization

Normalization:

Finiteness of Integral closure: Let L be a finite extension of the quotient field $Q(R)$ of an affine domain R. If S is the integral closure of R in L then S isa finitely generated R-module. In particular, S is an affine domain; and so the operation of normalization is well defined for algebraic varieties.

Normalization is an important tool in Resolution of Singularities of a variety.

Main Theorem of Elimination Theory
These are algebraic analogues of the fact that projective varieties over the complex numbers $\mathbb{C}$ are compact and Hausdorff in the classical topology.

Chevallen (1958) and Grothendieck isolated and studied the algebraic property that projective space has under the names **proper** (a kind of relative compactness) and **separated** (a relative form of Hausdorff property).

Main Theorem of Elimination Theory: If X is any variety over an algebraically closed field k, and Y is a Zariski closed subset of $X \times \mathbb{P}^n_k$, then the image of Y under projection to X is closed.

Projective version of Hilbert's Nullstellensatz

Hilbert's Nullstellensatz (Projective version): Let X be a d-dimensional subvariety of $\mathbb{P}^n_{\overline{k}}$. Then there exists a linear subspace L of dimension $n-d-1$ such that $L \cap X = \emptyset$. For all such L, the projection p_L restricts to a finite-to-one surjective closed map $p_L : X \longrightarrow \mathbb{P}^1_{\overline{k}}$ and the homogeneous coordinate ring $\overline{k}[X_0, \ldots, X_n]/I(X)$ of X is a finitely generated module over $\overline{k}[Y_0, \ldots, Y_d]$.

Corollary: The image of a projective variety under a morphism is closed. More precisely, if Y is a projective variety over a field k and $\pi : Y \longrightarrow X$ is a morphism to a projective variety X, then $\pi(Y)$ is a closed subset of X in the Zariski topology.

Density of Zariski Open Sets: $X \subset \mathbb{P}^n_k$ be a d-dimensional variety and let X_0 be a Zariski open set in X. Then the closure of X_0 in the classical topology is X.

Birational correspondences between projective varieties

One of the main activities of classical geometry was to find non-trivial birational correspondences between projective varieties; e.g. by projecting from a smooth point one gets a birational correspondence from an irreducible quadric to projective space. To find these correspondences, the principle method used was by using Linear systems.

Corollary: Compete Linear systems are finite dimensional.

Generic Flatness Lemma

Noether normalization is useful to establish:

Grothendieck's Generic Flatness Lemma: Let R be a noetherian domain and S a finitely generated R-algebra. Let M be a finitely generated S-module. Then there exists an element $a \neq 0$ in R such that M_a is a free R_a-module.

The proof of this is an example of a technique which Grothendieck called "devissage" (or "unscrewing" – after one application of the recursive step of the argument, you are back to the same spot but one dimension lower).

Fibre dimension: Upper semi-continuous

If $\varphi : R \longrightarrow S$ is a ring homomorphism then the fibre of φ at a prime $\mathfrak{p}$ of R is $Q(R/\mathfrak{p}) \otimes_R S$. (It is the affine coordinate ring of the (scheme theoretic) fibre of ${\varphi^*}^{-1}(\mathfrak{p})$.)

An invariant is said to be **upper semi-continuous** if for each integer e the set on which the invariant takes values $\geq e$ is closed.

Corollary: Let X be a variety over $\overline{k}$ and Y be a Zariski closed subset of $X \times \mathbb{P}^n$. For any number e, if X_e is the set of points p of X such that the fibre of Y over p has dimension $\geq e$, then X_e is closed in X.

Reciprocity and Fermat: Genesis of Algebraic Number Theory

The **Quadratic Reciprocity Law** was enunciated by Euler in the Opuscula, and also by Legendre's work in 1785. Both gave incomplete proofs. Gauss discovered and gave a proof of this law in 1796 when he was just 19.

Gauss arrive at a **law of cubic reciprocity** in papers from 1808 to 1817; and a **biquadratic reciprocity law** in papers of 1828 and 1832. He made use of the complex integers $\mathbb{Z}[i]$, now called the ring of Gaussian integers in the latter case; and the ring $\mathbb{Z}[\omega]$, where ω is a primitive cube root of unity, for the cubic case.

Biqudratic Reciprocity laws was proved by Gauss in 1828 by introducing the Gaussian integers $\mathbb{Z}[i]$. He established it is a unique factorization domain; and explited this property. He also established the cubic reciprocity law by similarly studying $\mathbb{Z}[\omega]$, where ω is a primitive third root of unity.

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Euler, Dirichlet, and Kummer all used this idea of adjoining a n-th root of unity η to $\mathbb{Z}$. They studied the unique factorization property of the ring $\mathbb{Z}[\eta]$ to prove special cases of Fermat's last theorem.
Neither Euler, Lagrange or Gauss envisioned the rich possibilities which their work on complex integers opened up. The theory of algebraic numbers grew out of the attempts to solve Fermat's assertion. Ernst Eduard Kummer (1810-1883), a pupil of Gauss and Dirichlet, extended the Gauss' theory to consider complex integers as those that satisfied the equation $a_0 + a_1\alpha + \ldots + a_{p-2}\alpha^{p-2}$ for $a_i \in \mathbb{Z}$.
He made the mistake to assume that unique factorization holds in this class of complex numbers; and claimed Fermat. Dirichlet pointed out this mistake in 1843. (Cauchy and Lamé made a similar mistake.)
$n = 23$ is the smallest case when $\mathbb{Z}[\eta]$, with η a primitive 23-rd root of unity, is not a unique factorization domain.

Ideal vs real

To restore unique factorization Kummer created a theory of ideal numbers in a series of papers starting 1844. Ideal numbers restore some of the unique factorization property; and Kummer's methods eventually could establish Fermat till $n = 256$!

Richard Dedekind (1851-1916), a pupil of Gauss, who spent fifty years of his life as a teacher at a technical high school in Germany, approached the problem of unique factorization in an entirely new and fresh manner. Instead of the number n he consider the set of all multiples of n, the ideal $n\mathbb{Z}$. He sought to find conditions under which in his number rings D these ideals nD had a unique factorization into prime ideals. Today these rings are called Dedekind domains.

Dedekind created the modern theory of algebraic numbers in his Zahlentheorie and its supplements (1871, 1876-77, 1879, 1894).

He introduced algebraic integers, the concept of a number field. the concept of a **ring of numbers**, the concept of **an ideal in a ring of numbers** (in fact, four versions of his theory of ideals).

Roots of Commutative Algebra

Emmy Noether studied and was influenced by the works of Dedekind; and she often attributed her ideas as being already in the work of Dedekind. She saw more deeply than others, how his ideas were really applicable in a wider context.
She also took into account the work of Kummer's student Kronecker who in 1881 put the notion of adjoining a root of a polynomial $f(x) = 0$ to a field; and introduced a theory for these polynomial rings equivalent to Dedekind's theory of ideals.

There is no way to factorize ideals in polynomial rings multiplicatively, as in Dedekind's theory, but in 1905 Lasker showed how to generalize unique factorization property into primary decomposition of ideals.

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Noether was deeply influenced by the work of Steinitz in field theory; and wanted to establish Ring Theory similarly on the foundations of axioms. She did this in her famous 1921 paper.
Both Dedekind's and Lasker's theories were throroughly reformulated and axiomatized by Emmy Noether in the 1920's, initiating the modern development of commutative algebra.

Rings and ideals

Emmy Noether introduced the concept of a ring, and an ideal, and a module over the ring. And formalized how one can play with ideals: addition, intersections, etc. and modules: direct sums, products, etc.

Noether's groundbreaking work in algebra began in 1920. In collaboration with W. Schmeidler, she published a paper about the theory of ideals in which they defined left and right ideals in a ring.

The following year she published a landmark paper called Idealtheorie in Ringbereichen, analyzing ascending chain conditions with regard to (mathematical) ideals. Noted algebraist Irving Kaplansky called this work "revolutionary"; the publication gave rise to the term "Noetherian ring" and the naming of several other mathematical objects as noetherian. Noether developed the theory of ideals in commutative rings into a tool with wide-ranging applications. She made elegant use of the ascending chain condition, and objects satisfying it are named noetherian in her honor.

Ascending Chain Conditions (ACC)

She was motivated to do everything possible in rings which were possible in number theory. **She recognized the importance of the ascending chain condition of ideals** (which first appeared in the works of Dedekind and Lasker). (She used to modestly say that whatever she has done is already in the work of Dedekind!)

Rings in which ideals satisfy the ascending chain condition (ACC) are now called noetherian rings in her honour.

She tied ACC up with the Lasker theory of primary ideals. Every ideal is the intersection of primary ideals. Lasker did this for polynomial rings over a field, and convergent power series rings; by using complicated arguments from elimination theory to make an induction on the number of variables.

She connected ACC to Hilbert's basis theorem: In such rings every ideal has a finite set of generators. Again: a greatly simplified approach! And conversely. Noether induction began here too.

She showed that such rings satisfy the descending chain condition on prime ideals. In her 1927 paper she connected it to the number rings studied by Dedekind, and by adding the property of integrally closed, showed the Dedekind property that every ideal in those rings were a product of prime ideals.

First Year course in T.I.F.R.

When I began teaching Commutative Algebra in the second semester first year course in T.I.F.R. I used to teach mainly about Noether's and Krull's works: noetherian rings, its applications to geometry via noether normalization, primary decomposition, Krull's work on it, dedekind domains, number rings and japanese rings, completions, in the first half and then some quadratic forms, brauer groups, leading up to the Hasse–Brauer–Noether theorem.
Today, the main shift is to include more of Krull's dimension theory, and study and charactere regular local rings. This has been possible by defocusing on primary decomposition and instead to only work with the associated primes. This shift is possible due to the hindsight of Kaplansky and to refined uses of the localization techniques. Incidentally, this localization techniques began in domains in the work of a doctoral student of Emmy Noether, Heinrich Grell in 1927; and later attained maturity in the works of Chevalley and Uzkov in the 1940's.

Papers on Dedekind rings and orders

By an order over an integrally closed noetherian domain R we mean a subring A of a central simple algebra C over the quotient field K of R such that A is a finitely generated R-module which spans C over K.

Noether wrote a number of other important papers on Dedekind rings and orders in algebraic number fields and function fields:

[1] Der Diskriminantensatz für die Ordnungen eines algebraischen Zahl-oder Funktionenkörpers (1927)

[2] Normalbasis der Körpern ohne höhere Verzweigung (1932)

[3] Ideal differentiation und Differente (1950)

In [1] she used the theory of finite dimensional commutative algebras over a field F to get an extension to a more general class of orders Dedekind's classical theorem that the rational primes that are ramified in the maximal order of integers of a number field are precisely the ones that divide the discriminant of the order. She defined separable algebras over commutative rings in this paper; and we also see the use of a localization argument for the first time.

Different ideal, tensor products, etc.

The paper [3] was started in 1927 but was only published posthumously in 1950, fifteen years after her death. She begins by noting the analogy between the main theorem of ramification theory of algebraic number theory with the elementary theorem on multiple roots of a polynomial. She then says that there is more than a formal analogy, since the different can be defined by a type of "ideal differentiation" that in special cases is related to differentiation of polynomials. The definition of the different and related concepts are based on the concept of **tensor products** of algebras over commutative rings; and this paper may be the first place in which tensor products are defined in this generality. She defined the notion of the different ideal of an R-algebra A relative to R. She studied the different of an order relative to R in a finite dimensional separable extension of the quotient field of a normal domain R. She obtained complementary basis, as in the classical case. She had wanted to pursue the classical theory in this more general set up.

Thus, in her last recorded work, she indicated the development of a Galois theory for algebras; and made substantial progress in the separable case. (See Auslander–Buchsbaum (1959), and Auslander–Goldman (1960) for further developments.)

Non-commutative algebra and Representation theory

Emmy Noether's impact on non-commutative algebra via Representation theory was equally powerful. The following two papers were fundamental:
[1] Hypercomplex Grössen und Darstellungstheorie (1929)
[2] Nichtkommutative Algebra (1933)
[1] deals with her approach to representation of algebras based on the Wedderburn's theorem, and results of E. Artin generalizing Wedderburn's theorem for rings satisfying DCC. She laid the foundations of representation theory of algebras.
She had already begun to write versions in 1924; and the final version was an edited version of the set of lecture notes taken by van der Waerden during the Winter session of 1927/28. She began the paper by saying that the most important general theorems about algebras go back to Molien; who had been the first to investigate representations of algebras, and finite groups, based on a structure theory. She also quoted an independent approach starting with Dedekind's concept of the group determinant, and Frobenius's result that the irreducible factors of the group determinant corresponded to the equivalence classes of the irreducible representations

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The theme of her article [1] was that the basic results about the representations of finite groups and algebras were all special cases of a general theory of non-commutative rings satisfying certain finiteness conditions (the ACC and DCC for chains of left ideals). The classification of irreducible modules and the proof of complete reducibility are achieved by a study of one-sided ideals in semisimple rings (or rings without radicals).

In this paper she began by introducing very basic tools which she would be using: she introduced the homomorphism theorems, Jordan–Holder theorem and theorems about complete reducibility for groups with operators satisfying ACC and DCC for chains of subgroups. (Note: Groups with operators were first introduced by W. Krulll (1925) and O. Schmidt (1928) to prove Remak's theorem (for groups with operators) about uniqueness properties of decompositions of finite groups into a direct product of indecomposable subgroups.

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Noether presented modules over non-commutative rings as examples of groups with operators, and they played a central role in her approach to representation theory. An algebra over a field k was a ring A with the additional structure of a finite dimensional (left) vector space over k, satisfying the condition $\lambda(ab) = (\lambda a)b = a(\lambda b)$, for $\lambda \in k$, $a, b \in A$. The identity element acted as the identity operator. The group algebra $k[G]$ was an example.

Noether showed the equivalence of writing a ring R as a decomposition of left ideals I_j, and writing 1 as a sum of idempotents $\sum e_j$, with $e_j^2 = e_j$, $e_i e_j = 0$, for $i \neq j$, and $I_j = Re_j$. She proved that the summands of R in a decomposition of R into a direct sum of indecomposable two sided ideals, are uniquely determined. (Extending such results of Wedderburn, Dickson).

Nilpotent ideals

She made the fundamental observation that in such a decomposition of R a left ideal contained in one summand was never isomorphic (as a left R-module) to a left ideal contained in another summand.
She developed the theory of nilpotent ideals and the radical ideal of a ring R. In a ring satisfying ACC for left ideals she showed that there is a unique maximal nilpotent left ideal N; and moreover N is a two sided ideal which contains all right nilpotent ideals. (This ideal is called the radical of R.)
A ring satisfying ACC and DCC is said to be completely reducible if is a finite direct sum of simple, or minimal, left ideal. Noether proved that a ring (satisfying ACC and DCC) is completely reducible if and only if its radical is zero. The proof uses an idea of Wedderburn (1908) that a non-zero left ideal in such rings contains a non-zero idempotent element. (Wedderburn's paper is the first recorded use of the ACC condition.)

Semi-simple rings and modules

Rings satisfying the two equivalent conditions above are called semisimple rings today. They can be decomposed as a direct sum of simple rings; which are uniquely determined. In view of Wedderburn's theorem on simple rings, one has the complete structure theorem of semi-simple rings. Thus, Noether's approach to representation theory of algebras was based on the idea that these representations correspond to a left R-module in such a way that two representations are equivalent if and only if the correspond left modules are isomorphic. Thus the classification theory becomes the construction and classification of left R-modules; which she did when the algebra was semi-simple However, she also included a few results in the non semi-simple case.

She thus managed to recover the theorems of Masche, Frobenius and Burnside by an elegant argument which placed them in a much more general background; she worked with linear transformations instead of matrix theory. She also reproved the Frobenius–Schur theorem (1906) extending Maschke's theorm to an infinite group of matrices which is not assumed to be reducible. H.Weyl, in 1928, used her ideas in his work on applications of group representations in quantum mechanics in his book.

Brauer and Noether's approaches

Brauer, and independently, Emmy Noether, realized that Schur's theory of index, combined with Wedderburn's theorem, provided a link between representation theory and the theory of simple algebras. Brauer's study of irreducible matrix groups, and their behaviour under extension of the groud field, continuing the ideas of Schur, led him to a theory of factor sets for irreducible groups of matrices, and to the concept of the Brauer group. He made the transiion from the representation theory of matrix groups to the theory of simple algebras.
Noether took a direct step to the theory of simple algebras. There were a series of letters between Noether and Brauer between 1927 to 1934. This led to the Brauer–Noether theorem (1927) on existence of splitting fields of simple algebras and to characterize the splitting fields of a division algebra as maximal commutative subfields of the algebra itself or of a full matrix ring over this algebras. This imbedding of the splitting field provides deep insight into the structure of the algebra: which can be interpreted as a crossed product of the splitting field with its Galois group. Later in 1931 it led to the announcement of the Brauer–Hasse–Noether theorem that every central division algebra over a number field is cyclic.

Generalized airthmetic of hypercomplex systems

The Brauer–Hasse–Noether theorem is that a central simple algebra over an algebraic number field K which splits over every completion K_v is a matrix algebra over K. This is the local-global part of the theory. As a consequence, it leads to a complete description of finite dimensional division algebras over algebraic number fields in terms of their local invariants. Together with the Grunwald–Wang theorem, the Brauer–Hasse–Noether theorem implies that every central simple algebra over an algebraic number field is cyclic, settling a conjecture of Dickson.

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There are three different proofs of this theorem.
The proof by Brauer–Hasse–Noether used the theory of division algebras over a p-adic field, and the theory of the norm-residue symbol, which Hasse developed in 1932. It was Hasse inspired approach!
Albert and Hasse independently gave a different proof in 1932,
The idea of factor sets appeared as early as Hölder's 1893 paper; again in Schur's 1904 study or projective representations; and yet again in Dickson's construction of cyclic algebras. Brauer's theory of factor sets attaches a factor set to each central simple algebra A over k with a splitting field $k(\theta)$ of finite degree over k. When $k(\theta)/k$ is Galois, Noether developed a simpler version of the factor set, and a type of central simple algebra called a crossed product. Dickson's cyclic algebras were a special case of crossed product when the Galois group is cyclic.
In the second edition of his book, van der Waerden gave a self contained proof of Brauer–Noether theorem, etc. leading to the Brauer–Hasse–Noether theorem, based on the theory of crossed products. M. Deuring in his survey volume (1925) and A. Albert in his Colloquium Publication of 1938 gave a proof the Brauer–Hasse–Noether theorem continuing these ideas.

Her ICM address in Zürich, 1932

In the 1943 paper Der Hauptgeschlechtssatz für relativ-galoissche Zahikörpher Emmy Noether initiated the idea of applicability of non-commutative algebra, especially, the theory of central simple algebras, to commutative algebra. This was the theme of her address to the 1932 International Congress of Mathematicians at Zürich. In Nichtkommutative Algebren (1933) she applied non-commutative methods to derive the Galois theory of fields. In her Zürich address she emphasized the applicability of these methods to arithmetic. Under her influence Hasse (1932) and Chevalley (1933) applied these methods to obtain some of the main results of global and local class field theory. In her 1943 paper she gave an extension to Galois extensions of a known result of class field theory on cyclic extensions of a number field. The principal tool she used was the theory of crossed products and Hasse's local-global principle. Before proving the main theorem she gave her generalization of Hilbert's Theorem 90 that the cohomology group $H^1(G, E^*)$ is trivial, if E/F is Galois with group G. She proves this by applying the Skolem–Noether theorem to the crossed product of E and G with trivial factor set. Jacobson also noted that in the proof of the main theorem, Noether used a connecting homomorphism: first seen here!