Noether's Theorems:

their

Growing Physical Relevance

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Introduction: The Noether Theorems

- Noether's work on symmetries and conservation laws for variational problems (1918)
- Forgotten within a few years
- Reappeared in truncated form in the 1950's
- Only in 1960's 70's and 80's did physicists appreciate the full power and generality of her work.
- Why did it take so long?
- Summary of talk

Introduction: why so little impact?

- Brief answer: the power of variational principles were not fully appreciated by physicists a that time
- There were two theorems
  symmetries of the first kind: transformations depending on parameters
  symmetries of the second kind: transformations depending on arbitrary functions
- Edward Hill presented Noether's work in simplified form (1951) to mathematical physicists
- Physicists of the time latched on to theorem 1 and quoted it as The Noether Theorem.

Introduction: Linguistics

- George Bernard Shaw/Oscar Wilde two peoples divided by a common language
- Mathematicians and Physicists need better understanding of each other
- Michael Atiyah urges mathematicians to understand physics
  But also presents them with the harder task of understanding physicists
- The rigour and generality of mathematical treatments is often a barrier to physicists
- Mathematicians are often mystified by the language used by physicists

Introduction: Dialects

- For example the words "Action, Field Theory" means different things
- In physics Fields are sections of vector bundles over spacetime. (more general fields exist, connections, or maps between manifolds)
- Action provides the variational principle that determines the equations governing the fields
- A physicist's *theory* is determined by a set of fields, and an action principle. The challenge is to work out its predictions
- The final objective is to test these predictions by comparison with experiments (eg. collider experiments, spectroscopy)

Variational Principles in Physics

- Can express mechanical laws in terms of forces (Newton) or a variational principle (Lagrange).
- Optics Laws of reflection (Hero of Alexandria) or Refraction (Fermat) Laws seem disparate until they are unified by a variational formulation
- soap films either force balance, or minimisation of energy
- In the 1910's it was not at all obvious that physical laws could/should be expressed as variational principles.

Variational Principles: believers and sceptics

- Einstein (GTR) was not convinced of the need for a variational formulation. He was looking for equations of motion in tensorial form in fact, he first wrote his equations in the incorrect form
  Ricci Tensor=Matter Tensor
- Neither was Pauli a believer in variational principles
- Resistance may have been due to dissipative systems (those generating entropy, more later). However Newton's planetary system has neglibible dissipation
- Atomic physics is also well described without dissipation

Variational Principles: Hilbert´s approach to GTR

- Hilbert believed in a variational formulation. He was working on a variational formulation of Einstein´s theory of GTR
- but F. Klein dismissed Hilbert´s "fanatical belief in the variational principles, the view that one can describe the reality of Nature by purely mathematical considerations".
- Wigner: "The unreasonable effectiveness of mathematics in physics.

Variational Principles: an elegant restatement

- Variational principles provide an elegant restatement of the laws of physics
- statics the minimisation of energy
- mechanics the Lagrangian formulation
- optics Fermat's principle of least time
- Field theory The Action principle
- Equilibrium Statistical Mechanics Minimise Free energy
- non-equilibrium still open

Symmetries: Lie Groups

- Hermann Weyl: Symmetry, as wide or narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.
- Entered mathematics through Sophus Lie's study of symmetries of differential equations
- with Quantum theory (Eugene Wigner), the mathematical machinery of Group theory entered physics in a big way. (Gruppenpest).
- Noether's work deals with the intersection between Variational Principles and Lie Group theory. Neither of these paradigms had been assimilated into the physicists' thinking.

Symmetries:Erlangen

- Felix Klein Erlangen Program: classify geometries according to their invariance group(connected component)
  plane geometry ISO(2) Euclidean group E(2)
  spherical geometry SO(3) Rotation group
  Hyperbolic geometry PSL(2,R) or SO(2,1)
- similar attitude in physics emerged about symmetries of spacetime
- Aristotelian spacetime $E^3 \times E^1$ Aristotelian group $ISO(3) \times R$, dimension 7
- Galilei spacetime E^3 bundle over E^1 Galilei group dimension 10
- Minowski spacetime ISO(3,1) or Poincare group

Symmetry: Conservation Laws

- Characterise a physical theory by its symmetries
- Use symmetries to write down a Variational formulation (Action)
- Symmetries plus Noether's theorems give us conservation laws which can be used to simplify the analysis of the equations of motion.

Symmetries: some history

- special instances of conservation laws known earlier Newton momentum, angular momentum, Lagrange Energy
- Noether's theorems formalise the connection between symmetries and conservation laws
- Symmetries of spacetime are those transformations that preserve the background structure

Noether´s first theorem

- For a variational problem with symmetries of the first kind.
- associates to every continuous symmetry of the first kind a conservation law
- The Lagrangian formulation naturally induces a symplectic formulation on the cotangent bundle.
- The conserved quantity is a function on the cotangent bundle
- Its Hamiltonian vector field is the infinitesimal generator of the symmetry transformation.

Noether´s first theorem: examples

- All the spacetime symmetries are generated by such conserved quantities
- Energy, Momentum, Angular Momentum, Boost Generators
- For electrodynamics the action is invariant under the 15 parameter conformal group. These led to new conservation laws not known before (Bessel-Hagen).

Noether's second theorem

- For a variational problem with symmetries of the second kind
- the corresponding conserved quantity vanishes identically
- The Lagrangian defines a map of the tangent bundle into the cotangent bundle.
- This gives rise to a two form ω which is degenerate and closed. The kernel of this two form are the generators of symmetries of the second kind
- This was worked out in the 1930's by Dirac in his theory of constrained dynamics
- occurs in theories with gauge invariance or diffeo invariance

Nother's second theorem: Toy Model

- Consider the following simple model
- $\vec{x} \in R^3$ and $I = \int L dt$, where $\vec{v} = \frac{d\vec{x}}{dt}$

$$L = (\vec{v}.\vec{v})^{1/2}$$

- L is homogeneous of degree one in the velocities
- Action is invariant under arbitrary reparametrisations
- Find that the Hamiltonian vanishes
- example of diffeo invariance on the world line. Example of a gauge symmetry.

Noether's second theorem: gauge theories

- Hermann Weyl put forward a theory by generalising parallel transport to allow for non integrable length changes.
- He was trying to unify GR with electromagnetism. The theory turned out to be not in accord with experiments. But it had a mathematical structure that remained in physics, the idea of gauge invariance.
- Weyl's idea was that lengths could only be compared at the same location in space time
- The same mathematical idea survives in today's electrodynamics. It is the phase of the wave function which is non integrable.

Symmetries of the second kind: importance in physics

- Have come to dominate physics
- Einstein's GTR threw up diffeo invariance
- in the 70's non Abelian gauge theories
- in the 80's string theories These are described by a map from the 2dimensional world sheet to space time. Diffeos internal to the world sheet are symmetries of the second kind.

Gravitational Energy

- The genesis of Noether's paper was Hilberts attempts to understand General Relativity
- Translations are a subgroup of diffeo group
- It appeared that there was no conservation law for energy
- Noether's work showed clearly that this is a general consequence of diffeo invariance

Gravitational Energy: its elusive nature

- Took many years to digest this
- Since there is no background structure in GR all diffeos preserve the action and we are in the purview of theorem 2.
- for a closed Universe, this is in fact true, the total energy, momentum, charge angular momentum all vanish.
- for asymptotic flatness, we have to recognise that there is an asymptotic structure at infinity. So one can define the total energy of an isolated system. Came out of the Hamiltonian formulation ADM Energy

- Other attempts Komar integrals
- impossible to localise gravitational energy. Nevertheless, gravitational energy is real. It can be made to vanish at any point but not everywhere.
- Described by pseudotensors not tensors
- non abelian gauge theories

Conclusion: Summary

- It appears that Noether's theorems were neglected because physics was not prepared at the time for two important ideas
- Variational Principles and Symmetries
- Many of the development of modern physics can be traced to these important ideas
- gauge theories, conserved currents
- in the 70's graded Poincare Algebras (Supersymmetry) were discovered. Square root of translations.
- were soon elevated to symmetries of the second kind (Supergravity)

Conclusion: Emmy Noether

- Emmy Noether was the daughter of a mathematician. Initially studied languages (English and French) but switched to mathematics
- Her work on symmetries is a preamble to her main life's work in mathematics, modern algebra.
- She didn't consider herself a physicist, yet contributed hugely to physics.
- refers in her paper to the physicists' General Relativity

referee report

- last para of a referee report (Zaycoff, Bessel-Hagen) written by Noether to Einstein: I cannot appreciate to what extent the integration of the conservation laws is interesting from the point of view of physics. If that were the case, it might be possible to induce a physics journal to accept this limited part, with a reference to Bessel-Hagen;

referee report

- it would also be possible to introduce in it a reference to the statement of my theorems in Courant-Hilbert (Yellow Collection), p. 216, one of the most recent volumes, with an explanatory text. But for this, I must leave it to the physicists to judge the value. With my best wishes for 1926 and my best regards, Your devoted, Emmy Noether.

thrice discriminated

- Noether had visited Moscow, was generally supportive of the October revolution and like many academics had Marxist sympathies.
- she was thus at the intersection of three discriminations Woman, Marxist, Jew.
- In fact, she was once evicted from her lodgings because some German student leaders who lived in the same building wrote to the landlord that they didn't want to share their living quarters with a Marxist-leaning Jewess.

Conclusion: unpaid lecturer

- She completed her dissertation (1907) at the University of Erlangen and went on to teach there unpaid for seven years
- She was then invited (1915) by Hilbert and Klein to join the mathematics department at Gottingen.
- Philosophical faculty (philologists and historians) objected vehemently. spent four years lecturing under Hilbert's name. (Unpaid)
- 1919 she finally got the rank of privatdozentin (also unpaid)
- only started getting paid for her work in 1923

Conclusion: Baden am Göttingen

- (1915) One faculty member protested: "What will our soldiers think when they return to the university and find that they are required to learn at the feet of a woman?"
- Hilbert responded with indignation, stating, "I do not see that the sex of the candidate is an argument against her admission as privatdozent. After all, we are a university, not a bath house.

Conclusion:lecturing style

- Her lectures were spontaneous discussions with the students and demanded their active involvement. Not all students could stomach this style of lecturing, but many mathematical results emerged from the lectures, subsequent discussions and lecture notes.
- She was generous with her ideas and many of them were developed and published by others, students and colleagues
- she was very popular with students at the University and was naturally warm and friendly, occasionally critical but always nurturing.

Emmy Noether

Emmy Noether with her students and colleagues