Role of Symmetries in Condensed matter physics

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The Legacy of Emmy Noether, ICTS, Aug. 29-30, 2016

Emmy Noether

1882–1935

Memorial address: Emmy Noether

by Hermann Weyl

http://celebratio.org/Noether_E/article/111/

Indeed, two traits determined above all her nature: First, the native productive power of her mathematical genius. She was not clay-pressed by the artistic hands of God into a harmoni ous form, but rather a chunk of human primary rock into which He had blown His creative breath of life. Second, **her heart knew no malice; she did not believe in evil** — indeed it never entered her mind that it could play a rôle among men. This was never more forcefully apparent to me than in the last stormy summer, that of 1933, which we spent together in Göttingen. The memory of her work in science and of her personality among her fellows will not soon pass away. She was a great mathematician, the greatest, I firmly believe, that her sex has ever produced and a great woman.

Noether's Theorem

Continuous Symmetries

$\updownarrow$

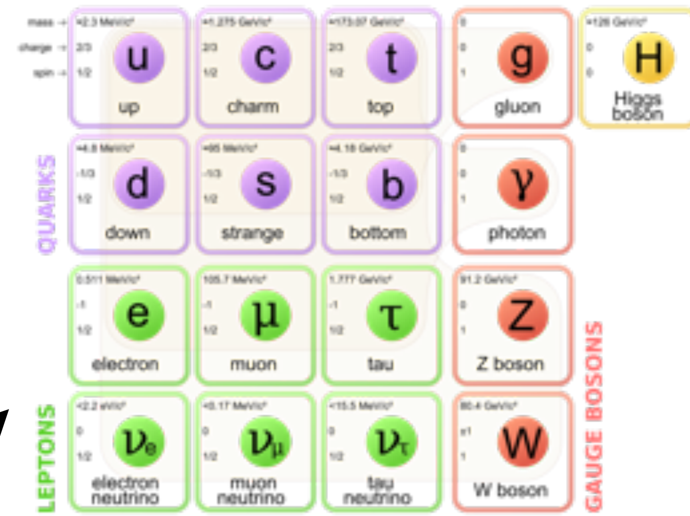
Conservation Laws

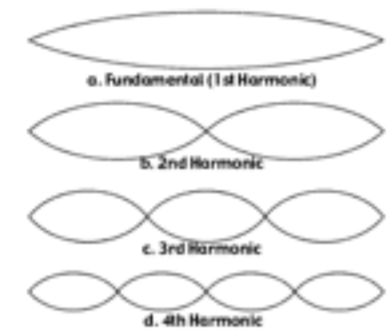
$\updownarrow$

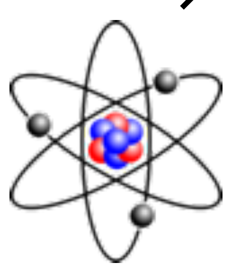
Invariants (of motion)

Symmetries

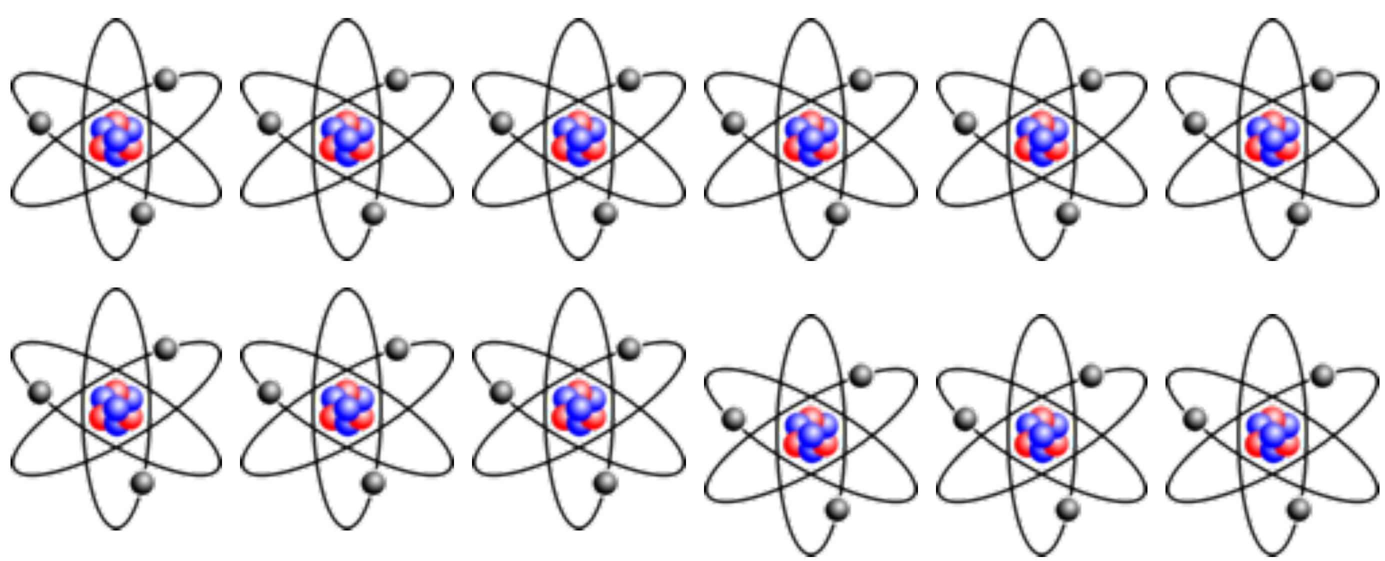
(in condensed matter physics)







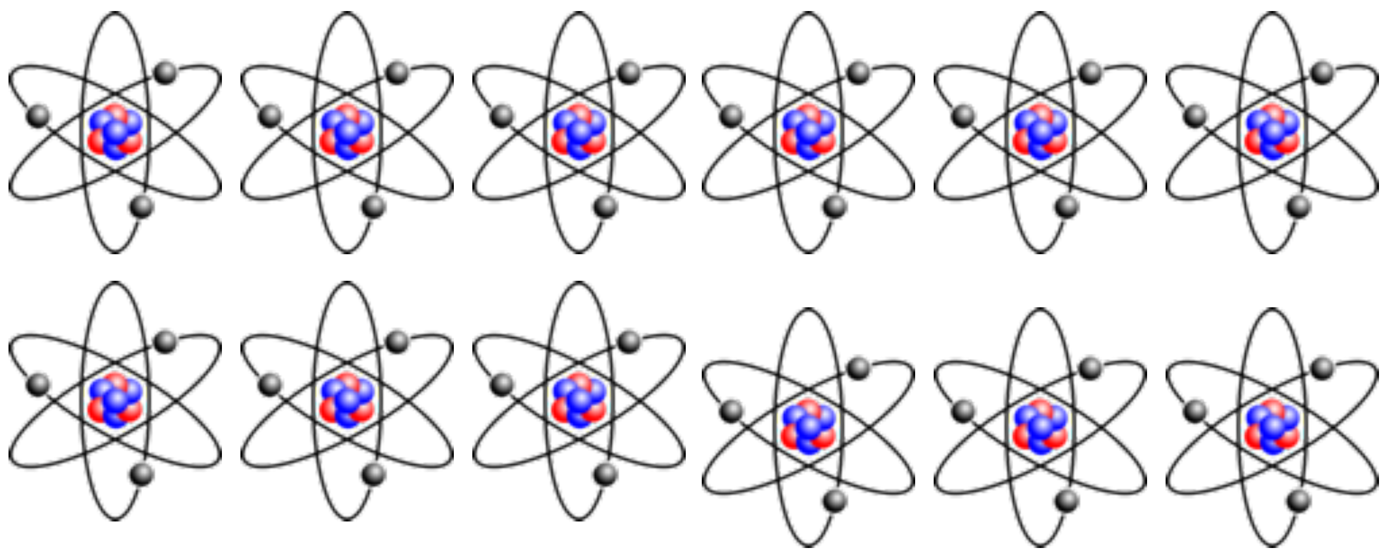
Degrees of freedom depend of length/energy scale



Emergent Phenomena

- Many particles give rise to qualitatively new behaviour

Materials

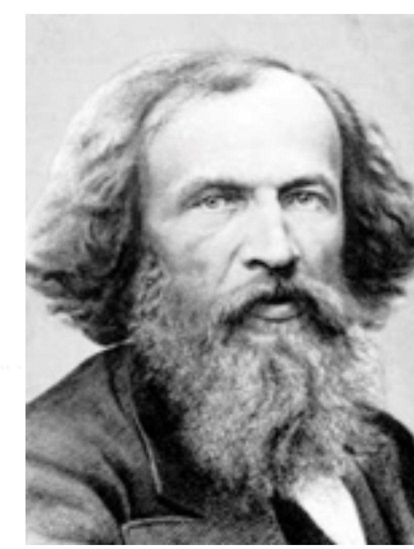


$$\sim 10^{23}$$

Periodic Table

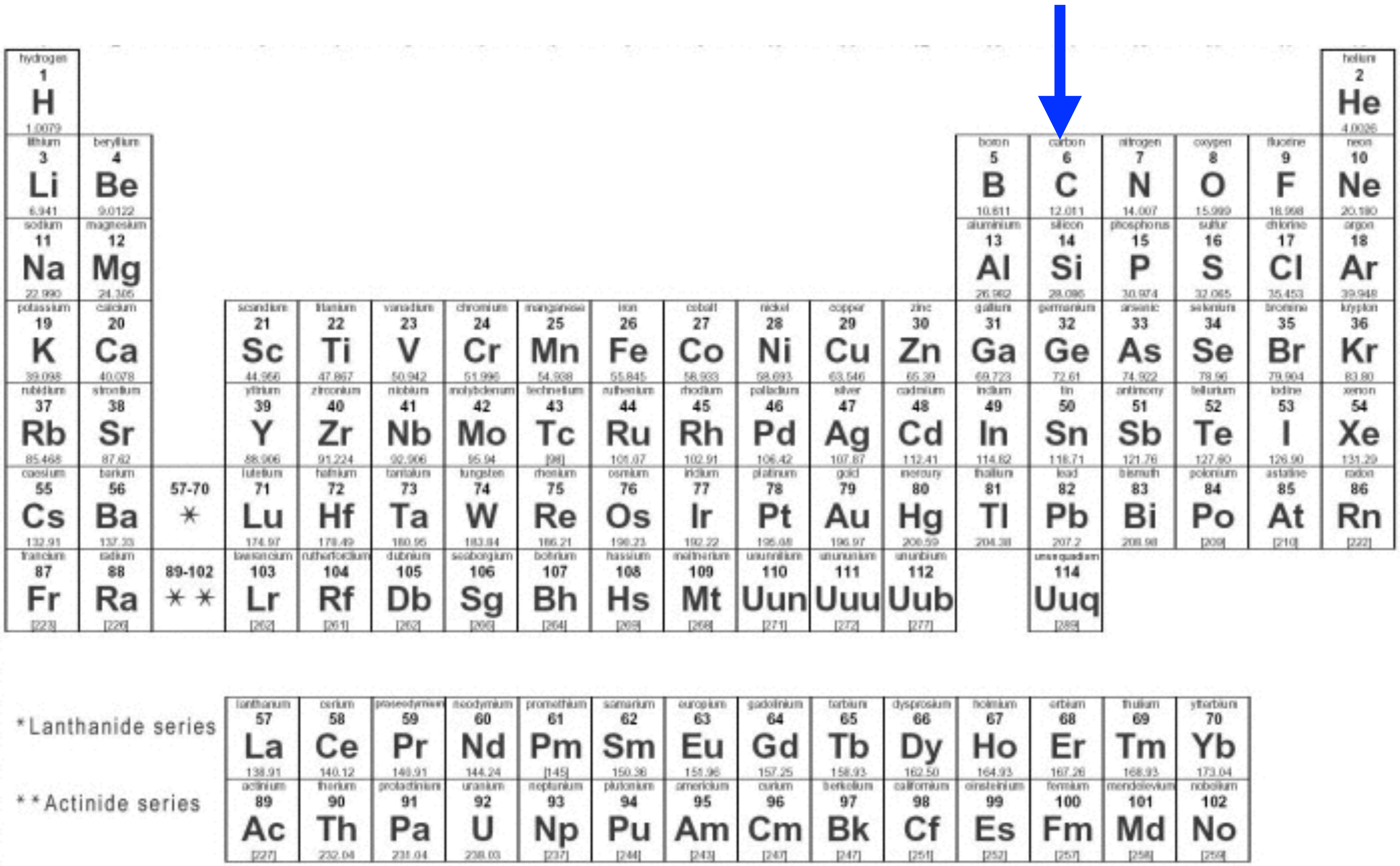
hydrogen 1 H 1.0079																		helium 2 He 4.0026
lithium 3 Li 6.941	beryllium 4 Be 9.0122												boron 5 B 10.811	carbon 6 C 12.011	nitrogen 7 N 14.007	oxygen 8 O 15.999	fluorine 9 F 18.998	neon 10 Ne 20.180
sodium 11 Na 22.990	magnesium 12 Mg 24.305												aluminium 13 Al 26.982	silicon 14 Si 28.086	phosphorus 15 P 30.974	sulfur 16 S 32.065	chlorine 17 Cl 35.453	argon 18 Ar 39.948
potassium 19 K 39.098	calcium 20 Ca 40.078		scandium 21 Sc 44.956	titanium 22 Ti 47.867	vanadium 23 V 50.942	chromium 24 Cr 51.996	manganese 25 Mn 54.938	iron 26 Fe 55.845	cobalt 27 Co 58.933	nickel 28 Ni 58.693	copper 29 Cu 63.546	zinc 30 Zn 65.39	gallium 31 Ga 69.723	germanium 32 Ge 72.61	arsenic 33 As 74.922	selenium 34 Se 78.96	bromine 35 Br 79.904	krypton 36 Kr 83.80
rubidium 37 Rb 85.468	strontium 38 Sr 87.62		yttrium 39 Y 88.906	zirconium 40 Zr 91.224	niobium 41 Nb 92.906	molybdenum 42 Mo 95.94	technetium 43 Tc [98]	ruthenium 44 Ru 101.07	rhodium 45 Rh 102.91	palladium 46 Pd 106.42	silver 47 Ag 107.87	cadmium 48 Cd 112.41	indium 49 In 114.82	tin 50 Sn 118.71	antimony 51 Sb 121.76	tellurium 52 Te 127.60	iodine 53 I 126.90	xenon 54 Xe 131.29
caesium 55 Cs 132.91	barium 56 Ba 137.33	57-70 *	lutetium 71 Lu 174.97	hafnium 72 Hf 178.49	tantalum 73 Ta 180.95	tungsten 74 W 183.84	rhenium 75 Re 186.21	osmium 76 Os 190.23	iridium 77 Ir 192.22	platinum 78 Pt 195.08	gold 79 Au 196.97	mercury 80 Hg 200.59	thallium 81 Tl 204.38	lead 82 Pb 207.2	bismuth 83 Bi 208.98	polonium 84 Po [209]	astatine 85 At [210]	radon 86 Rn [222]
francium 87 Fr [223]	radium 88 Ra [226]	89-102 * *	lawrencium 103 Lr [262]	rutherfordium 104 Rf [261]	dubnium 105 Db [262]	seaborgium 106 Sg [266]	bohrium 107 Bh [264]	hassium 108 Hs [269]	meitnerium 109 Mt [268]	ununnilium 110 Uun [271]	unununium 111 Uuu [272]	ununbium 112 Uub [277]		ununquadium 114 Uuq [289]				

*Lanthanide series	lanthanum 57 La 138.91	cerium 58 Ce 140.12	praseodymium 59 Pr 140.91	neodymium 60 Nd 144.24	promethium 61 Pm [145]	samarium 62 Sm 150.36	europium 63 Eu 151.96	gadolinium 64 Gd 157.25	terbium 65 Tb 158.93	dysprosium 66 Dy 162.50	holmium 67 Ho 164.93	erbium 68 Er 167.26	thulium 69 Tm 168.93	ytterbium 70 Yb 173.04
* *Actinide series	actinium 89 Ac [227]	thorium 90 Th 232.04	protactinium 91 Pa 231.04	uranium 92 U 238.03	neptunium 93 Np [237]	plutonium 94 Pu [244]	americium 95 Am [243]	curium 96 Cm [247]	berkelium 97 Bk [247]	californium 98 Cf [251]	einsteinium 99 Es [252]	fermium 100 Fm [257]	mendelevium 101 Md [258]	nobelium 102 No [259]



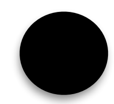
D. Mendeleev, 1834–1907

Degree of freedom : carbon



Symmetries

1 C atom



- The laws of physics governing a single atom is completely symmetric under translation and rotation

2 C atoms

- The interactions between two atoms depend on their relative position (symmetric under translation and rotation)
- This extends to many carbon atoms

Specifying condensed matter systems

1. Degrees of freedom (Carbon atom)

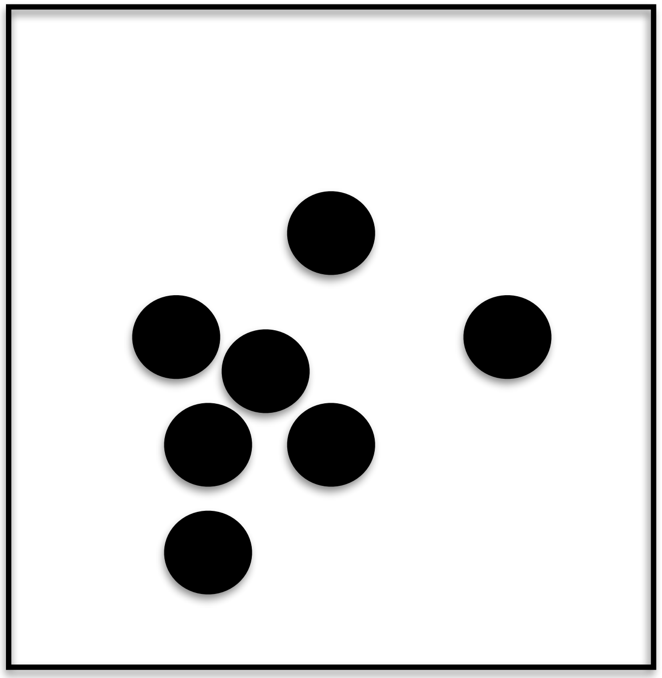
2. Symmetries (Translation and Rotation)

3. Interaction (Hamiltonian)

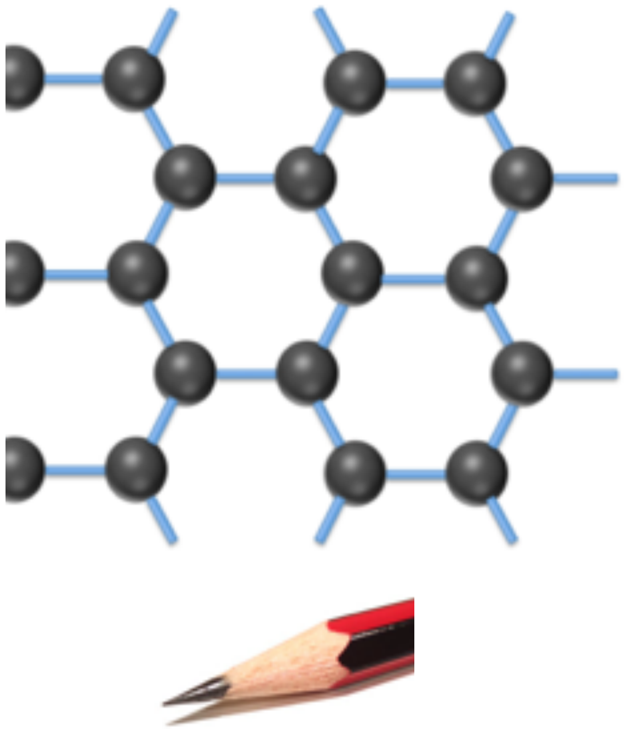
$$\mathcal{H} = \sum_i \frac{\mathbf{P}_i^2}{2m} + \sum_{\langle ij \rangle} \mathcal{V}(|\mathbf{r_i} - \mathbf{r_j}|)$$

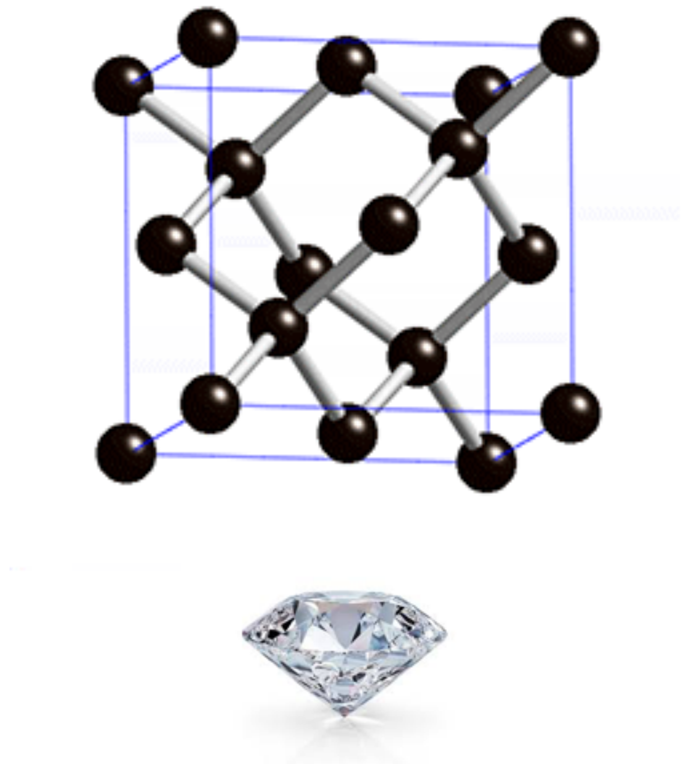
What is the state of the system of many carbon atoms ?

- High temperature fluid state : Indeed the state respects translation and rotation symmetries



- Low temperatures : The system breaks the translation symmetry **spontaneously** to form a lattice.



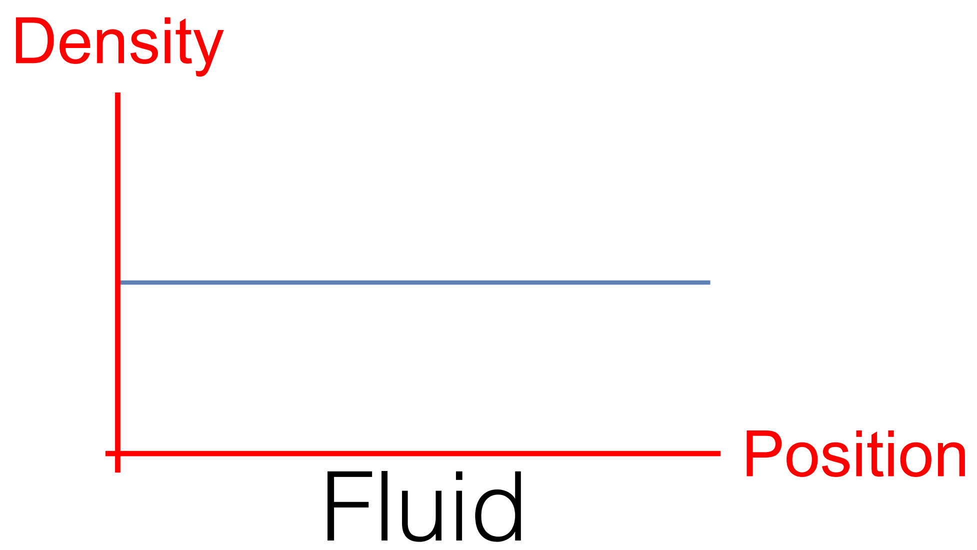


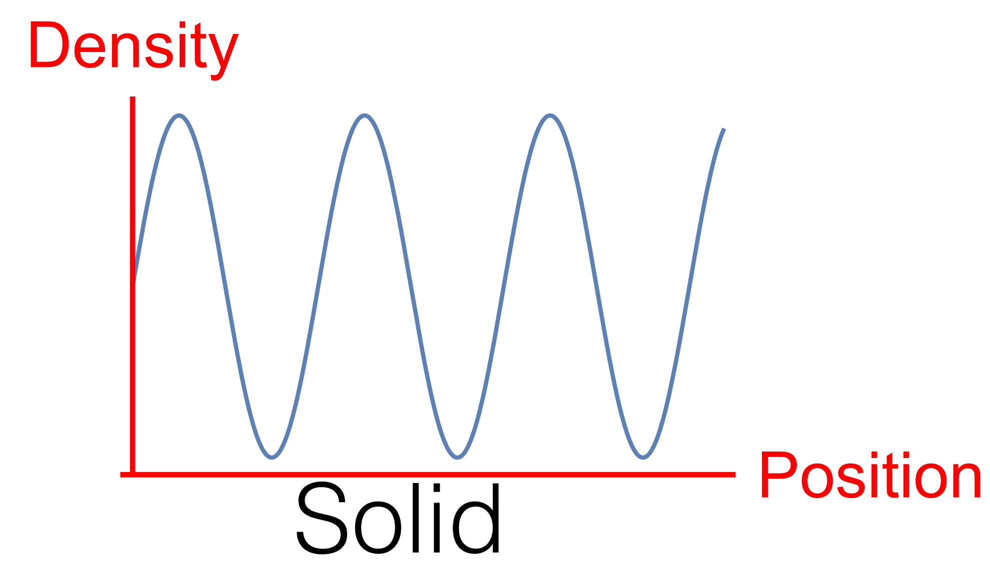
- Physical properties drastically differ depending on the pattern of symmetry breaking.

- Example of emergent phenomena

Characterising spontaneous symmetry breaking

Average density profile $\langle \rho(x) \rangle \sim \rho_0 + \rho_{\mathbf{k}} \Re \left[e^{i\mathbf{k}\cdot\mathbf{x}} \right]$



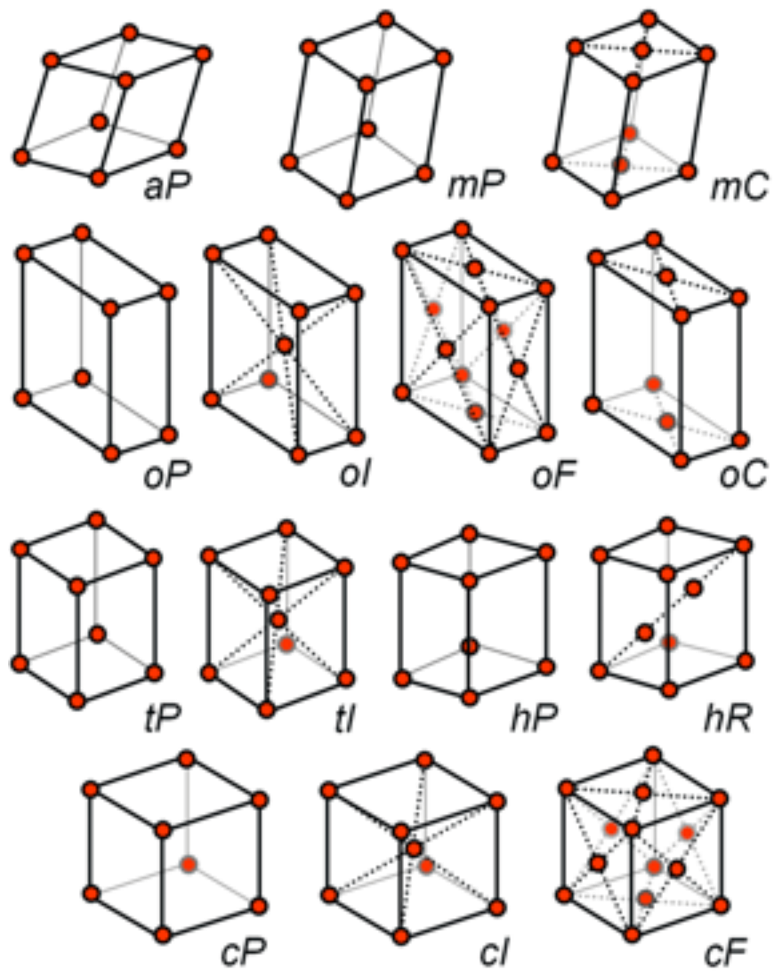


$\rho_{\mathbf{k}} = 0$ $\rho_{\mathbf{k}} \neq 0$

$\rho_{\mathbf{k}}$ is an example an "order parameter"

Particular values of $\mathbf{k}$ determines the lattice structure

3D crystals : (230 space groups)



A. Bravais, 1811-1863

Almost all of them found in nature

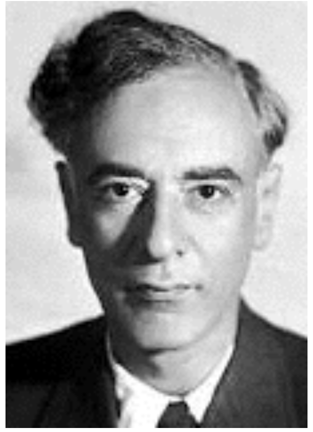
Order parameter : Generalisation

- Order parameter quantifies the symmetry breaking

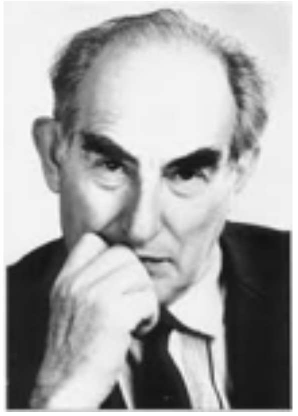
- The structure of the order parameter manifold determines the low energy excitations in a phase

- This features are generalised to study all phases like ferromagnets, superconductors etc.

- Basis of classifying condensed matter phases over the better part of last century



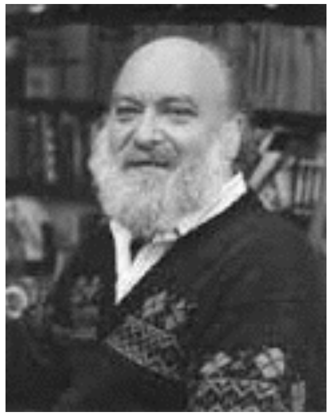
Landau, 1908-1968



Ginzburg, 1916-2009

Wilson, 1936-2013

Anderson, 1923-



Kadanoff, 1937-2015

Order parameter : Generalisation

- Order parameter quantifies the symmetry breaking
- The structure of the order parameter manifold determines the low energy excitations in a phase

Noether's theorem

$$S = \int d^d x \, \mathcal{L}[\phi_a, \partial_\mu \phi_a]$$

For invariance under

$$\phi_a \to \phi_a + \delta\phi_a$$

$$\partial_\mu J^\mu = 0$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a$$

Noether's Charge

$$Q = \int d^D\vec{x}\; J^0(\vec{x},t)$$

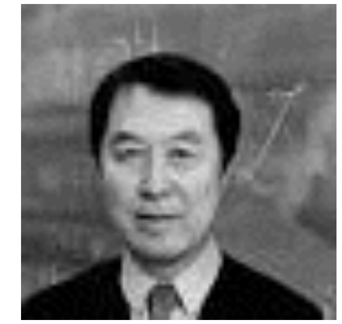
$$[Q, \phi_a(x)] = \int d^D\vec{x}'\,[J^0(x'), \phi_a(x)]$$

$$= \int d^D x' \left[\frac{\partial L}{\partial \dot{\phi}_b}\delta\phi_b , \phi_a(x)\right]$$

$$= -i\,\delta\phi_a(x)$$

$\therefore$ Symmetry Transformation generated by Q

Nambu - Goldstone Bosons



Nambu, 1921-2015

Goldstone, 1933

$$[H, Q] = 0$$

$|\psi_{GS}\rangle \rightarrow$ Ground state with $E = 0$

i.e, $\quad H|\psi_{GS}\rangle = 0$

If $|\psi_{GS}\rangle$ is invariant under symmetry transformation, then

$$e^{i\theta Q}|\psi_{GS}\rangle = |\psi_{GS}\rangle \Rightarrow Q|\psi_{GS}\rangle = 0$$

Nambu - Goldstone Bosons

However if $|\psi_{GS}\rangle$ Spontaneously Breaks the Symmetry, then

$$e^{i\theta Q}|\psi_{GS}\rangle \neq |\psi_{GS}\rangle$$

$$\Rightarrow \quad Q|\psi_{GS}\rangle \neq 0 = |\psi'_{GS}\rangle$$

$$H|\psi'_{GS}\rangle = HQ|\psi_{GS}\rangle = [H,Q]|\psi_{GS}\rangle = 0$$

$\rightarrow$ $|\psi'_{GS}\rangle$ is also a ground state.

Nambu - Goldstone Bosons

Consider the state

$$|k\rangle \sim \int d^D\vec{x} \; e^{i\vec{k}\cdot\vec{x}} \; J^0(\vec{x}) \, |\psi_{GS}\rangle$$

This is a state with momentum $\vec{k}$

$\epsilon_{\vec{k}} \to 0$ as $\vec{k} \to 0$ as $H Q |\psi_{GS}\rangle = 0$

These gapless modes (Nambu-Goldstone bosons) are protected by Symmetry breaking

Phonon as Nambu-Goldstone Boson

→ Solid : Spontaneous breaking of Translation Symmetry

→ Gapless excitations : Lattice Vibrations. (Phonons)

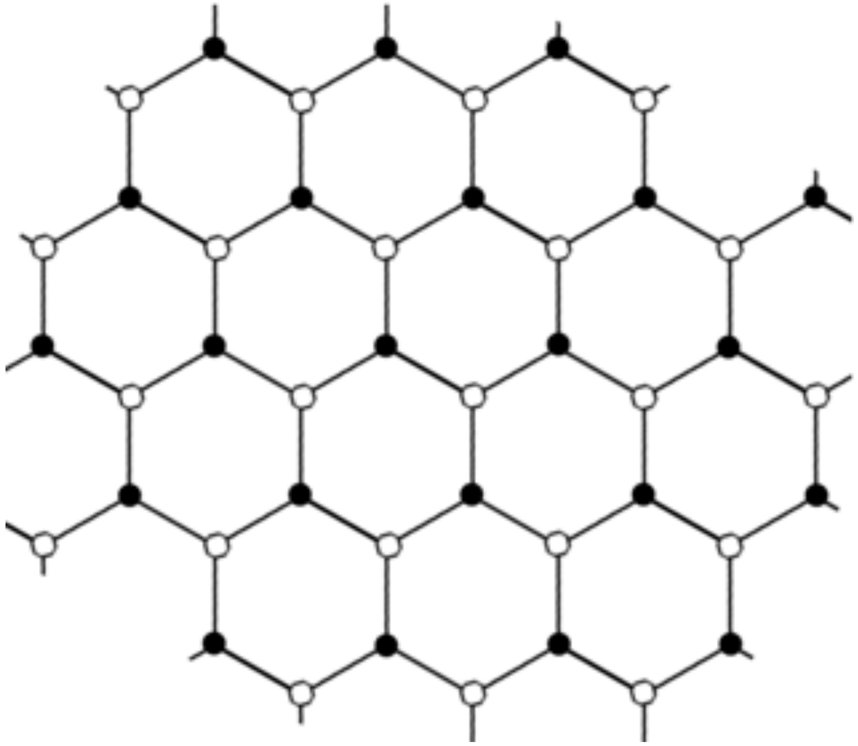
Phonons have bosonic statistics and can be considered as a low energy "particle" that emerges due to Spontaneous Sym. breaking

Role of symmetry beyond spontaneous breaking

- Over the last 20-30 years, it has emerged that symmetry can play a more subtle role in quantum condensed matter systems beyond spontaneous breaking.
- While we are presently trying to understand the general structure (some of it already done), let us discuss two examples

Electrons in a crystal

- Degrees of freedom : Electrons hopping on a honeycomb lattice

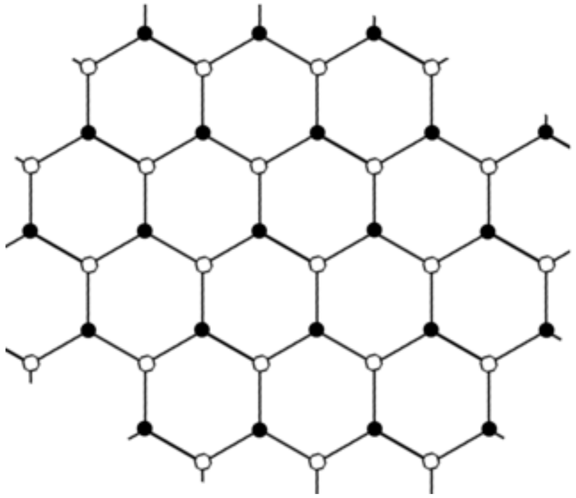


- At each site, 4 possible states $|0\rangle_i, \ |\uparrow\rangle_i, \ |\downarrow\rangle_i, \ |\uparrow\downarrow\rangle_i$

Fermions

Electrons in a crystal : Symmetries

- Translation and rotations of the Honeycomb lattice
- Charge Conservation : Electron numbers should not change
- Time reversal symmetry

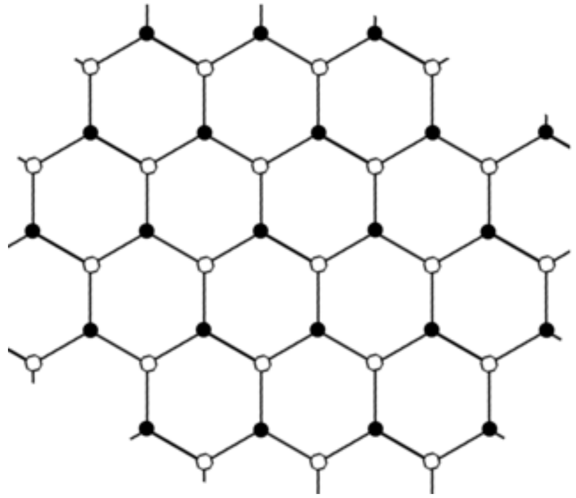


Under Time Reversal

$$|\uparrow\rangle_i \Rightarrow |\downarrow\rangle_i$$
$$|\downarrow\rangle_i \Rightarrow -|\uparrow\rangle_i$$

Electrons in a crystal : Hamiltonian

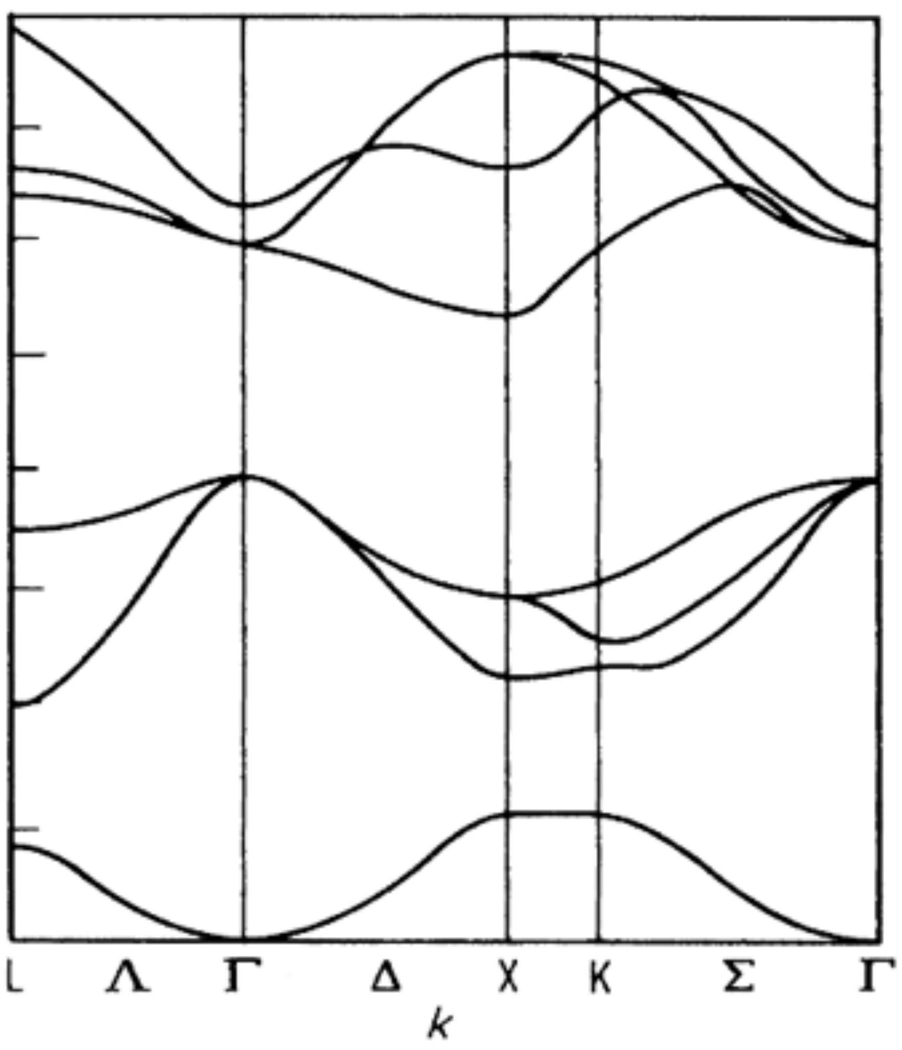
$$\mathcal{H} = \sum_{ij} t_{i-j}^{\sigma\sigma'} |\sigma\rangle_i \langle\sigma'|_j + \mathrm{h.c.}$$



- Due to Translation symmetry the Hamiltonian is diagonalised by Fourier transform

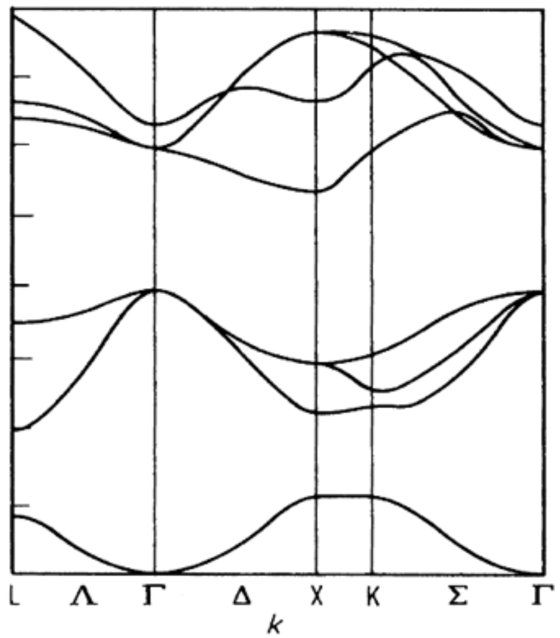
$$\mathcal{H} = \sum_{\mathbf{k}} \varepsilon_n(\mathbf{k}) |u_n(\mathbf{k})\rangle \langle u_n(\mathbf{k})|$$

Band insulators



- Energy bands and energy gaps
- When all states below a gap is filled in the ground state (lowest energy state): Band insulator
- Require finite energy to excite an electron
- Notice: The ground state does not break any of the symmetries (lattice, charge conservation, time reversal)

Are all such Band Insulators identical ?



$$\mathcal{H}_1 \qquad\qquad \mathcal{H}_2$$

$$\mathcal{H}_\lambda = (1-\lambda)\mathcal{H}_1 + \lambda\mathcal{H}_2$$

$\mathcal{H}_1$ $\mathcal{H}_2$, $\mathcal{H}_\lambda$ have lattice, charge conservation and time reversal symmetries.

Is it possible to tune λ from 0 to 1 without closing the energy gap

for arbitrary $\mathcal{H}_1$ and $\mathcal{H}_2$?

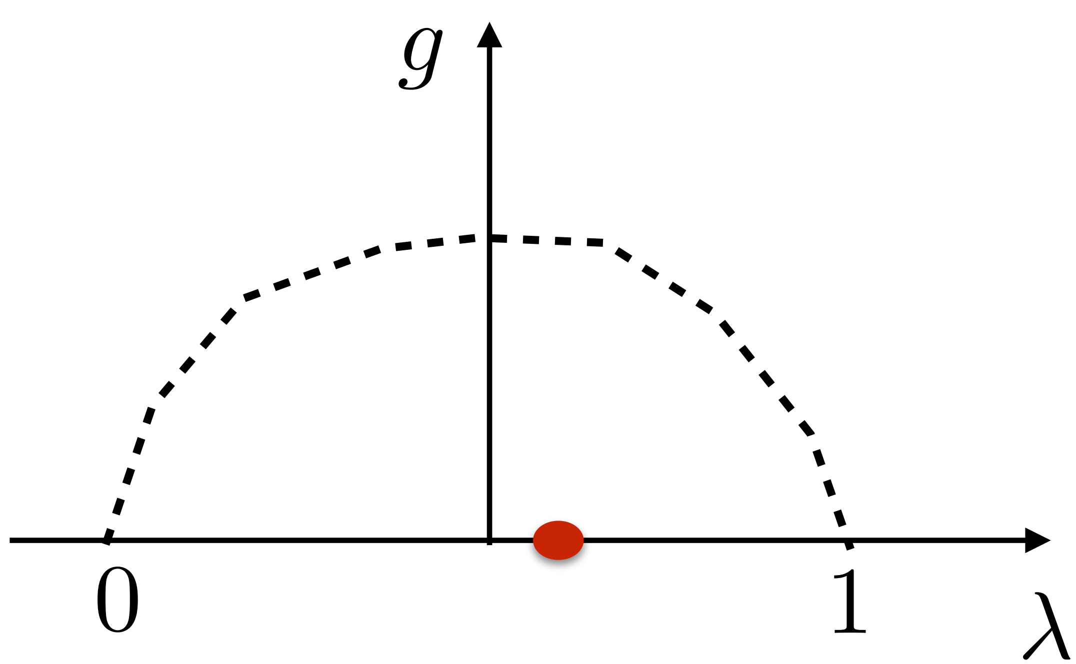
$|\psi_{GS}(\lambda)\rangle$ evolves continuously without without gap closing

Answer : No ! Because of the symmetries !

Symmetry based characterisation beyond spontaneous symmetry breaking

Liang Fu, C. Kane, E. Mele, R. Roy, L. Balents, J. Moore (2006)

- Consider a case where the two ground states cannot be connected by tuning λ



$$\mathcal{H}_{g,\lambda}$$

$$\mathcal{H}_{g=0,\lambda=0} = \mathcal{H}_1$$

$$\mathcal{H}_{g=0,\lambda=1} = \mathcal{H}_2$$

However we now allow the Hamiltonian to break time reversal at intermediate stage

Now I can connect the two ground states without closing the gap

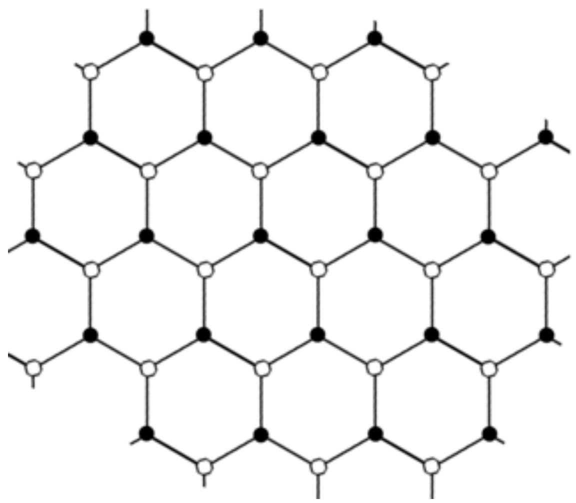
Symmetry Protected Topological Phases

X. G. Wen et. al. (2011)

- Symmetry protection is apparent
- **Topological ?**
    - Robustness to perturbations as long as symmetries are preserved
    - Related to the topological structure of the electronic wave-function
    - Invariants
- The Invariants have different values in the ground states cannot be deformed continuously without closing the energy gap

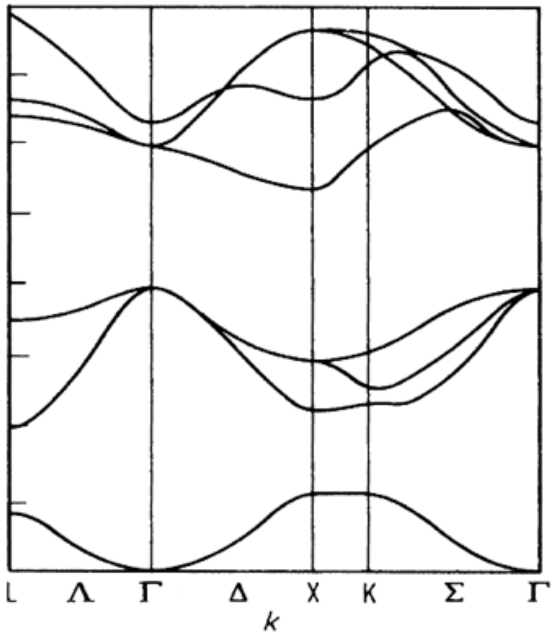
Topological Band Insulator

$$\mathcal{H} = \sum_{ij} t_{i-j}^{\sigma\sigma'} |\sigma\rangle_i \langle\sigma'|_j + \text{h.c.}$$



- Due to Translation symmetry the Hamiltonian is diagonalised by Fourier transform

$$\mathcal{H} = \sum_{\mathbf{k}} \varepsilon_n(\mathbf{k}) |u_n(\mathbf{k})\rangle\langle u_n(\mathbf{k})|$$



Topological (Z_2) invariant for Topological Band Insulator

$$\mathcal{H} = \sum_{\vec{k}} |\phi_{k,\sigma}\rangle \, \mathcal{H}^{\sigma\sigma'}(\vec{k}) \, \langle \phi_{k,\sigma'}|$$

$\mathcal{H}(\vec{k}) \rightarrow$ matrix (2x2, for us)

under Time reversal

$$\mathcal{H}(\vec{k}) \rightarrow \sigma^2 \mathcal{H}^*(-\vec{k}) \sigma^2$$

$$\mathcal{H}(-\vec{k}) \rightarrow \sigma^2 \mathcal{H}^*(\vec{k}) \sigma^2$$

$\mathbb{Z}_2$ Invariant

$$\mathcal{H} = \sum_{\mathbf{k}} \varepsilon_n(\mathbf{k}) |u_n(\mathbf{k})\rangle\langle u_n(\mathbf{k})|$$

$$T\,|u_n(\vec{k})\rangle = i\sigma^2\,|u_n(\vec{k})\rangle^*$$

Is an eigenstate of $\mathcal{H}(-k)$

with eigenvalue $\epsilon_n(\vec{k})$

Construct the Matrix (overlap of T.R states)

$$G_{nm}(\vec{k}) = \langle u_n(\vec{k})|\,T\,|u_m(\vec{k})\rangle$$

$\mathbb{Z}_2$ Invariant

Focus on the zeros of the eigenvalues of $G_{nm}(\vec{k})$

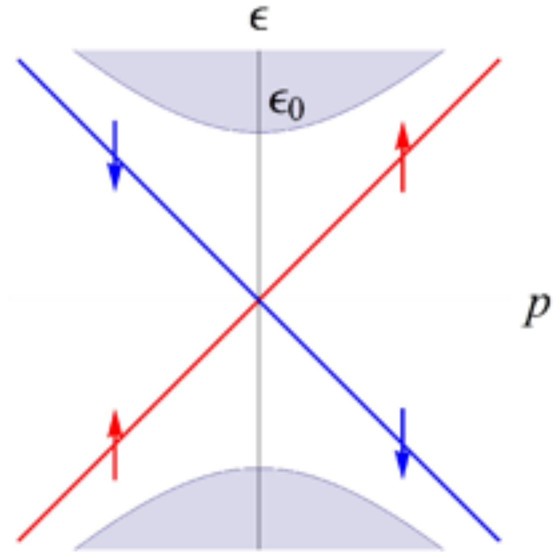
$$\nu = \left(\frac{\#\text{ of Zeros}}{2} \right) \pmod 2$$

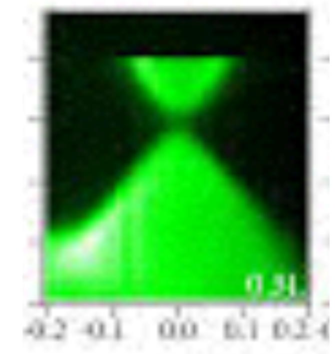
→ Even pairs of Zeros $\Rightarrow \nu = 0$
→ odd pairs of Zeros $\Rightarrow \nu = 1$

} $\mathbb{Z}_2$ Inv

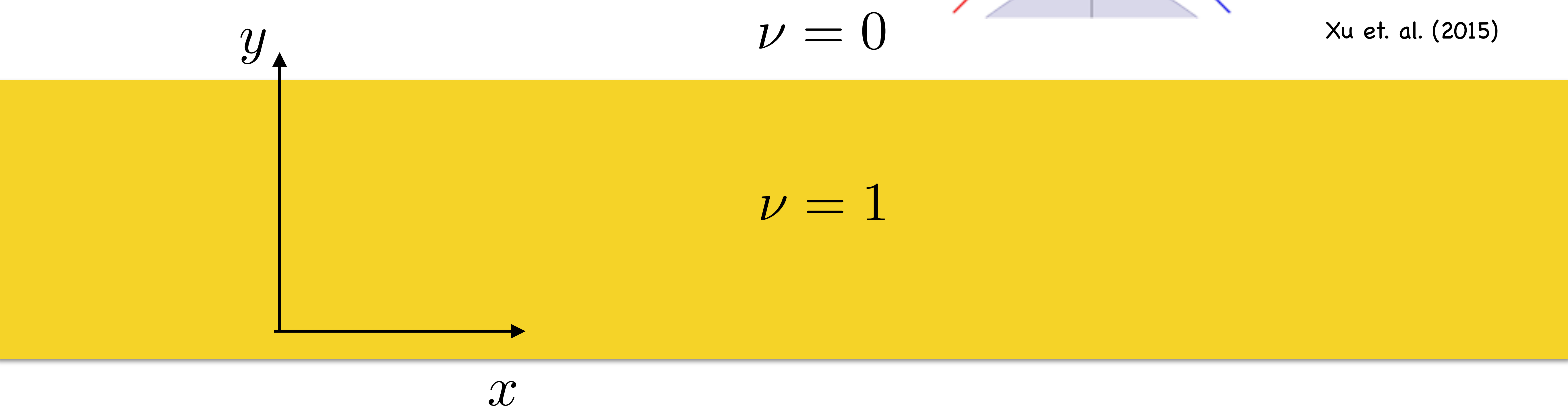
THE TWO CASES CANNOT BE CONTINUOUSLY CONNECTED

Edge states





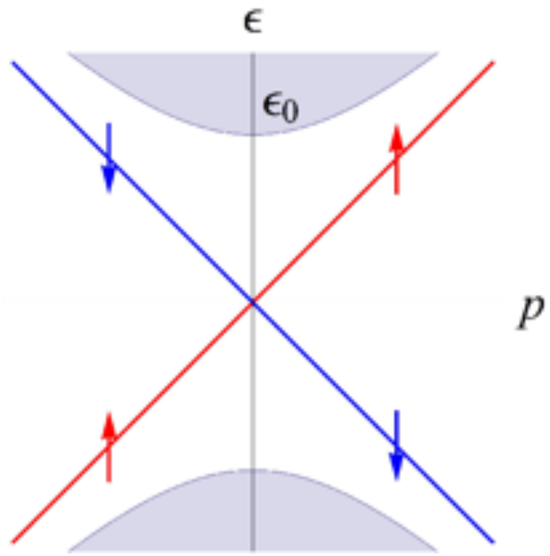
Xu et. al. (2015)

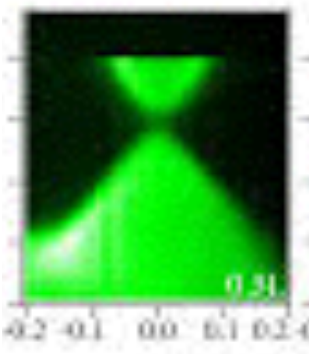


- The Integer can change only by closing the gap at the edge

- Robust Gapless edge states

Edge states





- Robust Gapless edge states : Odd number of Dirac Fermions
- Anomalous theory
- Require the second edge to cancel the anomaly. Thus can only be realised as an edge of a 2 dimensional system

SPT phases and interactions

- Topological Band insulator is an example where we have free electrons.
- By now a systematic way to classify such topological phases are known
- What about Interactions ? seems very rich, partially known

Interacting SPT : Integer quantum Hall effect of bosons

- Integer quantum Hall effect of fermions do not require interactions

- Bosons do

- Such Integer quantum Hall effect of Bosons is a SPT protected by total charge conservation of bosons, i.e. a U(1) symmetry

$$\sigma_{xy}^{\text{charge}} = \pm 2n, \quad \sigma_{xy}^{\text{Thermal}} = 0$$

No symmetries

- What if the system has no symmetries ?
- Can their be still different phases in a quantum many-body system ? yes
- How will we characterise them ?
  Pattern of quantum entanglement

Summary

- Symmetries play a central role in classifying condensed matter phases
- Spontaneous symmetry breaking
- Symmetry protected topological phases : symmetry protects topological invariants
- Phases beyond symmetries