Origin and Development of Valuation Theory

Sudesh K. Khanduja

Indian Institute of Science Education and Research Mohali

skhanduja@iisermohali.ac.in

Valuations have been around in Mathematics since ancient times. When Euclid had proved the uniqueness of prime decomposition of a natural number, then this result permitted to code the natural numbers by the exponents with which the various primes p occur in these numbers. These exponents in fact represent p-adic valuations used in number theory. However, Valuation Theory as a separate and systematic mathematical research, based on a set of axioms started in the 20^{th} century only, in the year 1912 when the Hungarian mathematician JOSEF KÜRSCHÁK (1864-1933) announced at the Cambridge International Congress of Mathematicians the first abstract structure theorem on valued fields.

A mapping $\phi : K \longrightarrow \mathbb{R}$ is called a valuation of a field K if it satisfies the following four axioms:

$$\phi(a) > 0 \text{ if } a \neq 0, \phi(0) = 0,$$

$$\phi(a+b) \leq \phi(a) + \phi(b),$$

$$\phi(ab) = \phi(a)\phi(b),$$

$$\exists\ a \text{ such that } \phi(a) \neq 0,\ 1.$$

Kürschák used the name "Bewertung" which is translated in English as "valuation". These valuations are also called absolute values. The pair (K, ϕ) is called a valued field.

Example: (p-adic absolute value)

Let p be a prime number and $0 < c < 1$ be a real number. For any non-zero integer a, write a as $p^r a', r \geq 0, p \nmid a'$. Define

$$\phi_p(a) = c^r.$$

It can be easily checked that $\phi_p(a+b) \leq \max\{\phi_p(a), \phi_p(b)\}$ for all $a, b \in \mathbb{Z}$. The extension of ϕ_p to $\mathbb{Q}$ is called the p-adic absolute value of $\mathbb{Q}$.

As to the choice of his axioms (1)-(4), Kürschák refers to Hensel's article of 1907, where HENSEL in the case of p-adic numbers, had already defined some similar valuation function. The properties of that function were used by Kürschák as his axioms. Although the formal definition of a valuation has been given by Kürschák, the ideas which governed valuation theory in its first phase all came from Hensel. Hence Hensel may be called the Father of Valuation Theory.

Before Kürschák, Hensel had defined p-adic numbers through their power series expansions with respect to a prime element. This procedure was quite unusual since Hensel's power series don't converge in the usual sense and they don't represent "numbers" in the sense understood at that time, i.e., they are not complex numbers. Kürschák's paper was written to give a solid foundation for p-adic algebraic numbers, in a similar way as Cantor had given for real numbers. Thus we see that the main motivation to introduce Valuation Theory came from algebraic number theory, while the model for axioms and the method of reasoning was taken from analysis. **Valuation theory forms a solid link between algebra, number theory and analysis.**

Kürschák's main theorem:

Every valued field (K, ϕ) admits a valued field extension $(\mathbb{C}_K, \psi)$ which is algebraically closed and complete with respect to the metric given by $d(x, y) = \psi(x - y)$.

When Kürschák published his valuation theory paper in 1913, he was 48. This is the only paper of Kürschák on Valuation theory. He held a position at the Technical University in Budapest and his list of publications comprises of about 80 papers between the years 1887 and 1932 on a wide variety of subjects, including analysis, calculus of variations and elementary geometry.

After Kürschák had started the theory of valued fields, it was ALEXANDER OSTROWSKI (1893-1986) who took over and developed it to a considerable degree. Ostrowski was born in Kiev and had come to Marburg in 1911 at the age of 18 in order to study with Hensel. A. Fraenkel, who had been in Marburg at that time recalls in his memoirs that Ostrowski showed unusual talent and originality (eine ungewöhnliche Begabung und Originalität). In Marburg, the home of p-adics, Kürschák's paper was thoroughly studied and discussed.

In 1918, Ostrowski proved two fundamental theorems. In the first theorem all possible absolute values of the rational number field $\mathbb{Q}$ are determined. Ostrowski proved that upto equivalence, these are precisely the usual absolute value and the p-adic absolute values ϕ_p for various primes p. Two absolute valuess ϕ and ϕ_1 of a field K are said to be equivalent if one is a power of the other, i.e., $\phi_1(a) = \phi(a)^r \ \forall \ a \in K$ for some real $r > 0$. An equivalence class of absolute values is called a **prime** of the field usually denoted by a symbol like $\wp$ and the corresponding completion is denoted by $K_\wp$.
Ostrowski's result can be expressed by saying that every prime $\wp$ of the rational number field $\mathbb{Q}$ either corresponds to a prime number p or $\wp = \wp_\infty$ is the prime containing the ordinary absolute value.

The classification of absolute values into archimedean and non-archimedean was given by Ostrowski in 1917. An absolute value ϕ is said to be non-archimedean if it satisfies the ultrametric inequality: $\phi(a+b) \leq max\{\phi(a), \phi(b)\}$. If K, L are fields with absolute values ϕ and ψ, then (K, ϕ) is said to be isomorphic to (L, ψ) if there exists a field isomorphism f from K onto L preserving absolute values i.e. $\psi(f(a)) = \phi(a)$ for all $a \in K$.

Ostrowski, 1918

Let K be a complete field with respect to an achimedean absolute value ϕ. Then (K, ϕ) is isomorphic to $(\mathbb{R}, |\cdot|^\lambda)$ or to $(\mathbb{C}, |\cdot|^\lambda)$ for some $\lambda > 0$.

Emmy Noether's interest in Ostrowski's work

One of the first readers of Ostrowski's paper was EMMY NOETHER. In a postcard written to Ostrowski in 1916, she writes

"Ihre Funktionalgleichungen habe ich angefangen zu lesen, und sie interessieren mich sehr. Kann man wohl den al lgemeinsten Körper charakterisieren, der einem Teiler des Körpers al ler reel len Zahlen isomorph ist ? $\cdots$ "

"I have started to read your functional equation[1] *and I am very interested in it. Is it perhaps possible to characterize the most general field which is isomorphic to a part of the field of real numbers?"*

[1] The title of Ostrowski's paper was "Über einige Lösungen der Fukhonalgeichung $\phi(x)\phi(y) = \phi(xy)$ (on some solutions of the functional equation $\phi(x)\phi(y) = \phi(xy)$)."

Emmy Noether does not only express her interest in Ostrowski's work but immediately poses the correct question: Which fields can be isomorphically embedded into $\mathbb{R}$? Her question was answered later in 1927 by Artin-Schreier's theory of formally real fields.
This postcard has been cited in order to put into evidence that Emmy Noether has shown interest in the development of valuation theory right from the beginning. Later in 1930-31, she actively participated, together with Richard Brauer and Helmut Hasse, in the proof of the Local-Global Principle for algebras.

Emmy Noether
March 23, 1882 - April 14, 1935

In 1934, Ostrowski wrote a paper of 136 pages adding several significant results to valuation theory. In this paper, he introduced Henselian valued fields i.e. a valued field K in which Hensel's lemma holds. He studied the properties of these fields and proved that any algebraic extension L of a henselian valued field K is again henselian.. A simple looking result proved in this paper is the description of all absolute values of algebraic number fields.

Absolute values of an algebraic number field

Theorem (1934). Let K be an algebraic number field and $\mathcal{O}_K$ be the ring of algebraic integers of K.
(i) An archimedean absolute value of K is equivalent to the absolute value given by $\alpha \to |\sigma(\alpha)|$, where σ is an embedding of K into $\mathbb{C}$.
(ii) If ϕ is a non-archimedean absolute value of K, then there exists a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ and a positive real number $c < 1$ such that $\phi(\alpha) = c^{\nu(\alpha)}$, where $\nu(\alpha)$ denotes the power of $\mathfrak{p}$ occuring in the prime ideal factorization of $\alpha\mathcal{O}_K$.
Absolute values defined by different prime ideals are non-equivalent and two absolute values defined by different embeddings of K into $\mathbb{C}$ are equivalent if and only if these embeddings are complex conjugates.

Rychlík and Hensel's Lemma

Between 1918-1924, the Czech mathematician Karel Rychlík (1885-1968) wrote several papers presenting Hensel's ideas of algebraic number theory. For a complete field he gave a simple proof for the prolongation of any non-archimedean absolute value to an algebraic extension. Of particular interest to us is his 1923 paper which appeared in Crelle's Journal and which contains a proof of what is called the Hensel-Rychlík Lemma.

Definition.

Let ϕ be a non-archimedean absolute value of a field K. The set $R = \{x \in K | \phi(x) \leq 1\}$ is a subring of K, called the valuation ring of ϕ. The set $M = \{x \in R | \phi(x) < 1\}$ is an ideal of R which consists of all non-units of R. M is a maximal ideal of R. The field R/M is called the residue field of ϕ.

Example. The valuation ring of a p-adic absolute value ϕ_p is the subring $\{\frac{a}{b} | a, b \in \mathbb{Z}, p \nmid b\}$ and the residue field of ϕ_p is $\mathbb{Z}/p\mathbb{Z}$.

Hensel-Rychlik Lemma.

Let $F(x)$ be a polynomial with coefficients in the valuation ring R of a complete non-Archimedean valued field (K, ϕ). If $\exists$ $a_0 \in R$ such that $\phi(F(a_0)) \leq \epsilon < \phi(F'(a_0))^2$, then there exists a root $a \in R$ of $F(x)$ with $\phi(a - a_0) \leq \epsilon\phi(F'(a_0))^{-1} < \phi(F'(a_0))$.

In the special case when $K = \mathbb{Q}_p$ with the p-adic absolute value and valuation ring denoted by $\mathbb{Z}_p$, we obtain:

Classical Hensel's Lemma.

Let $F(x) \in \mathbb{Z}_p[x]$ be a polynomial. If $\exists$ $a_0 \in \mathbb{Z}$ such that $F(a_0) \equiv 0 (mod\ p), F'(a_0)$ is not congrunt $0\ mod\ p$, then there exists a root $a \in \mathbb{Z}_p$ of $F(x)$ with $a_0 \equiv a(mod\ p)$.

In 1999, Jayanti Saha also formulated a version of Hensel's Lemma. In 2013/16, Sanjeev Kumar/Bablesh Jhorar extended Hensel's Lemma and applied it to obtain certain irreducibility criteria for polynomials.

Hasse: The Local-Global Principle

Consider the year 1920. In that year Helmut Hasse (1898-1971), a young student of 22, decided to leave his home university of Göttingen in order to go to Marburg and continue his studies with Hensel. As pointed out by Hasse in the forward to his collected papers, the motivation for going to Marburg was Hensel's book "Zahlentheorie."

Hasse registered at Marburg University in May 1920. Already at the end of this month, Hensel suggested to him a subject for his doctoral thesis: *Quadratic Forms over* $\mathbb{Q}$ *and* $\mathbb{Q}_p$. In May 1921, one year after his moving to Marburg, Hasse had completed his thesis and proved the famous Local-Global Principle. The principle states that a rational number a is representable by a quadratic form $f(x_1, \ldots, x_n)$ with coefficients in $\mathbb{Q}$ over the fields of rationals if and only if it is representable by f over the p-adic completions $\mathbb{Q}_p$ for all primes p and over $\mathbb{R}$.

It may be pointed out that Local-Global Principle does not hold for higher degree forms. In 1951 Selmer showed that $3x^3 + 4y^3 + 5z^3$ has non-trivial zeros in every completion of $\mathbb{Q}$ but not in $\mathbb{Q}$.

After his thesis, Hasse published in quick succession six other papers elaborating on the subject. In the first of these, he developed a Local-Global Principle for the equivalence of quadratic forms with rational coefficients. Given two quadratic forms with coefficients in $\mathbb{Q}$, Hasse proved that they are equivalent over $\mathbb{Q}$ if and only if they are equivalent over $\mathbb{Q}_p$ for all primes p (including $p = \infty$).

Let us briefly mention that Hasse's Local-Global Principle has turned out to be of importance far beyond the application to quadratic forms over algebraic number fields. Given a field K equipped with a set V of valuations and given any field theoretic statement A over K, one can ask whether the following is true: A holds over K if and only if A holds over the completion $K_\wp$ for all primes $\wp \in V$.

An important example of the validity of this principle is the Hasse Norm Theorem proved by Hasse in 1931. Its earlier special cases were proved by Hilbert and Furtwängler.

Hilbert-Furtwängler-Hasse Norm Theorem.

Let L/K be a cyclic extension of number fields. For a prime $\mathfrak{p}$ of K, let $K_\mathfrak{p}$ denote completion of K at $\mathfrak{p}$ and $L_\wp$ denote the completion of L w.r.t. a prime $\wp$ of L lying over $\mathfrak{p}$. If a non-zero element a of K is a norm in $L_\wp/K_\mathfrak{p}$ for all $\mathfrak{p}$, then a is a norm in L/K.

Emmy Noether's creative power towards abstraction

During the late 1920's and early 1930's there was growing awareness that the theory of non-commutative algebras could be used to obtain essential information about the arithmetic structure of algebraic number fields. This view was forcibly and repeatedly brought forward by Emmy Noether. As written by Van der Waerden " The influence of Emmy Noether, on Hasse as well as on others, was based on her ability to formulate their problems in abstract form which, in her opinions, clarified the situation. She did not solve mathematical problems but she led the way to the solution by putting them on the abstract track which, in her opinion, would lead to the solution by simplification."

Just as Emmy Noether showed that a Dedekind ring can be treated by means of its localizations, it was proved in the non-commutative case, the arithmetic of a maximal order can be similarly described by its localizations with respect to the prime ideals of the center. Moreover, by including the infinite primes belonging to the archimedean valuations, it was feasible to proceed much further towards non-commutative foundation of commutative number theory following the desideratum of Emmy Noether. Thus once more it turned out that valuation theory provides for useful and adequate methods to deal with questions of higher algebraic number theory. As an example let us cite Emmy Noether, in a postcard to Hasse of June 25, 1930: *"Ihre hyperkomplexe p-adik hat mir sehr viel Freude gemacht $\cdots$"* *Your hypercomplex p-adics has given me much pleasure $\cdots$.*

On December 29, 1931 Kurt Hensel, the mathematician who had discoverd p-adic numbers, celebrated his seventieth birthday. On this occasion a special volume of Crelle's Journal was dedicated to him. The dedication volume contains the paper, authored jointly by Richard Brauer, Helmut Hasse and Emmy Noether, with the title "Beweis eines Hauptsatzes in der Theorie der Algebren (Proof of a Main Theorem in the theory of algebras)".

The Main Theorem allows a complete classification of division algebras over a number field leading to the structure of the Brauer group of an algebraic number field.

Brauer-Hasse-Noether Theorem

Every central division algebra over a number field is cyclic.

Definition.

Let L/K be a cyclic field extension of degree n and let σ denote a generator its Galois group G. Given any a in the multiplicative group $K^{\times}$, consider the K-algebra generated by L and some element u with the defining relations:

$$u^n = a, zu = uz^{\sigma} \ (z \in L).$$

This is a central simple algebra of dimension n^2 over K and and is called a cyclic algebra denoted by $(L/K, \sigma, a)$. The field L is a maximal commutative subalgebra of $(L/K, \sigma, a)$.

When Artin heard of the proof of the Main Theorem he wrote:

> *. . . Sie K onnen sich gar nicht vorstellen, wie ich mich über den endlich geglückten Beweis für die cyklischen Systeme gefreut habe. Das ist der grösste Fortscritt in der Zahlentheorie der letzten Jahre. Meinen herzlichen Glückwunsch Beweis . . .*
>
> *. . . You cannot imagine however so pleased I was about the proof, finally successful, for the cyclic systems. This is the greatest advancement in number theory of the last years. My heartfelt congratulations for your proof . . .*

In November 1932, Noether delivered a plenary address on "Hyper-complex systems in their relations to commutative algebra and to number theory" at the International Congress of Mathematicians in Zürich. The congress was attended by 800 people, including Noether's colleagues Hermann Weyl, Edmund Landau, and Wolfgang Krull.

A Glimpse of Krull Valuations

For non-archimedean absolute values, Ostrowski introduced the additive notation of valuation by defining $v(a) = -log\phi(a)$. Then the mapping

$$v : K \to \mathbb{R} \cup \{\infty\}$$

satisfies
(v1) $v(a) = \infty \Leftrightarrow x = 0$
(v2) $v(ab) = v(a) + v(b)$
(v3) $v(a + b) \geq \min\{v(a), v(b)\}$.
Such a valuation is called classical or real valuation.

Remark.

Two real valuations v, v' are said to be equivalent if there exists a real number ρ such that $v'(a) = \rho v(a)$ for each $a \in K$.

Remark.

Let K be a field. There is a natural one-to-one correspondence between the set of equivalence classes of real valuations of K and the set of equivalence classes of non-archimedean absolute values of K given by $v \to \phi = e^{-v}$; $\phi \to v = -\log_e \phi$.

In 1932, Wolfgang Krull gave a more general universal definition of valuation as follows:

Definition.

A mapping $v : K \to G \cup \{\infty\}$, where G is a totally ordered additively written abelian group, is called a valuation of K if for all $a,\ b \in K$, $(v1)$, $(v2)$, $(v3)$ are satisfied. The set $\{a \in K : v(a) \geq 0\}$ is called the valuation ring of K.

Example.

Let R be U.F.D. with quotient field K and π be a prime element of R. We denote the π-adic valuation of K defined for any non-zero $\alpha \in R$ by $v_\pi(\alpha) = m$, where $\alpha = \pi^m\beta,\ \beta \in R$, π does not divide β. It can be extended to K in a canonical manner.

Example.

The polynomial ring $\mathbb{Q}[x]$ is a U.F.D. Let v_x denote the x-adic valuation of the field $\mathbb{Q}(x)$ of rational functions in x corresponding to the irreducible element x of $\mathbb{Q}[x]$. For any non-zero polynomial $g(x)$ belonging to $\mathbb{Q}(x)$ we shall denote g^* the constant term of the polynomial $g(x)/x^{v_x(g(x))}$. Let p be any rational prime. Let v be the mapping from non-zero elements of $\mathbb{Q}(x)$ to $\mathbb{Z} \times \mathbb{Z}$ (lexicographically ordered) defined on $\mathbb{Q}[x]$ by

$$v(g(x)) = (v_x(g(x)), v_p(g^*)).$$

Then v gives a valuation on $\mathbb{Q}(x)$.

Emmy Noether and Krull valuations

Several ideas used in the results regarding Krull valuations are inspired by ring-theortic viewpoint given by Emmy Noether. Deuring and Krull were stimulated by her and gave an elegant proof for the existence of arbitrary Krull valuations which are centred at a given prime ideal of a given domain. She worked closely with Wolfgang Krull, who greatly advanced commutative algebra with his Hauptidealsatz[a] and his dimension theory for commutative rings.

[a] Principal Ideal Theorem : If R is a Noetherian ring and I is a principal, proper ideal of R, then I has height at most one.

Tributes to Emmy Noether

Herman Weyl while delivering a speech at the funeral ceremony of Emmy Noether in the house of President Park in Bryn Mawr on April 17, 1935 said
"...Justifiably proud, for you were a great woman mathematician - I have no reservations in calling you the greatest that history has known. Your work has changed the way we look at algebra, and with your many gothic letters you have left your name written indelibly across its pages. No-one, perhaps, contributed as much as you towards remoulding the axiomatic approach into a powerful research instrument, instead of a mere aid in the logical elucidation of the foundations of mathematics, as it had previously been. Amongst your predecessors in algebra and number theory it was probably Dedekind who came closest... "

In his memorial address, Alexandrov named Emmy Noether "the greatest woman mathematician of all time"

On her death, Albert Einstein wrote in a letter to the New York Times which was published on May 5, 1935 with the heading "*Professor Einstein Writes in Appreciation of a Fellow-Mathematician*", "...In the judgement of the most competent living mathematicians, Fraulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began. In the realm of algebra, in which the most gifted mathematicians have been busy for centuries, she discovered methods which have proved of enormous importance inthe development of the present-day younger generation of mathematicians..."

E. Noether and H. Weyl (with hat) among a group of mathematicians

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Thank You