Noether's theorem and Particle Physics.

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Number fields

Calculus of variations

Noncommutative algebras

University of Erlangen 1907, mathematics

1915-1933 University of Göttingen (David Hilbert, Felix Klein)

Plenary address, International Congress of Mathematicians, 1932

Bryn Mawr College, Pennsylvania, 1933.

INVARIANT VARIATIONAL PROBLEMS

(For F. Klein, on the occasion of the fiftieth anniversary of his doctorate)

by **Emmy Noether** in Göttingen

Presented by F. Klein at the session of 26 July 1918*

We consider variational problems which are invariant[A] under a continuous group (in the sense of Lie); the consequences that are implied for the associated differential equations find their most general expression in the theorems formulated in §1, which are proven in the subsequent sections. For those differential equations that arise from variational problems, the statements that can be formulated are much more precise than for the arbitrary differential equations that are invariant under a group, which are the subject of Lie's researches. What follows thus depends upon a combination of the methods of the formal calculus of variations and of Lie's theory of groups. For certain groups and variational problems this combination is not new; I shall mention Hamel and Herglotz for certain finite groups, Lorentz and his students (for example, Fokker), Weyl and Klein for certain infinite groups.[1] In particular, Klein's second note and the following developments were mutually influential, and for this reason I take the liberty of referring to the final remarks in Klein's note.

- Role the theorem played in development of theoretical description of particles and interactions among them!

- Deep insights offered by the theorem and related ideas in 'understanding' the observed structure of the theories, in this case the Standard Model of particle physics!

- Legacy of Emmy Noether: as a woman theoretical physicist and mathematician!

- Noether's theorem.

- Conservation laws and particle physics!

- Gauge invariance and Noether's theorem

- Legacy of Emmy Noether for women mathematicians and physicists

Symmetries of laws of nature have been the underpinning of the development of particle physics.

Noether's theorem (NT) revealed the great connection between continuous symmetries and conservation laws.

These symmetries could be space time symmetries or internal symmetries.

I: Proided an understanding of space time symmetries of laws of nature

⇒ Conservation laws which are sacrosanct and which **have** to be satisfied independent of the details of the dynamics *(example: energy conservation)*.

Removed the mystery in which energy conservation was shrouded, so much so ⇒ postulation of new particles and interactions *(example: neutrinos)*;

II:

The theorem provided deep understanding of observed facts about the dynamics.

Example: Hilbert's 'improper local energy conservation in general theory of relativity' was understood in terms of this relation between invariances and conservation!

(Application to General Relativity: will not be discussed in my talk)

Her result was completely general and we particle physicists have kept on finding special cases!

Einstein:

"Yesterday I received from Miss Noether a very interesting paper on invariant forms. I am impressed that one can comprehend these matters from so general a viewpoint. "

General path of last 60-70 years of development in Particle physics:

Use the theorem and observed conservation laws $\Rightarrow$ Symmetries of equations of motion $\Rightarrow$ symmetries of the dynamics. *Ex: $U(1)$ invariance of electromagnetic interactions, Ward identity, conserved vector charge in weak interactions.*

$\Rightarrow$ Invariances and 'well being' of theories *Ex: renormalisation*

$\Rightarrow$ Generalisation of the symmetries to new theories *Ex: electro weak unification*

$\Rightarrow$ new theoretical ideas. *Ex.: spontaneous symmetry breaking and the Higgs mechanism*

Theoretical short comings of current theory $\Rightarrow$ New symmetries as yet not discovered *Ex.: Supersymmetry*

Lagrangian Field Theory:

Classical Mechanics: Action, time integral of Lagrangian L, gives equations of motion by least action principle.

In a local field theory L can be written as spatial integral of the Lagrangian density $\mathcal{L}$ called Lagrangian in the rest of the talk

$$S = \int L dt = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x$$

The equations of motion for fields are obtained by principle of least action demanding $\delta S = 0$.

This leads to Euler-Lagrange equations of motion (EOM):

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

Noether's theorem in this case speaks about conserved current associated with a continuous transformation of the fields ϕ.

In infinitesimal form the transformation is:

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$$

α is an infinitesimal parameter and $\Delta\phi$ is the deformation in configuration space.

This is a symmetry transformation if the EOM's are unchanged $\Rightarrow$ Lagrangian $\mathcal{L}$ is invariant upto a 4-divergence.

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\ \partial_\mu \mathcal{J}^\mu(x),$$

for some $\mathcal{J}_\mu$.

How to get $\mathcal{J}_\mu$?

$\alpha\Delta\mathcal{L}$ obtained by varying fields is

$$\begin{aligned}\alpha\Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\alpha\Delta\phi + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\alpha\Delta\phi) \\ &= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) - \alpha\Delta\phi\left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi}\right] \\ &= \alpha\partial_\mu\mathcal{J}^\mu\end{aligned}$$

This then means a conserved Noether current

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu$$

and a Noether charge

$$Q \equiv \int j^0(x)d^3x$$

constant in time!, generator of the symmetry transformation. Existence of this charge depends on asymptotic behavior of fields!

We can find various conserved Noether charges for different transformations as in classical dynamics.

Scalar fields and space-time translation:

Symmetry transformation:

$$x_\mu \to x_\mu - a_\mu.$$

For a scalar field ϕ

$$\phi(x) \to \phi(x + a_\mu) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

Conserved Noether current

$$T^\mu_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \mathcal{L}\delta^\mu_\nu$$

$\mu = \nu = 0 \Rightarrow$ Conserved charge is Hamiltonian; for $\mu = 0, \nu = i$ it is the linear momentum P^i!

Complex scalar fields: ϕ, ϕ^*

Consider phase transformation : $\phi(x) \rightarrow e^{i\alpha}\phi(x)$ and

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2|\phi|^2,$$

Conserved Noether current (upto a constant factor) is

$$j^\mu = i\left[\partial^\mu \phi^* \phi - \phi^* \partial^\mu \phi\right]$$

In a gauge theory this field is coupled to an electromagnetic field.

If $\alpha = \alpha(x)$, then the Lagrangian is invariant under $\phi(x) \to e^{i\alpha}\phi(x)$ only if ϕ couples to a spin 1 vector field A_μ called a gauge field given by

$$\mathcal{L} = |D_\mu \phi|^2 - m^2 |\phi|^2,$$

where $D_\mu = \partial_\mu + ieA_\mu$ and $A_\mu \to A_\mu - \frac{1}{e}\alpha$

Now the Conserved Noether current is identified with the electromagnetic current density and the corresponding charge as the electromagnetic charge.

For fermions the conserved Noether current is $j^\nu = \bar{\psi}\gamma^\nu\psi$ where ψ is the fermion field and γ^ν are Dirac γ matrices!

Continuous symmetry operation: Gauge transformation.

Conserved Noether current: Electromagnetic current density

Gauge invariance $\sim$ conserved current.

Look at quantum field theory. Current conservation implies interesting properties for amplitudes.

A_μ : creates a photon or annihilates a photon and the interaction Lagrangian is $j^\mu A_\mu = \bar{\psi}\gamma^\mu\psi A_\mu$

In a quantum theory amplitude for (say) photon creation $i \rightarrow f + \gamma(k)$ can be computed

$$\mathcal{M}^\mu(k) = \int d^4x e^{ikx} < f|j^\mu|i >$$

If Noether current is conserved in Quantum theory too, we get

$$k_\mu \mathcal{M}^\mu = 0$$

This is called Ward identity. Consequence of gauge invariance/current conservation.

A physical photon has only two polarisations. A_μ has 4 independent components. So what is happening?

Gauge invariance/Ward identities ensure that only the physical degrees of freedom contribute!

Better still:

Consider electron vertex at tree level from $\mathcal{L}_{int} = ie\bar{\psi}\gamma^\mu\psi$ is

$$\bar{u}(p')\gamma^\mu u(p)$$

p, p' being the momenta of the incoming and outgoing electron. For a real photon $q^2 = 0$ where $q = p - p'$.

Higher order correction can change the vertex to $\bar{\psi}\Gamma^\mu\psi$.

Using Lorentz invariance we can show that

$$\Gamma_\mu(p,p') = \gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m} F_2(q^2)$$

where F_i are arbitrary functions of q^2.

To lowest order $F_1(0) = 1, F_2(0) = 0$

Ward identity guarantees that $F^1(0) = 1$ to all orders in $\alpha_{em} = \frac{e^2}{4\pi}$!

Since the arguments use gauge invariance and current conservation, even for the proton one predicts $F_1(0) = 1$.

Since observed charge of electron and proton are equal, it tells us that structure of proton and its strong interactions do NOT renormalise the charge!

This observation was used later to great effect in development of theory of weak interactions!

Energy conservation was known to hold in all the theories except for the issues in the context of General theory of relativity, mentioned already.

But to quote Feza Gursey (from an article by Nina Byer)

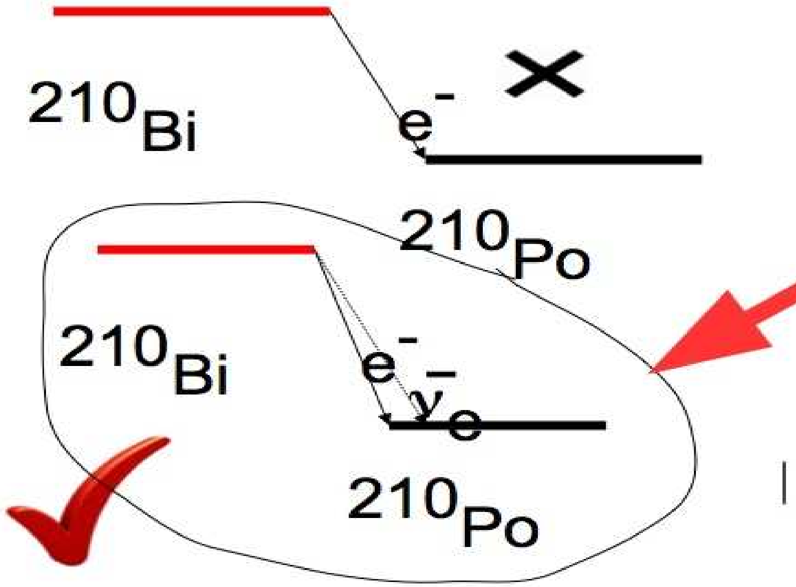
"Before Noether' Theorem the principle of conservation of energy was shrouded in mystery, leading to the obscure physical systems of Mach and Ostwald. Noether's simple and profound mathematical formulation did much to demystify physics."

Her theorem said that for all theories which have finite continuous symmetry group 'energy conservation' was non negotiable!

Did physicists know of any place where energy is not conserved?

- 1918: Chadwick noted an apparent violation of energy conservation in radioactive beta decay

$$^{210}Bi \rightarrow {}^{210}Po + e^-$$



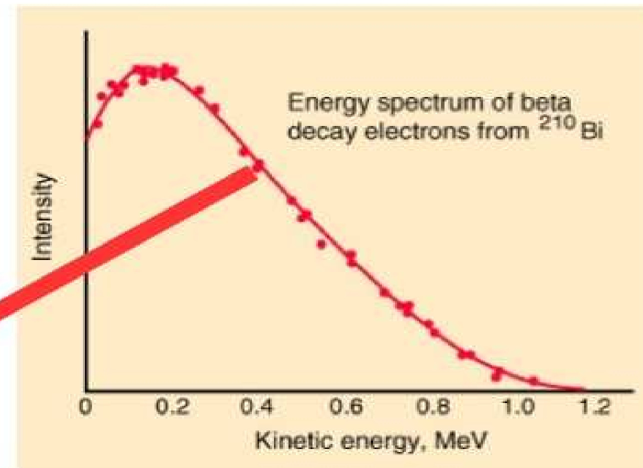


Figure 1.4 The continuous energy spectrum for β decay of ^{210}Bi, from G. J. Neary, Roy. Phys. Soc. (London), A175, 71 (1940).

Apparent non conservation of energy!

Physicists were desperate.

Bohr was even making proposals that energy conservation is violated.

Remember Noether proved both the theorem and its inverse!

So if energy is not conserved then theories are not invariant under time translation!

One should appreciate the desperation!

Pauli postulated neutrino production in the process of radioactive decay to 'preserve'

1)Conservation of energy and angular momentum

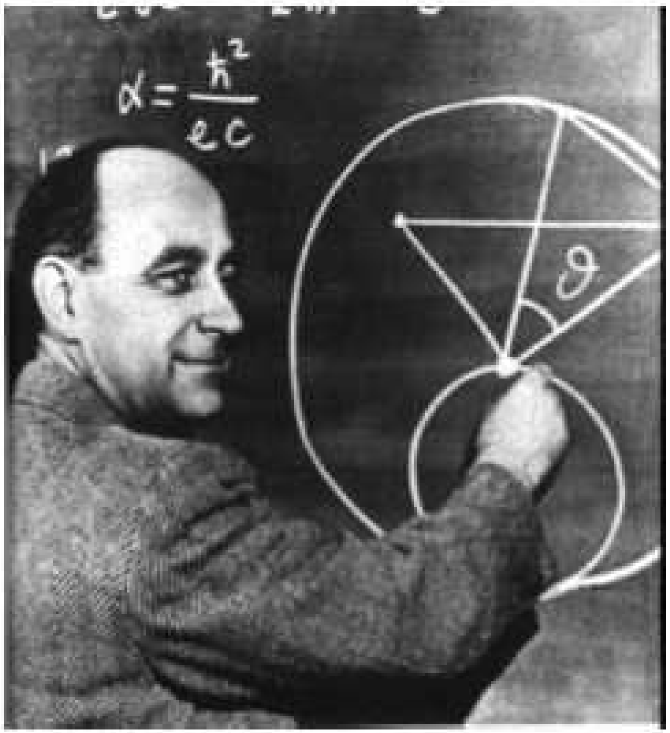
2)'exclusion principle' , explain obesrved spin, parity of even A (odd Z and odd N) nuclei

Pauli had postulated it to be a spin ½, neutral particle.

No statement about its mass

Observed end point of beta spectrum meant that mass if any is small

1934: Enrico Fermi gave a theory of β decay where he used Pauli's idea and christened his particle the 'small neutron ' **(Neutrino)**



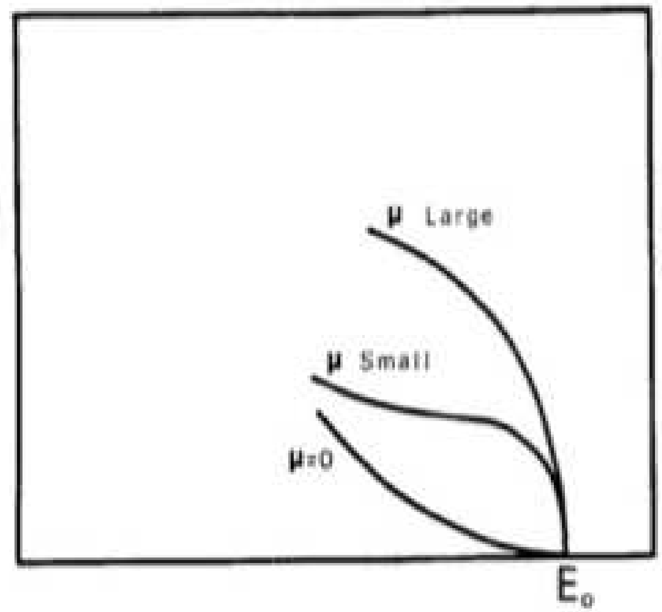


Figure 1.10 Behaviour of the energy spectrum of β decay electron near the end point.

E. Fermi

Knowing structure of Quantum Electordynamics one could derive the Ward identities and also the conserved currents.

Fermi described weak interactions, about which nothing was known, in terms of currents very similar to the Noether current.

Data forced that the current be both vector and axial vector

I.e. $\bar{\psi}\gamma_\mu\psi$ and parity violating $\bar{\psi}\gamma_\mu\gamma_5$

Experiments also showed that the weak charge multiplying the vector current of proton was the same as that for point like leptons

(analogus to $F_1(0)$ being 1 both for electron and proton!)

This was among the first hints that there might be a gauge theory description of weak interactions and that there might be a 'unified' description of both weak and electromagnetic!

In the Standard Model of particle physics quarks and leptons have the electromagnetic charges they have to avoid ill effects of current (non) conservation !

Technical name 'Anamoly Cancellation' !

Essential for the gauge theoretic description of the Standard Model to be renormalisable!

Requirement of anomaly cancellation uniquely fixes charges of u and d quark in terms of the e!

Let us see what it is!

Gauge invariance also requires the gauge bosons and ALL the fermions to have zero mass.

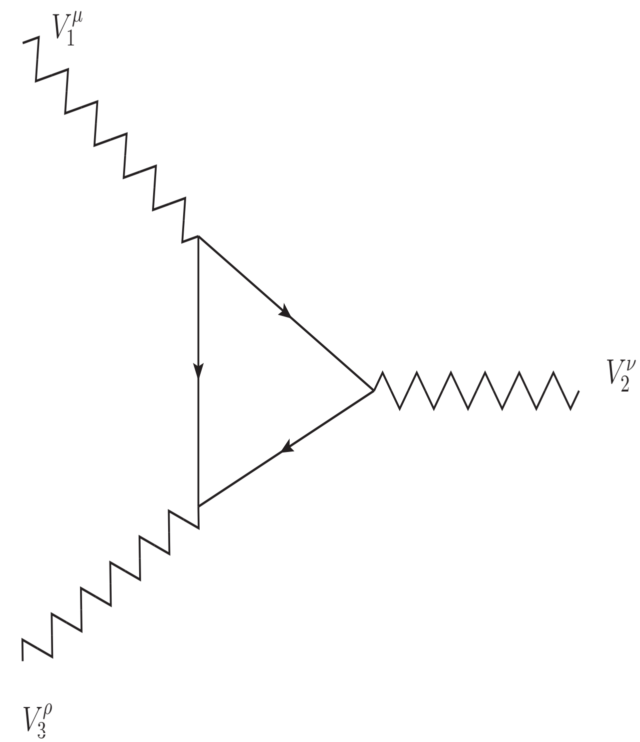
Physics properties of weak interactions required these bosons to have large masses and fermion masses are from from zero!

Explicit masses for fermions and W/Z bosons would spoil the renormalisability of gauge theory of weak interactions!

So Spontaneous Symmetry Breaking (SSB) of gauge invariance was suggested.

In fact in this scheme Noether charge is not well defined. Discussion very subtle. But in the end SSB gives scattering amplitudes which satisfy gauge invariance. Nature seems to have used it.

But the SM requires different treatment of Left handed(L) and Right handed fermions.
This causes problem
The triangle loop diagram has infinities as soon as there is a γ_5 at one of the vertices.
When V_1, V_2, V_3 are weak bosons these triangle diagrams may have at least one vertex with γ_5
Hence is infinite and has a structure which is not present at the tree level. Can not be then 'renormalised'



If we sum over all the quarks and leptons of a given family, i.e. three pairs of u, d quarks and e^-, ν_e as well as the antiparticles, the coefficient of this infinite term is zero.

By demanding that this coefficient be zero, one finds in the framework of $SU(2)_L \times U(1)_Y$ that the quarks should have fractional electric charges that they are seen to have! Unique solution! One caveat: no ν_R

Cancellations require that all the members of a given generation of leptons and quarks should be present in the spectrum if some members are already observed in nature!

So when top and ν_τ were not seen they were predicted from above arguments. The anomalies are independent of the fermion mass. So can not give clue as to what the mass would be!

Cancellation of Anomalies is a deep and fundamental requirement!

These infinities will spoil the renormalisability! They arise because a current which is conserved at the classical level is not conserved at the quantum level!

Consider the phase transformations $\psi \to e^{i\alpha}\psi$ and $\psi \to e^{i\alpha\gamma_5}\psi$

For the SM Lagrangian classically the corresponding conserved currents are

$$j_\mu = \bar{\psi}\gamma_\mu\psi \quad \text{and} \quad j^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$$

Unfortunately in the quantum theory one finds

$$\partial^\mu j^5_\mu \propto \epsilon^{\alpha\nu\beta\rho} F_{\alpha\nu} F_{\beta\rho}$$

I.e it is non zero. The technical reason can be most clearly understood in the language of Functional Integral formulation of theory in terms of non invariance of the measure under the transformation.

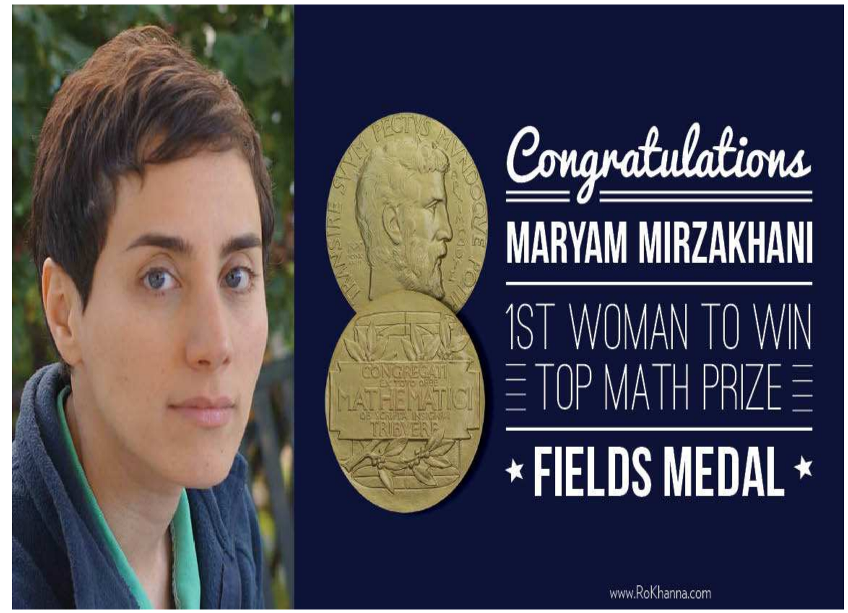
So this underlies the importance of Noether's theorem once again.

So we see that one can not overemphasize Emmy Noether's contributions to particle physics!

They have guided the development of our current understanding of particles and interactions among them at all steps!

There is yet another legacy of Emmy Noether!

$\Rightarrow$



Ackerman-Teubner Memorial Prize (1932) Fields Medal (2014)

She is of course source of inspiration to all the women theoretical physicists and mathematicians where they are still somewhat small in number!

It was very lucky for us that Hilbert, Klein, Einstein recognized her merit and mentored her. Her story underlies the importance of mentoring.

Apart from Hilbert who told the colleagues that gender was not relevant as this was an appointment to a University and not a Public bath house, there was Klein who said 'I am ashamed to occupy such a preferred position beside Emmy who is far superior to me!

Her story also underlies the importance of strong family support! She worked without any payment for a long time!

It is quite interesting that even with support of the doyens like Einstein, Emmy Noether had to be content with a position in a College away from Princeton and travelled every week to give lectures there!

Have things changed much?

Yes to quite some extent in the mindset..but there is still a long way to go!

It is still not uncommon that it is a choice either/or for women scientists unless they have a chunk of luck!

We will all do well to enhance this legacy of Emmy Noether as well!