On Noether's Theorem in Classical Dynamics: Continuous Symmetries and Conservation Laws for Physical Systems

Discussion Meet on the 'Legacy of Emmy Noether',

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The 1918 paper: 'Invariante Variationsprobleme',

Nachr. d. Konig. Gesellsch. d. Wiss. zu Gottingen,

Math.-Phys. Klasse., 235–257 (1918), by Emmy Noether.

Sophus Lie's work – given system of partial differential equations, study continuous group of transformations leaving them invariant.

Noether aim – if the equations come from a variational principle, study the consequences.

Framework classical local field theory.

Independent 'space time variables' x^μ, 'local fields' $u^\alpha(x)$.

Domain $D \to$ action

$$I_D = \int_D \mathrm{d}x \; f\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots\right) \tag{1}$$

Stationarity of $I_D \to$ Euler–Lagrange 'equations of motion' (EOM):

$$\begin{aligned} \psi_\alpha\left(x, u, \frac{\partial u}{\partial x}, \cdots\right) &\equiv \text{Lagrange expressions} \\ &\equiv \frac{\partial f}{\partial u^\alpha} - \frac{\partial}{\partial x^\mu}\frac{\partial f}{\partial \frac{\partial u^\alpha}{\partial x^\mu}} + \cdots = 0. \end{aligned} \tag{2}$$

Two kinds of transformations preserving I_D:

$$x, u(x) \to y = \text{functions of } x, u, \frac{\partial u}{\partial x}, \cdots; v(y) = \text{functions of } x, u, \frac{\partial u}{\partial x}, \cdots; \tag{3}$$

dependent on finite number of independent parameters (i)

or

involving some number of independent arbitrary functions of x (ii) (4)

Particular simple case mentioned:

$$y = \text{functions of } x; \quad v(y) = \text{functions of } x, u(x). \tag{5}$$

Two theorems (and converses) proved:

Type (i): there exist 'divergence theorems', as many as independent parameters in transformation.

Type (ii): linear relationships among Lagrange expressions, so EOM(2) not mutually independent.

Simplify from field theory to mechanics:

$x^\mu \to$ time t; $\quad u^\alpha(x) \to q^j(t), \quad j = 1, 2, \ldots, n;$

Lagrangian dependent only on $q^j(t), \dot{q}^j(t)$: $L(q, \dot{q})$. Connect Noether's analysis to concepts of Hamiltonian dynamics, and to Dirac's Generalized Hamiltonian Dynamics.

No constraints $\leftrightarrow$

$$H_{jk}(q,\dot{q}) = \frac{\partial^2 L(q,\dot{q})}{\partial \dot{q}^j \partial \dot{q}^k}, \quad \det(H_{jk}(q,\dot{q})) \neq 0. \tag{6}$$

Lagrange expressions

$$L_j(q,\dot{q},\ddot{q}) = \frac{\partial L}{\partial q^j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^j}, \quad j = 1, 2, \ldots, n; \tag{7}$$

$$\text{EOM:} \quad L_j(q,\dot{q},\ddot{q}) = 0. \tag{8}$$

Infinitesimal transformations of two types:

$$q^j(t) \to q'^j(t) = q^j(t) + \delta q^j(t),$$

$$\delta q^j(t) = \varepsilon \phi^j(q,t) \quad \text{or} \quad \varepsilon \phi^j(q,\dot{q},t). \tag{9}$$

$$\text{Infinitesimal symmetry } \Leftrightarrow \delta L(q,\dot{q}) = \text{ total time derivative}$$

$$= \varepsilon \frac{\mathrm{d}}{\mathrm{d}t}(F(q,t) \text{ or } F(q,\dot{q},t)), \text{ some } F. \tag{10}$$

$$\text{Lagrangian mechanics rule: } \delta\dot{q}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\delta q(t). \tag{11}$$

Effect on action:

$$I(t_1,t_2) = \int_{t_1}^{t_2} \mathrm{d}t \; L(q(t),\dot{q}(t)),$$

$$\delta I(t_1,t_2) = \varepsilon F(q,\dot{q},t)|_{t_1}^{t_2} \tag{12}$$

List of important consequences:

(i) EOM $\Rightarrow$ existence of a constant of motion (COM) $G(q,p,t)$:

$$\varepsilon G(q,p,t) = p_j \delta q^j - \varepsilon F(q,\dot{q},t). \tag{13}$$

(ii) $$q, p \rightarrow q + \delta q, \quad p + \delta p \text{ is an infinitesimal canonical transformation (CT).} \tag{14}$$

(iii) this is generated by the COM:

$$\delta q^j = \varepsilon\{q^j, G(q,p,t)\}, \delta p_j = \varepsilon\{p_j, G(q,p,t)\}. \tag{15}$$

(iv) This CT preserves the EOM in Hamiltonian form.

Symmetry triangle

$$\delta q \text{ a symmetry if } \delta L = \varepsilon \frac{\mathrm{d}F}{\mathrm{d}t}$$

$$\swarrow \qquad \searrow$$

$$\varepsilon G = p\delta q - \varepsilon F = COM \longrightarrow \delta q, \delta p \text{ is a CT generated by } G$$

Gauge type symmetries, constrained systems

Infinitesimal parameter ε in (9) $\to \varepsilon(t)$.

P.A.M. Dirac, Can. J. Math. **2**, 129–148 (1950); Yeshiva Lectures 1964.

Sketch of Dirac treatment

$$\text{Legendre map} \qquad p_j = \frac{\partial L(q,\dot{q})}{\partial \dot{q}^j}, \quad j = 1, 2, \ldots, n \tag{16}$$

leads to **primary constraints**

$$\phi_\rho(q,p) \approx 0, \rho = 1, 2, \ldots, k = n - \text{rank } H(q,\dot{q}). \tag{17}$$

Initial phase space form of Euler–Lagrange EOM:

$$\text{initial Hamiltonian } H_0(q,p) = p_j \dot{q}^j - L(q,\dot{q}) \tag{18}$$

EOM:

$$\begin{aligned} \dot{q}^j &\approx \{q^j, H_0\} + \{q^j, \phi_\rho\} v_\rho, \\ \dot{p}_j &\approx \{p_j, H_0\} + \{p_j, \phi_\rho\} v_\rho, \\ \frac{\mathrm{d}}{\mathrm{d}t} f(q,p,t) &\approx \{f, H_0\} + \{f, \phi_\rho\} v_\rho + \frac{\partial f}{\partial t}. \end{aligned} \tag{19}$$

Consistency analysis: first step, add

$$\{\phi_\sigma, H_0\} + \{\phi_\sigma, \phi_\rho\} v_\rho \approx 0 \quad \begin{cases} \diagup \ \text{secondary constraints } \chi_{..}(q,p) \approx 0 \\ \diagdown \ \text{restrictions on } v_\rho \end{cases} \tag{20}$$

Second step: add

$$\{\chi_{..}(q,p), H_0\} + \{\chi_{..}(q,p), \phi_\rho\} v_\rho \approx 0. \tag{21}$$

End of analysis: final Hamiltonian $H(q,p)$, complete set of constraints:

$$H(q,p) = H_0(q,p) + a_\rho(q,p)\phi_\rho(q,p),$$

$$\phi_\rho(q,p) \approx 0, \quad \chi_{..}(q,p) \approx 0. \tag{22}$$

$$\text{General EOM:} \quad \frac{\mathrm{d}}{\mathrm{d}t} f(q,p,t) \approx \{f,H\} + \{f,\phi_\alpha\}v_\alpha + \frac{\partial f}{\partial t} \tag{23}$$

First class and second class functions:

$$f(q,p) \text{ first class } \Leftrightarrow \{f, \phi_\rho \text{ or } \chi_{..}\} \approx 0, \text{ modulo } \phi_\rho \approx \chi_{..} \approx 0 \tag{24}$$

otherwise second class

Facts

final Hamiltonian $H(q,p)$ is first class.

Set $\phi_\alpha(q,p)$ = maximum number of primary first class constraints

Now we can describe the Noether results: for any infinitesimal continuous symmetry,

(i) the COM $G(q,p)$ is first class;

(ii) the symmetry transformation is a CT generated by

$$G(q,p) + \phi_\alpha(q,p)u_\alpha. \tag{25}$$

If the symmetry is of gauge type, more specific results:

$$\begin{aligned} \text{Type I:} \quad & \delta q^j(t) = \varepsilon(t)\phi^j(q,\dot{q},t); \\ \text{Type II:} \quad & \delta q^j(t) = \varepsilon(t)\phi^j(q,\dot{q},t) + \dot{\varepsilon}(t)\phi'^j(q,\dot{q},t), \end{aligned} \tag{26}$$

then for the COM,

$$\begin{aligned} \text{Type I:} \quad & G(q,p,t) = \varepsilon(t) \times \text{primary first class constraints;} \\ \text{Type II:} \quad & G(q,p,t) = \varepsilon(t) \times \text{combination of primary and secondary first class constraints} \\ & + \dot{\varepsilon}(t) \times \text{primary first class constraints.} \end{aligned} \tag{27}$$

Thank you