Interplay of Symmetries and Other Integrable Quantifiers in Finite Dimensional Nonlinear Dynamical Systems

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Objectives

- Integrable finite dimensional nonlinear dynamical systems are intimately related to the existence of continuous symmetries (Noether, Lie (point/contact), λ- symmetries, adjoint symmetries, etc.).
- There exist other integrability methods which also help to identify sufficient number of integrals and obtain solutions.
- These include methods of Darboux polynomials, Jacobi last multipliers and modified Prelle-Singer procedure.
- We bring out the interconnections between all these methods.

Plan of talk

- Dynamical systems
- Lie/ Noether symmetries
- Generalized symmetries
- Darboux polynomials
- Jacobi last multiplier
- Modified Prelle-Singer method
- Interconnections
- Conclusions

Dynamical Systems:

Consider the integrability of the dynamics of an n-th order nonlinear system:

$$\frac{d^n x}{dt^n} = x^{(n)} = \phi(x^{(n-1)}, x^{(n-2)}, ...x^{(2)}, x^{(1)}, x, t)$$

or a system of n first order ODEs:

$$\dot{x}_i = \phi_i(x_1, x_2, ...x_n, t), \; i = 1, 2, ...n$$

or equivalent second order systems, etc.

M. Lakshmanan and S. Rajasekar, Nonlinear Dynamics: Integrability, Chaos, and Patterns, New York: Springer, (2003)

Dynamical System: Introduction

- We look for n- independent analytic integrals of motion so that the solutions can be expressed in terms of analytic/ single valued functions.

Example: 1

1) Modified Emden Equation (V. K. Chandrasekar, M. Senthilvelan, M. Lakshmanan, Phys. Rev. E **72**, 066203 (2005))

$$\ddot{x} + 3x\dot{x} + x^3 + \omega_0^2 x = 0, \tag{1}$$

Eq. (1) admits two time dependent integrals:

$$I_1 = e^{-2\sqrt{-\omega_0^2}t}\left(\frac{\dot{x} + x^2 + \sqrt{-\omega_0^2}x}{\dot{x} + x^2 - \sqrt{-\omega_0^2}x}\right),$$

$$I_2 = -2e^{\sqrt{-\omega_0^2}t}\left(\frac{\omega_0^2 + \dot{x} + x^2}{\dot{x} + x^2 + \sqrt{-\omega_0^2}x}\right).$$

When $\omega_0^2 = 0$

$$\Rightarrow I_1 = -t + \frac{x}{x^2 + \dot{x}}, \quad I_2 = \frac{1}{2}t^2 + \frac{1 - tx}{x^2 + \dot{x}}.$$

Example: 2

2) Mathews-Lakshmanan oscillator: (P. M. Mathews and M. Lakshmanan, Q. Appl. Math, 32, 215 (1974))

$$\ddot{x} - \frac{kx\dot{x}^2}{1 + kx^2} + \frac{\omega^2 x}{1 + kx^2} = 0.$$

$$\Rightarrow I_1 = \frac{k\dot{x}^2 - \omega^2}{1 + kx^2}, \quad I_2 = \tan^{-1}\left(\frac{x\sqrt{\frac{\omega^2 - k\dot{x}^2}{1 + kx^2}}}{\dot{x}}\right) - t\sqrt{\frac{\omega^2 - k\dot{x}^2}{1 + kx^2}}.$$

II Continuous symmetries:

(a) Point symmetries

- One way is to look at the continuous symmetries associated with the system as advocated by Emmy Noether and earlier by Sophus Lie.
- For example for a second order ODE, look for infinitetesimal symmetries:

$$
\begin{aligned}
& t \to T = t + \varepsilon\,\xi(t,x), \\
& x \to X = x + \varepsilon\,\eta(t,x), \quad \epsilon \ll 1, \\
& \dot{x} = x^{(1)} \to \frac{dX}{dT} = \dot{x} + \varepsilon\,\eta^{(1)}(t,x,\dot{x}), \quad \eta^{(1)} = \dot{\eta} - \dot{x}\dot{\xi} \\
& \ddot{x} = x^{(2)} \to \frac{d^2X}{dT^2} = \ddot{x} + \varepsilon\,\eta^{(2)}(t,x,\dot{x}), \quad \eta^{(2)} = \dot{\eta}^{(1)} - \dot{x}\dot{\xi}
\end{aligned}
$$

- Vector field

$$v = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}$$

- P. J. Olver, *Applications of Lie Groups to Differential Equations*, (Springer-Verlag, New York, 1986)
- G. W. Bluman and S. Anco, *Symmetries and Integration Methods for Differential Equations* (Springer-Verlag, New York, 2002)

Point symmetries: Contd...

The invariance of

$$\ddot{x} = \phi(\dot{x}, x, t)$$

can be written as

$$v^{(2)}(\ddot{x} - \phi(\dot{x}, x, t))|_{\ddot{x}-\phi(\dot{x},x,t)=0} = 0, \tag{3}$$

where $v^{(2)}$ is the second prolongation

$$v^{(2)} = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \eta^{(1)}\frac{\partial}{\partial \dot{x}} + \eta^{(2)}\frac{\partial}{\partial \ddot{x}}. \tag{4}$$

Applying (4) in (3)

$$\xi\frac{\partial \phi}{\partial t} + \eta\frac{\partial \phi}{\partial x} + \eta^{(1)}\frac{\partial \phi}{\partial \dot{x}} - \eta^{(2)} = 0. \tag{5}$$

Point symmetries: Contd...

- By introducing the characteristics

$$Q = \eta(t, x) - \dot{x}\xi(t, x)$$

the invariance condition can be rewritten in terms of the characteristics Q as

$$D^2[Q] - \phi_{\dot{x}} D^1[Q] + \phi_x Q = 0$$

- For the n-th order equation, the invariance condition becomes

$$D^n[Q] - \phi_{x^{(n-1)}} D^{(n-1)}[Q] + ... - \phi_{x^{(1)}} D[Q] + \phi_x Q = 0$$

where $D = \frac{\partial}{\partial t} + x^{(1)} \frac{\partial}{\partial x} + ... + \phi \frac{\partial}{\partial x^{(n-1)}}$

Finding integrals

- The explicit form of the invariance condition (5) is

$$\xi\frac{\partial\phi}{\partial t}+\eta\frac{\partial\phi}{\partial x}+(\eta_t+(\eta_x-\xi_t)\dot{x}-\xi_x\dot{x}^2)\frac{\partial\phi}{\partial\dot{x}}-\eta_{tt}-(2\eta_{tx}-\xi_{tt})\dot{x}$$
$$-(\eta_{xx}-2\xi_{tx})\dot{x}^2-\xi_{xx}\dot{x}^3-(\eta_x-2\xi_t-3\xi_x\dot{x})\phi=0. \quad (6)$$

- From this ξ, η can be obtained (if they exist)
- Similarity variables/Integrals can be constructed
- Solutions can be found
- Integrability requires at least n-parameter symmetries
- Linearizing transformations can be constructed.
- For second order 2/8 symmetries are required for integrability/linearization.

Example:

The modified Emden equation (S. N. Pandey, P. S. Bindu, M. Senthilvelan and M. Lakshmanan, *J. Math. Phys.* **50**, 102701 (2009))

$$\ddot{x} + 3x\dot{x} + x^3 = 0,$$

$$\begin{aligned}
\xi &= x\left(a_2 + a_1 t - \frac{c_2 + b_1}{2}t^2 - \frac{c_1 + d_2}{2}t^3 + \frac{d_1}{4}t^4\right) \\
&\quad -\frac{d_1}{2}t^3 + \left(c_1 + \frac{3}{2}d_2\right)t^2 + b_1 t + b_2, \\
\eta &= -x^3\left(a_1 t + a_2 + \frac{d_1}{4}t^4 - \left(\frac{c_1 + d_2}{2}\right)t^3 - \frac{c_2 + b_1}{2}t^2\right) \\
&\quad +x^2\left(d_1 t^3 - 3\left(\frac{c_1 + d_2}{2}\right)t^2 - (c_2 + b_1)t + a_1\right) \\
&\quad +x\left(-\frac{3}{2}d_1 t^2 + c_1 t + c_2\right) + d_1 t + d_2,
\end{aligned}$$

where a_i, b_i, c_i and d_i, $i = 1, 2$, are real arbitrary constants.
Eight vector fields - Integrable and linearizable.

(a) Noether symmetries

Let a physical system be described by a Lagrangian $L(x, \dot{x}, t)$.
Action integral $\Rightarrow$

$$A = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt,$$

where the integration is along a curve $x = x(t)$.
Noether's theorem: Whenever the action integral is invariant under a continuous symmetry group such that $\delta A = 0$, the solutions to Euler's equation admit the conserved quantity

$$\Phi = (\xi \dot{x} - \eta) \frac{\partial L}{\partial \dot{x}} - \xi L + f, \quad (7)$$

where f is a function of q and t.

Noether symmetries: Contd...

The functions ξ, η and f can be determined by differentiating (7) with respect to t.

$$\Rightarrow v^{(1)}L = \dot{\xi}L + \dot{f}, \tag{8}$$

where

$$v^{(1)} = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \eta^{(1)}\frac{\partial}{\partial \dot{x}},$$

and Φ is a constant of motion.

- Noether symmetries form a sub-class of Lie symmetries.
- Noether theorem yields the conserved quantities in a direct manner.

- M. Lutzky, *J. Phys. A: Math. Gen.* **11**, 249 (1978); **12**, 973 (1979)
- R. Sahadevan and M. Lakshmanan, J, Phys. A, **19**, 2949 (1986); J. Math. Phys. **32**, 75, (1992)

Noether symmetries: Contd...

Example: The modified Emden equation:

$$\ddot{x} + 3x\dot{x} + x^3 = 0, \tag{9}$$

possesses nonstandard Lagrangian of the form

$$L = \frac{1}{3(\dot{x} + x^2)}$$

Solving (8) with the above Lagrangian, we find

$$v_1 = x\frac{\partial}{\partial t} - x^3\frac{\partial}{\partial x}, \quad v_2 = xt\frac{\partial}{\partial t} + \left(x^2 - tx^3\right)\frac{\partial}{\partial x},$$
$$v_3 = \left(t - \frac{3t^2x}{2}\right)\frac{\partial}{\partial t} + \left(2x - 3tx^2 + \frac{3t^2x^3}{2}\right)\frac{\partial}{\partial x},$$
$$v_4 = \left(\frac{t^3x}{2} - \frac{t^2}{2}\right)\frac{\partial}{\partial t} + \left(1 - 2tx + \frac{3t^2x^2}{2} - \frac{t^3x^3}{2}\right)\frac{\partial}{\partial x}, \ v_5 = \frac{\partial}{\partial t}.$$

Noether symmetries: Example Contd...

The associated conserved quantities are

$$I_1 = t - \frac{x}{x^2+\dot{x}}, \quad I_2 = \frac{(-x+tx^2+t\dot{x})^2}{(x^2+\dot{x})^2} = I_1^2,$$

$$I_3 = \frac{-9t^2x^3+3t^3x^4-3x(2+3t^2\dot{x})+2tx^2(6+3t^2\dot{x})+9t\dot{x}(6+3t^2\dot{x})}{(x^2+\dot{x})^2} = I_1 I_4,$$

$$I_4 = \frac{3(2-2tx+t^2x^2+t^2\dot{x})}{(x^2+\dot{x})},$$

$$I_5 = \frac{2\dot{x}+x^2}{3(x^2+\dot{x})^2} = \frac{1}{9}(I_4 - 3I_1^2) = -H = L - p\dot{x}, \quad \frac{dI_i}{dt} = 0, \ i = 1,2,3,4,5.$$

(b) Contact symmetries:

- Not all systems admit adequate number of Lie point symmetries
- One has to look for generalizations involving derivatives of dynamical variables $\Rightarrow$ contact symmetries
- One such generalization

$$
\begin{aligned}
T &= t + \varepsilon\,\xi(t, x, x^{(1)}), \quad (x^{(1)} = \dot{x}) \\
X &= x + \varepsilon\,\eta(t, x, x^{(1)}), \quad \epsilon \ll 1, \\
\dot{X} &= x^{(1)} + \varepsilon\,\eta^{(1)}(t, x, x^{(1)}) + O(\epsilon^2), \quad \epsilon \ll 1,
\end{aligned}
$$

where

$$
\eta^{(1)} = \dot{\eta} - x^{(1)}\dot{\xi} \quad \eta^{(2)} = \dot{\eta}^{(1)} - x^{(1)}\dot{\xi}
$$

Contact symmetries: Contd...

The associated vector field

$$\Omega = \xi(t, x, x^{(1)})\frac{\partial}{\partial t} + \eta(t, x, x^{(1)})\frac{\partial}{\partial x}.$$

Introducing the characteristics

$$W = \eta(t, x, x^{(1)}) - x^{(1)}\xi(t, x, x^{(1)})$$

the invariance condition can be written as

$$D^2[W] - \phi_{\dot{x}} D[W] + \phi_x W = 0. \tag{10}$$

Solving (10), we get W which in turn yields ξ and η.

Contact symmetries: Contd...

Example:

Modified Emden Equation $\ddot{x} + 3x\dot{x} + x^3 = 0$. Solving (10), we find two particular solutions

$$W_1 = \frac{x^2(1-tx)}{x^2+\dot{x}} - t^2\dot{x}, \quad W_2 = -\frac{x\dot{x}}{\sqrt{x^2+2\dot{x}}} - \frac{x\left(x^2+\dot{x}\right)}{\sqrt{x^2+2\dot{x}}}.$$

The infinitesimal vector fields read

$$\Omega_1 = t^2\frac{\partial}{\partial t} + \frac{x^2(1-tx)}{\dot{x}+x^2}\frac{\partial}{\partial x}, \quad \Omega_2 = \frac{x}{\sqrt{2\dot{x}+x^2}}\frac{\partial}{\partial t} - x\frac{(\dot{x}+x^2)}{\sqrt{2\dot{x}+x^2}}\frac{\partial}{\partial x}.$$

- Unlike Lie point symmetries it is difficult to construct the integrals from contact symmetries.

(c) λ- Symmetries: (C. Muriel and J. L. Romero, *J. Phys. A* **42**, 365207 (2009))

- One can consider even more general symmetries (which are neither Lie point symmetries nor contact symmetries but satisfy a new prolongation formula)

$$T = t + \varepsilon\,\xi(t,x)e^{\int \lambda(\dot{x},x,t)dt},$$
$$X = x + \varepsilon\,\eta(t,x)e^{\int \lambda(\dot{x},x,t)dt}.$$

Let

$$v = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}$$

be a λ- symmetry. Then the characteristics satisfy

$$\tilde{D}^2[Q] - \phi_{\dot{x}}\tilde{D}[Q] + \phi_x Q = 0.$$

where $\tilde{D} = D + \lambda(\dot{x},x,t)$

λ- Symmetries: Contd...

The invariance condition in terms of ξ and η reads

$$\xi\phi_t + \eta\phi_x + \eta^{[\lambda,(1)]}\phi_{\dot{x}} - \eta^{[\lambda,(2)]} = 0,$$

where $\eta^{[\lambda,(1)]}$ and $\eta^{[\lambda,(2)]}$ are the first and second λ- prolongations.

$$\begin{aligned}
\eta^{[\lambda,(1)]} &= (D+\lambda)\eta(t,x) - (D+\lambda)(\xi(t,x))x^{(1)}, \\
\eta^{[\lambda,(2)]} &= (D+\lambda)\eta^{[\lambda,(1)]}(t,x,x^{(1)}) - (D+\lambda)(\xi(t,x))x^{(2)},
\end{aligned}$$

- C. Muriel and J. L. Romero, *IMA J. Appl. Math.* **66**, 111 (2001); *Theor. Math. Phys.* **133**, 1565 (2002)
- C. Muriel and J. L. Romero, *J. Lie Theory* **13**, 167 (2003); *J. Nonlinear Math. Phys.* **15**, 300 (2008); *J. Phys. A: Math. Theor.* **42**, 365207 (2009); *Proc. Appl. Math. Mech.* **8**, 10747 (2008)

λ- Symmetries: Contd...

Assume $\xi = 0$, $\eta = 1$ so that $\tilde{v} = \frac{\partial}{\partial x}$.
λ-symmetry:

$$\phi_x + \lambda \phi_{\dot{x}} = D[\lambda] + \lambda^2.$$

$$\Rightarrow \lambda = -\frac{D[Q]}{Q}$$

Examples:

Modified Emden Equation $\ddot{x} + 3x\dot{x} + x^3 = 0$:

$$v = \frac{\partial}{\partial t}, \quad Q = \dot{\eta} - \dot{x}\xi = -\dot{x}, \quad \lambda = -(3x + \frac{x^3}{\dot{x}}) \tag{12}$$

$$v = x\frac{\partial}{\partial t} - x^3\frac{\partial}{\partial x}, \quad Q = -x(\dot{x} + x^2), \quad \lambda = -(\frac{\dot{x}}{x} - x)$$

Mathews-Lakshmanan oscillator $\ddot{x} - \frac{kx\dot{x}^2}{1+kx^2} + \frac{\omega^2 x}{1+kx^2} = 0$:

$$v = \frac{\partial}{\partial t}, \quad \lambda_1 = \frac{\dot{x}}{x} - x, \quad \lambda_2 = -\frac{(2 - \dot{x}t^2 - 4tx + t^2x^2)}{t(-2 + tx)}.$$

(d) Adjoint Symmetries (Bluman and Anco, Eur. J. App. Math. 1998)

- The integrating factors are the solutions of adjoint equation of the linearized equation.
- If the adjoint equation coincides with the linearized equation, then the underlying system is self-adjoint and the symmetry characteristics become the integrating factors.

$$L[Q] = 0 \tag{13a}$$

$$L = D^n - \sum_{i=0}^{n-1} \phi_{x^{(i)}} D^i \tag{13b}$$

then

$$L(t, x, x^{(1)}, ..., x^{(n-1)})(x^{(n)} - \phi(t, x, x^{(1)}, ..., x^{(n-1)})) = \frac{d}{dt}(I_i),\ i = 1, 2, ..., n.$$

and $\Lambda = Q$.

If not self-adjoint adequate modifications can be made.

Example

Modified Emden Equation:

$$\ddot{x} + 3x\dot{x} + x^3 = 0,$$

Two particular solutions of (13) are given by

$$\Lambda_1 = \frac{x}{(\dot{x} + x^2)}, \quad \Lambda_2 = \frac{t(-2 + tx)}{2(t\dot{x} - x + tx^2)^2}.$$

Multiplying the given equation by each one of these integrating factors and rewriting the resultant expression as a perfect derivative and integrating them we obtain two integrals I_1 and I_2 which are of the form,

$$I_1 = -\frac{(\dot{x}t - x + tx^2)}{\dot{x} + x^2}, \quad I_2 = -\frac{(2 + \dot{x}t^2 - 2tx + t^2x^2)}{2(\dot{x}t - x + tx^2)}.$$

M. Senthilvelan, V. K. Chandrasekar, R. Mohanasubha, *Pramana-J. Phys*, **85, 755-787** (2015)

III Jacobi Last multiplier

Method of Jacobi Last multiplier (C. G. J. Jacobi, 1843)

$$x^{(n)} = \phi(t, x, x^{(1)}, x^{(2)}, ..., x^{(n-1)})$$

The Jacobi last multiplier:

$$D[\log M] + \phi_{x^{(n-1)}} = 0,$$

$$D = \frac{\partial}{\partial t} + x^{(1)}\frac{\partial}{\partial x} + \cdots + \phi\frac{\partial}{\partial x^{(n-1)}}$$

$$\Rightarrow \frac{D[M]}{M} + \phi_{x^{(n-1)}} = 0. \tag{15}$$

- Solving (15), we get the multipliers (M_i).
- From multipliers integrals and Lagrangians can be constructed.

M. C. Nucci, K. M, Tamizhmani, *J. Nonlinear Math. Phys.* **17** 167 (2010)

Jacobi Last multiplier: Properties

Property -1

The ratio of two Jacobi last multipliers yields an integral ($I = \frac{M_1}{M_2}$)

$$\frac{D[M_1]}{M_1} + \phi_{x^{(n-1)}} = 0, \quad \frac{D[M_2]}{M_2} + \phi_{x^{(n-1)}} = 0.$$

$$\frac{D[M_1]}{M_1} - \frac{D[M_2]}{M_2} = 0$$

$$D[\frac{M_1}{M_2}] = \frac{dI}{dt} = 0 \Rightarrow \mathrm{I} = constant$$

M. C. Nucci and P. G. L. Leach, *J. Math. Phys.* **48**, 123510 (2007)

A. G. Choudhury and P. Guha, *J. Phys. A*, **43**, 125202, (2010)

Jacobi Last multiplier

Property -2

The Lagrangian of the ODE, if it exists,

$$M^{\frac{2}{n}} = \frac{\partial^2 L}{\partial (x^{(\frac{n}{2})})^2}.$$

The Euler-Lagrange equation:

$$\sum_{k=0}^{\frac{n}{2}} (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial x^{(k)}} \right) = 0.$$

Jacobi Last multiplier: Example

Let us consider Modified Emden Equation $\ddot{x} + 3x\dot{x} + x^3 = 0$.
The determining equation for the Jacobi last multiplier is

$$\frac{\partial M}{\partial t} + \dot{x}\frac{\partial M}{\partial x} - (3x\dot{x} + x^3)\frac{\partial M}{\partial \dot{x}} - 3xM = 0. \tag{16}$$

Three particular solutions of (16) are given by

$$M_1 = -\frac{1}{(\dot{x} + x^2)^3}, \quad M_2 = \frac{1}{(-x + t(\dot{x} + x^2))^3}, \quad M_3 = \frac{8}{(2 - 2tx + t^2(\dot{x} + x^2))^3}.$$

$\Rightarrow \frac{M_1}{M_2} = I_1$ and $\frac{M_2}{M_3} = I_2$.

R. Mohanasubha, V. K. Chandrasekar, M. Senthilvelan, M. Lakshmanan, *Proc. R. Soc. A* **470**, 20130656 (2014); **471**, 20140720 (2015); **472**, 20150847 (2016).

IV Darboux polynomial approach (G. Darboux, 1878)

- A strategy to find integrals
- A given system is said to be Darboux integrable, if it possesses sufficient number of first integrals or integrating factors in terms of Darboux polynomials
- Darboux showed that if we have $\frac{n(n+1)}{2} + 1$ Darboux polynomials, then there exists a rational first integral which can be expressed in terms of these Darboux polynomials.

Let

$$I = \frac{f(t, x, ..., x^{(n-1)})}{g(t, x, ..., x^{(n-1)})},$$

- G. Darboux, *Bull. Sci. Math* **2**, 60 (1878).
- J. Llibre et. al, *Q. Theor. Dyn. Syst.*, **11**, 129 (2011)

Darboux polynomial approach: Contd...

$$\frac{dI}{dt} = \frac{d}{dt}\left(\frac{f}{g}\right) = 0$$

$$\frac{df}{dt} = \alpha(t, x, ..., x^{(n-1)})f$$

$$\Rightarrow \qquad D[f] = \alpha(t, x, ..., x^{(n-1)})f,$$

where $\alpha(t, x, ..., x^{(n-1)}) = \frac{D[g]}{g}$.

Darboux polynomials: Property-1

The ratio of two Darboux polynomials which share the same cofactor provides a first integral for the ODE.

$$D[f_1] = \alpha f_1, \quad D[f_2] = \alpha f_2.$$

$$\frac{dI}{dt} = \frac{d}{dt}\left(\frac{f_1}{f_2}\right) = 0 \quad \Rightarrow \quad \frac{\dot{f}_1}{f_2} - \frac{\dot{f}_2 f_1}{f_2^2} = \frac{\alpha f_1}{f_2} - \frac{\alpha f_1}{f_2} = 0.$$

Darboux polynomial approach: Contd...

Darboux polynomials: Property-2

- The combination of Darboux polynomials are also a Darboux polynomials.
- Let a function $G = \prod_i f_i^{n_i}$, where f_i's are Darboux polynomials and n_i's are rational numbers.
- If we can identify a sufficient number of Darboux polynomials (irreducible polynomials) f_i's, satisfying the relations $D[f_i]/f_i = \alpha_i$, where α_i's are their cofactors, then

$$\frac{D[G]}{G} = \sum_i n_i \frac{D[f_i]}{f_i} = \sum_i n_i \alpha_i .$$

$$\Downarrow$$

Cofactor

Darboux polynomial approach: Example

Modified Emden Equation $\ddot{x} + 3x\dot{x} + x^3 = 0$.

$$\frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} - (3x\dot{x} + x^3)\frac{\partial f}{\partial \dot{x}} = \alpha f. \tag{17}$$

It is straightforward to check that

$$f_1 = -(\dot{x} + x^2), \quad f_2 = -x + t(\dot{x} + x^2), \quad f_3 = 1 + \frac{\dot{x}t^2}{2} - tx + \frac{t^2x^2}{2},$$

are particular solutions of (17) with same cofactors

$$\alpha_1 = \alpha_2 = \alpha_3 = -x.$$

$\Rightarrow \frac{f_1}{f_2} = I_1$ and $\frac{f_3}{f_2} = I_2$.

V Extended Prelle-Singer procedure

Chronology

- M. Prelle and M. Singer, Trans. Am. Math. Soc. **279**, 215 (1989).
- L. G. S. Duarte et. al. J. Phys. A, **34**, 3015 (2001).
- V.K.Chandrasekar, M. Senthilvelan, M. Lakshmanan, Proc. Roy. Soc. Lond. Series A **461** 2451-2476 (2005).
- V.K.Chandrasekar, M. Senthilvelan, M. Lakshmanan, Proc. Roy. Soc. Lond. Series A **462** 1831-1852 (2006).
- V.K.Chandrasekar, M. Senthilvelan, M. Lakshmanan, Proc. Roy. Soc. Lond. Series A **465** 585-608 (2009).
- V.K.Chandrasekar, M. Senthilvelan, M. Lakshmanan, Proc. Roy. Soc. Lond. Series A **465** 609- 629 (2009).
- V.K.Chandrasekar, M. Senthilvelan, M. Lakshmanan, Proc. Roy. Soc. Lond. Series A **465** 2369-2389 (2009).

V Extended Prelle-Singer procedure for second order ODE

Let

$$\ddot{x} = \phi(t, x, \dot{x}). \tag{18}$$

Let (18) admits a first integral $I(t, x, \dot{x}) = C$ (C constant on the solutions) so that

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0. \tag{19}$$

Rewriting the ODE

$$\phi dt - d\dot{x} = 0$$

and adding a null form

$$S(t, x, \dot{x})\dot{x} dt - S(t, x, \dot{x}) dx = 0.$$

We get

$$\left(\phi + S\dot{x}\right) dt - S dx - d\dot{x} = 0. \tag{20}$$

Extended Prelle-Singer procedure for second order ODE: Contd...

Multiplying (20) by $R(t, x, \dot{x})$

$$dI = R(\phi + S\dot{x})dt - RSdx - Rd\dot{x} = 0. \tag{21}$$

Comparing (21) and (19), we obtain

$$I_t = R(\phi + \dot{x}S), \tag{22a}$$

$$I_x = -RS, \tag{22b}$$

$$I_{\dot{x}} = -R. \tag{22c}$$

Compatibility: $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$

Extended Prelle-Singer procedure for second order ODE

$$
\begin{aligned}
D[S] &= -\phi_x + S\phi_{\dot{x}} + S^2, &(23)\\
D[R] &= -R(S + \phi_{\dot{x}}), &(24)\\
R_x &= R_{\dot{x}}S + RS_{\dot{x}}, &(25)
\end{aligned}
$$

where

$$
D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial \dot{x}}. \qquad (26)
$$

Note that for every independent set (S, R), $I(t, x, \dot{x})$ defines an integral.
Integrating (22)

$$
\begin{aligned}
I(t,x,\dot{x}) &= \int R(\phi + \dot{x}S)dt - \int \left(RS + \frac{d}{dx}\int R(\phi + \dot{x}S)dt \right) dx \\
&\quad - \int \left\{ R + \frac{d}{d\dot{x}} \left[\int R(\phi + \dot{x}S)dt - \int \left(RS + \frac{d}{dx}\int R(\phi + \dot{x}S)dt \right) dx \right] \right\} d\dot{x}.
\end{aligned}
$$

Extended Prelle-Singer procedure for second order ODE

Examples: Modified Emden Equation $\ddot{x} + 3x\dot{x} + x^3 = 0$.
Two particular solutions of (23) are

$$S_1 = \frac{-\dot{x} + x^2}{x}, \quad S_2 = \frac{2 - \dot{x}t^2 - 4tx + t^2x^2}{t(-2 + tx)}.. \tag{27}$$

The associated integrating factors are

$$\Rightarrow R_1 = \frac{x}{(\dot{x} + x^2)^2}, \quad R_2 = \frac{t(-2 + tx)}{2(t\dot{x} - x + tx^2)^2}.$$

- (S_1, R_1) and (S_2, R_2) satisfies (25) as well
- $(S_1, R_1) \Rightarrow I_1$; $(S_2, R_2) \Rightarrow I_2$

$$\Rightarrow I_1 = -t + \frac{x}{x^2 + \dot{x}}, \quad I_2 = \frac{1}{2}t^2 + \frac{1 - tx}{x^2 + \dot{x}}.$$

VI Interconnections

(a) Transformations

Introduce a transformation

$$S = -\frac{D[V]}{V}$$

in $D[S] = -\phi_x + S\phi_{\dot{x}} + S^2$

$$\Rightarrow \boxed{D^2[V] = \phi_{\dot{x}} D[V] + \phi_x V.}$$

With $R = \frac{V}{F}$,

$$D[R] = -R(S + \phi_{\dot{x}}).$$

$$\Rightarrow \boxed{D[F] = \phi_{\dot{x}} F.}$$

- R. Mohanasubha, V.K.Chandrasekar, M. Senthilvelan, M. Lakshmanan, *Proc. R. Soc. A* **470**, 20130656 (2014).
- R. Mohanasubha, V.K.Chandrasekar, M. Senthilvelan, M. Lakshmanan, *Proc. R. Soc. A* **471**, 20140720 (2015).
- R. Mohanasubha, V.K.Chandrasekar, M. Senthilvelan, M. Lakshmanan, *Proc. R. Soc. A* **472**, 20150847 (2016).

(b) Interconnections Contd...

On comparing with the above two equations with other methods we can interlink all of them.

(i) Lie/Noether symmetry:

$$Q = V$$

and that the null form

$$S = -\frac{D[Q]}{Q}$$

(ii) Contact symmetries:

$$W = V$$

and

$$S = -\frac{D[W]}{W}$$

(iii) λ**- symmetries:** Choosing $\xi = 0$ and $\eta = 1$,

$$D[\lambda] = \phi_x + \lambda\phi_{\dot{x}} - \lambda^2, \quad \tilde{v} = \frac{\partial}{\partial x}$$

$$\lambda = -S = \frac{D[V]}{V}.$$

Once Lie point symmetries are known, the λ-symmetries can be constructed.
If they are not known?
(iv) Adjoint- symmetries:
The linearized symmetry condition for the second-order ODE is

$$L[Q] = D^2[Q] - \phi_{\dot{x}} D[Q] + \phi_x Q = 0.$$

The adjoint of the above linearized symmetry condition is given by

$$L^*[\Lambda] = D^2[\Lambda] + D[\phi_{\dot{x}}\Lambda] + \phi_x \Lambda = 0.$$

Interconnections Contd...

All integrating factors also act as adjoint-symmetries.
In those cases, the adjoint-symmetry Λ should satisfy the adjoint-invariance condition

$$\Lambda_{t\dot{x}} + \Lambda_{x\dot{x}}\dot{x} + 2\Lambda_x + \Lambda\phi_{\dot{x}\dot{x}} + 2\phi_{\dot{x}}\Lambda_{\dot{x}} + \phi\Lambda_{\dot{x}\dot{x}} = 0. \qquad (28)$$

Eq. (28) can be obtained from $R_x = R_{\dot{x}}S + RS_{\dot{x}}$ The integrating factor equation R can be in the form

$$D^2[R] + D[\phi_{\dot{x}}R] - \phi_x R = 0.$$

Then

$$\Lambda = R$$

Interconnections Contd...

(v) Darboux polynomial:

$$D[f] = \alpha(t, x, \dot{x}) f,$$

$$D[F] = \phi_{\dot{x}} F$$

We have $f = F = \frac{V}{R}$ with $\alpha(t, x, \dot{x}) = \phi_{\dot{x}}$.
(vi) Jacobi last multiplier:

$$\frac{D[M]}{M} + \phi_{\dot{x}} = 0$$

Then $F = \frac{1}{M}$, or $M = \frac{1}{F} = \frac{R}{V}$

For second order ODEs

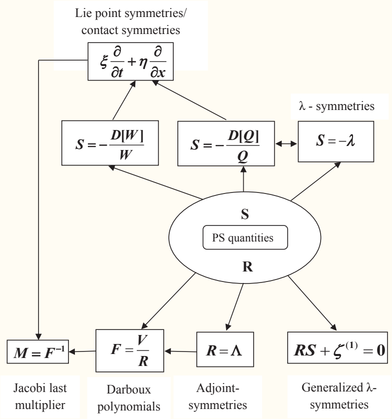

Figure : Flow chart connecting extended Prelle-Singer procedure with other methods for second order ODEs

From Lie point symmetries to other quantifiers

- We can start the procedure from anywhere.
- Instead of Prelle-Singer procedure, let us start from Lie point symmetries analysis.

$$\xi\frac{\partial\phi}{\partial t}+\eta\frac{\partial\phi}{\partial x}+\eta^{(1)}\frac{\partial\phi}{\partial\dot{x}}-\eta^{(2)}=0.\Rightarrow(\xi,\eta)$$

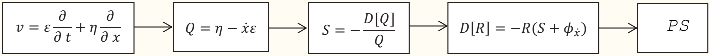

Figure : Determination of null forms and integrating factors of Eq. (9) from Lie point symmetries.

From contact symmetries to other quantifiers

- Now let us start from contact symmetries.

$$D^2[W] + 3xD[W] + 3(\dot{x} + x^2)W = 0.$$

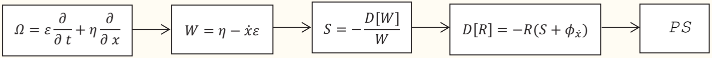

Figure : Determination of null forms and integrating factors of Eq. (9) from contact symmetries

From λ-symmetries to other quantifiers

- Now let us start from λ- symmetry analysis.

$$D[\lambda] + \lambda^2 + 3x\lambda + 3(\dot{x} + x^2) = 0.$$

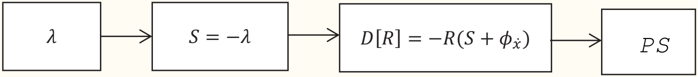

Figure : Determination of null forms and integrating factors of Eq. (9) from λ-symmetries.

From Darboux polynomials to other quantifiers

- Now let us start from Darboux polynomial approach.

$$\frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} - (3x\dot{x} + x^3)\frac{\partial f}{\partial \dot{x}} = \alpha f.$$

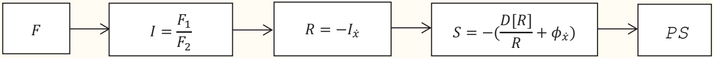

Figure : Determination of null forms and integrating factors of Eq. (9) from Darboux polynomials

From Jacobi last multipliers to other quantifiers

- Now we start from Jacobi last multiplier.

$$\frac{\partial M}{\partial t} + \dot{x}\frac{\partial M}{\partial x} - (3x\dot{x} + x^3)\frac{\partial M}{\partial \dot{x}} - 3xM = 0.$$

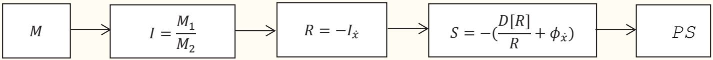

Figure : Determination of null forms and integrating factors of Eq. (9) from Jacobi last multipliers.

From adjoint-symmetries to other quantifiers

- Let us start the procedure from adjoint- symmetries.

$$\Lambda_{tt} + 2\Lambda_{tx}\dot{x} + \Lambda_{xx}\dot{x}^2 - 3\Lambda\dot{x} - 3x\Lambda_t - 3x\Lambda_x\dot{x} + (3\dot{x} + 3x^2)\Lambda$$
$$+(3x\dot{x} + x^3)\Lambda_x - (3x\dot{x} + x^3)\Lambda_{\dot{x}t} - (3x\dot{x} + x^3)\dot{x}\Lambda_{x\dot{x}} - (3\dot{x} + 3x^2)\Lambda_{\dot{x}}\dot{x} = 0.$$

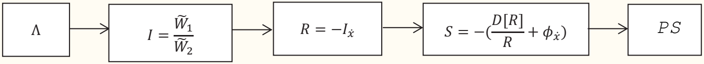

Figure : Determination of null forms and integrating factors of Eq. (9) from adjoint-symmetries.

VII Higher order ODEs

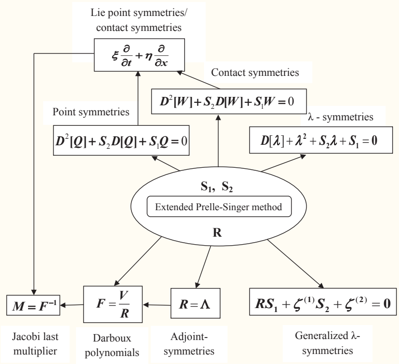

Figure : Flow chart connecting extended Prelle-Singer procedure with other methods for third order ODEs.

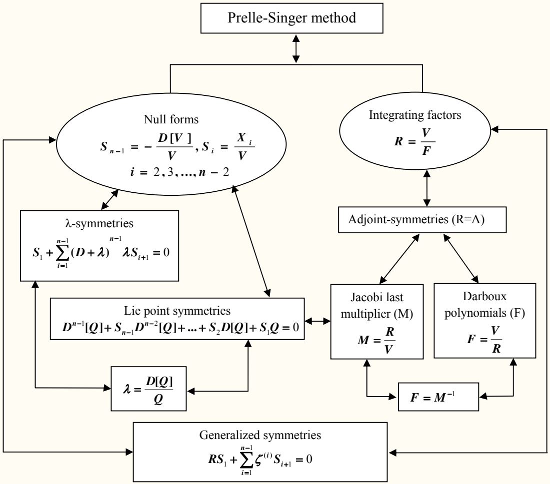

Figure : Flow chart connecting extended Prelle-Singer procedure with other methods for N-th order ODEs.

VIII Conclusions

- Symmetries play a crucial role in identifying integrable nonlinear dynamical systems, especially Noether and Lie symmetries.
- Other important methods of identifying integrable systems do exist.
- Fascinating interconnections can be identified.
- Much further work is required in the above analysis, PDEs, etc.