Top eigenvalue of a Gaussian random matrix: Large Deviations

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A Trivial Problem

$N \times N$ diagonal matrix with independent Gaussian entries

$$J = \begin{pmatrix} J_{11} & 0 & \dots & 0 \\ 0 & J_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_{NN} \end{pmatrix}$$

$$\mathrm{Prob.}[J_{ii}] \propto \exp\left[-\frac{1}{2}\, J_{ii}^2\right]$$

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- $P_N = \mathrm{Prob}[\lambda_1 \le 0,\ \lambda_2 \le 0,\ \dots,\ \lambda_N \le 0] = 2^{-N} = \exp[-(\ln 2)\, N]$

A Nontrivial Problem

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[R.M. May, Nature, 238, 413 (1972)——Ecosystems]

[Cavagna, Garrahan & Giardina 2000, Fyodorov 2004, Fyodorov & Williams 2007, Bray & Dean 2007, Auffinger, Ben Arous & Cerny 2010, Fydorov & Nadal 2012, ...——Glassy systems]

[Susskind 2003, Douglas et. al. 2004, Aazami & Easther 2006, Marsh, McAllister & Wrase 2011,—String theory & Cosmology]

[Beltrani 2007, Dedieu & Malajovich 2007, Houdre 2011, Chiani 2012—Random Polynomials]

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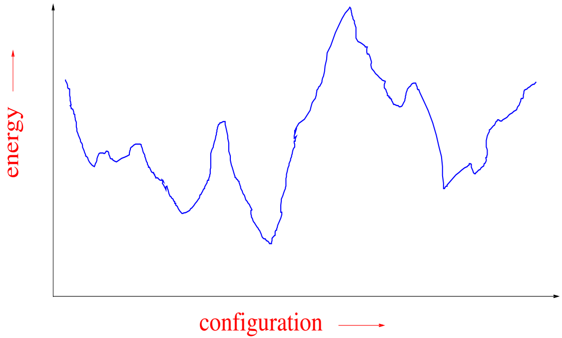
Complex Landscapes



Ex: Structural glasses, Supercooled liquids, Spin glasses, String landscapes

L. Susskind, "The anthropic landscape of string theory", hep-th/0302219

A. Sen, "Riding gravity away from doomsday", arXiv: 1503.08130

Stationary points of a complex landscape

A particle moving in a N-dimensional landscape: $V(y_1, y_2, \ldots, y_N)$

$$\frac{dy_i}{dt} = -\nabla_{y_i} V$$

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- Near a stationary point: $V(\{y_i\}) \approx \sum_{i,j} H_{i,j}(y_i - y_i^*)(y_j - y_j^*)$

Hessian matrix: $H_{i,j} \equiv \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$

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- Eigenvalues of the Hessian (stability) matrix determines the nature of the stationary point:

$\Longrightarrow$ Local Minimum, Local Maximum & Saddles

Eigenvalues of the Hessian Matrix

Examples:

- $N = 1$-dimensional surface: Hessian matrix $H = \frac{\partial^2 V}{\partial y^2}$

  If $\frac{\partial^2 V}{\partial y^2} < 0 \rightarrow$ Local Maximum; if $\frac{\partial^2 V}{\partial y^2} > 0 \rightarrow$ Local Minimum

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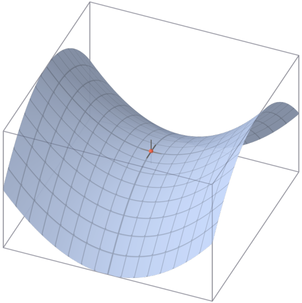
Two real eigenvalues: (λ_1, λ_2)

If $\lambda_1 < 0$ and $\lambda_2 < 0 \rightarrow$ Local Maximum

If $\lambda_1 > 0$ and $\lambda_2 > 0 \rightarrow$ Local Minimum

$$\left.\begin{array}{ll} \lambda_1 < 0, & \lambda_2 > 0 \\ \lambda_1 > 0, & \lambda_2 < 0 \end{array}\right\} \quad \rightarrow \text{Saddle}$$



Fraction of Local Maximum/Minimum in Random Hessian Model

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  $H \to (N \times N)$ real symmetric random matrix

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- In the random Hessian model

  $P_N =$ Fraction of Local Maxima/minima

  $= \text{Prob}[\lambda_1 < 0, \lambda_2 < 0, \ldots, \lambda_N < 0]$

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- $\longrightarrow$ Most of the stationary points $\to$ Saddles

I. Gaussian Random Matrices

Spectral Statistics in Random Matrix Theory (RMT)

Consider $N \times N$ Gaussian random matrix $J \equiv [J_{ij}]$

(i) real symmetric (ii) complex Hermitian (iii) complex quaternionic

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$$\mathrm{Prob.}[J] \propto \exp\left[-\beta \frac{N}{2} \sum_{i,j} |J_{ij}|^2\right]$$

$$\propto \exp\left[-\beta \, \tfrac{N}{2} \, \mathrm{Tr}\left(J^\dagger J\right)\right]$$

$\longrightarrow$ invariant under rotation

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N real eigenvalues: $\{\lambda_1, \lambda_2, \dots, \lambda_N\} \longrightarrow$ **strongly correlated**

Spectral statistics in RMT $\Rightarrow$ statistics of $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$

Joint distribution of eigenvalues

- Joint distribution of eigenvalues (Wigner, 1951)

$$\boxed{P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}}$$

where the Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE) or $\beta = 4$ (GSE)

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- Z_N = Partition Function

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

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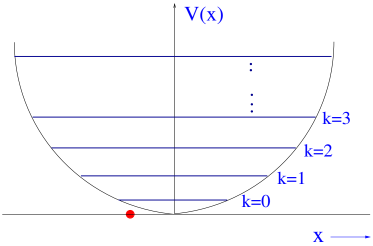
The Vandermonde term $\prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$ makes $\{\lambda_i\}'s$

$\Longrightarrow$ **strongly correlated**

A physical realization of GUE eigenvalues



A single quantum particle in a harmonic potential: $V(x) = \frac{1}{2} m\omega^2 x^2$

Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi_k}{dx^2} + \frac{1}{2} m\omega^2 x^2 \varphi_k(x) = \epsilon_k \varphi_k(x)$$

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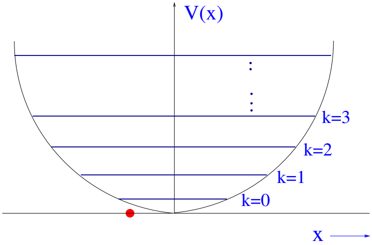

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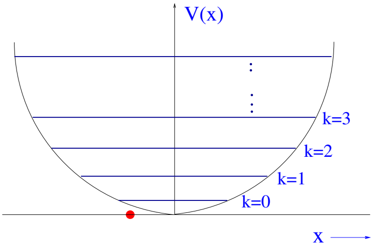

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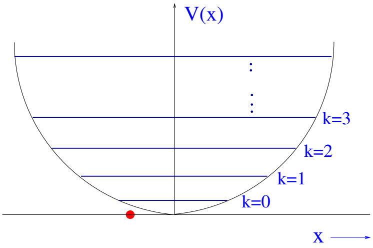

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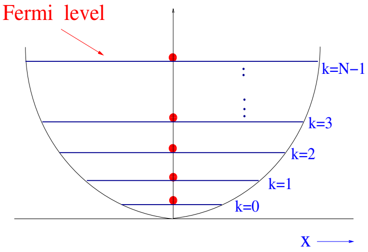
$H_k(x) \to$ Hermite polynomials

For example, $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, etc.

N spinless Fermions in a harmonic trap: T=0



ground state many-body wavefunction $\rightarrow$ Slater determinant

$\Psi_0(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)]$

with $0 \leq i \leq (N-1)$, $1 \leq j \leq N$

ground state energy: $E_0 = \hbar\,\omega\, N^2/2$

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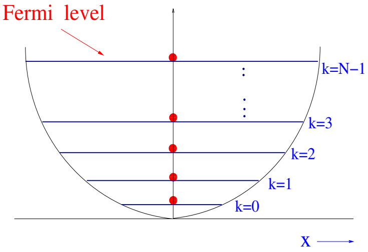

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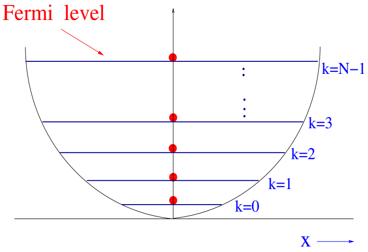
$$\Psi_0(\{x_i\}) \propto e^{-\frac{\alpha^2}{2}\sum_{i=1}^N x_i^2} \det_{1\le i,j\le N} [H_i(\alpha\, x_j)]$$

$$\propto e^{-\frac{\alpha^2}{2}\sum_{i=1}^N x_i^2} \prod_{j<k}(x_j - x_k)$$

N spinless Fermions in a harmonic trap: T=0



ground state many-body wavefunction $\rightarrow$ Slater determinant

$$\Psi_0(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)]$$

with $0 \le i \le (N-1)$, $1 \le j \le N$

ground state energy: $E_0 = \hbar\,\omega\, N^2/2$

$$\Psi_0(\{x_i\}) \propto e^{-\frac{\alpha^2}{2}\sum_{i=1}^N x_i^2} \det_{1\le i,j\le N}[H_i(\alpha\, x_j)]$$

$$\propto e^{-\frac{\alpha^2}{2}\sum_{i=1}^N x_i^2} \prod_{j<k}(x_j - x_k)$$

$$\Longrightarrow \qquad \boxed{|\Psi_0(\{x_i\})|^2 = \frac{1}{Z_N}\, e^{-\alpha^2 \sum_{i=1}^N x_i^2} \prod_{j<k}(x_j - x_k)^2}$$

Vandermonde determinant

Example: $N = 3$: $H_0(x) = 1,\ H_1(x) = 2\,x,\ H_2(x) = 4\,x^2 - 2$

$$\det \begin{pmatrix} H_0(x_1) & H_0(x_2) & H_0(x_3) \\ H_1(x_1) & H_1(x_2) & H_1(x_3) \\ H_2(x_1) & H_2(x_2) & H_2(x_3) \end{pmatrix}$$

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$$= 8\,\det\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

Vandermonde determinant

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$$= 8 \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

$$= 8\,(x_1 - x_2)\,(x_2 - x_3)\,(x_3 - x_1)$$

Free fermions at T=0 $\equiv$ GUE eigenvalues

- Fermions: squared many-body wave function at $T = 0$ (quantum probability density)

$$|\Psi_0(\{x_i\})|^2 = \frac{1}{Z_N} \exp\left[-\sum_{i=1}^{N} \alpha^2 x_i^2\right] \prod_{j<k} (x_j - x_k)^2 \text{ where } \alpha = \sqrt{m\omega/\hbar}$$

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- GUE eigenvalues: joint probability distribution

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^2$$

$\Rightarrow$ The positions of free fermions in a harmonic trap at $T = 0$ behave statistically as the eigenvalues of a GUE random matrix

$$\boxed{(\alpha\, x_1,\, \alpha\, x_2,\, \ldots, \alpha\, x_N) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_N)}$$

Dean, Le Doussal, S.M. & Schehr, PRL, 114, 110402 (2015); PRA, 94, 063622 (2016); J. Stat. Mech. P063301 (2017)

Joint distribution of eigenvalues

Joint distribution of eigenvalues (Wigner, 1951)

$$\boxed{P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N}\, \exp\left[-\frac{\beta}{2}\, N \sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}}$$

where the Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE) or $\beta = 4$ (GSE)

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Coulomb gas interpretration: (Dyson, 1962)

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \tfrac{1}{Z_N}\, \exp\left[-\tfrac{\beta}{2}\, \left(N \textstyle\sum_{i=1}^{N} \lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right)\right]$$

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$$\sim \exp\left[-\beta\, E[\{\lambda_i\}]\right]$$

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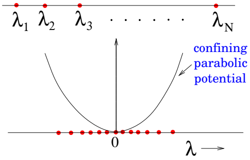
Boltzmann weight of a gas of N pairwise repelling charges (log-repulsion) in an external harmonic potential $V(\lambda) = \lambda^2$



Coulomb gas

- $Z_N = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N} N\,\lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right\}\right]$

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- Scaled energy of the Coulomb gas:

$$E\left(\{\lambda_i\}\right) = \frac{1}{2N}\sum_i \lambda_i^2 - \frac{1}{2N^2}\sum_{j\neq k} \log|\lambda_j - \lambda_k|$$

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- As $N \to \infty \to$ discrete sum $\to$ continuous integral:

$$\boxed{E[\rho(\lambda)] = \frac{1}{2}\left[\int \lambda^2 \rho(\lambda)\, d\lambda - \int \int \ln|\lambda - \lambda'|\, \rho(\lambda)\, \rho(\lambda')\, d\lambda\, d\lambda' \right]}$$

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  where the charge density: $\rho(\lambda) = \frac{1}{N}\sum_i \delta(\lambda - \lambda_i)$

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- Minimizing the energy subject to the constraint $\int \rho(\lambda)\, d\lambda = 1$ gives

$$Z_N \approx \exp\left[-\beta\, N^2\, E\left[\rho(\lambda)\right]\right]$$

Spectral Density: Wigner's Semicircle Law

- Av. density of eigenvalues (normalized to unity):

$$\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle$$

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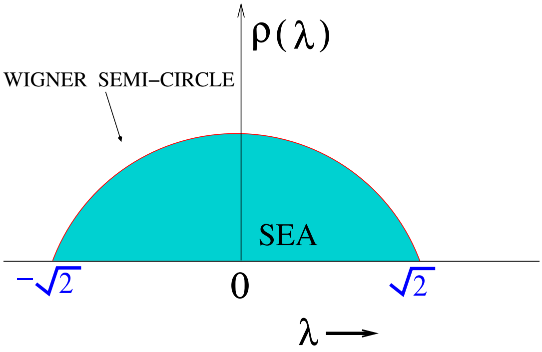
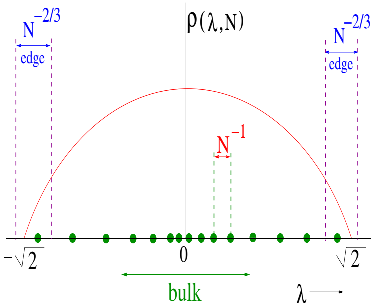
- Wigner's Semi-circle: $\rho(\lambda, N) \xrightarrow[N \to \infty]{} \rho(\lambda) = \frac{1}{\pi}\sqrt{2 - \lambda^2}$



Average density of eigevalues: Bulk and Edge



- Average density: $\rho(\lambda, N) \to \frac{1}{\pi}\sqrt{2-\lambda^2}$ with edges at $\pm\sqrt{2}$

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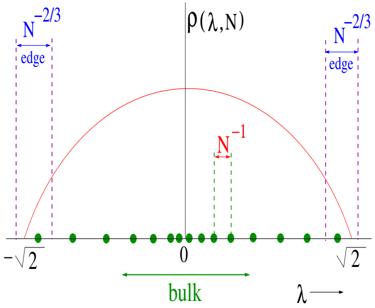

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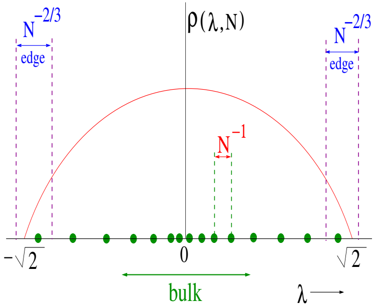

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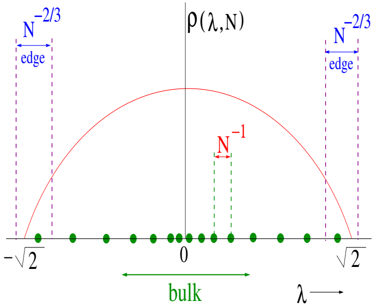

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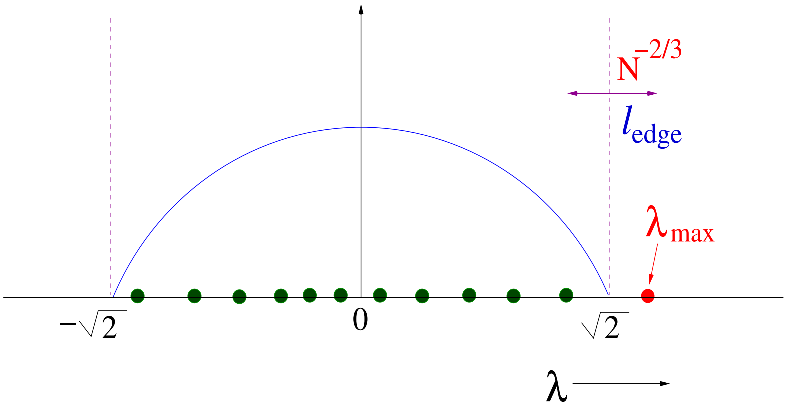
$$\boxed{l_{\rm edge} >> l_{\rm bulk}}$$

Average density at the edge: Bowick & Brezin '91, Forrester '93

II. Top eigenvalue: $\lambda_{\max}$

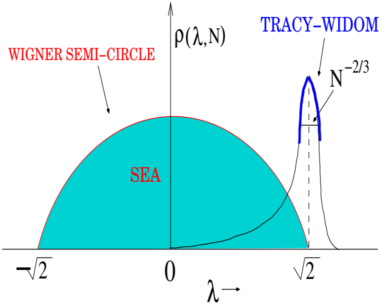
Top Eigenvalue of a random matrix $\lambda_{\max}$

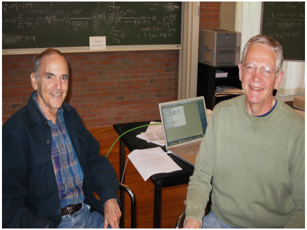


Recent excitements in statistical physics & mathematics on

$\lambda_{\max} \Rightarrow$ the top eigenvalue of a random matrix

Top Eigenvalue of a random matrix $\lambda_{\max}$



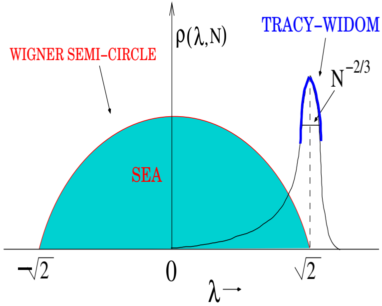


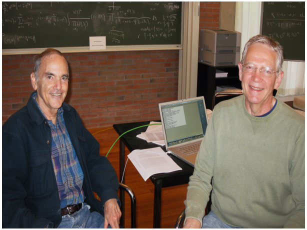
Average: $\langle \lambda_{\max} \rangle = \sqrt{2}$; Typical fluctuations: $|\lambda_{\max} - \sqrt{2}| \sim l_{\text{edge}} \sim N^{-2/3}$

typical fluctuations, for large N, are distributed via Tracy-Widom ('94)

$$P(\lambda_{\max}, N) \sim \sqrt{2}\, N^{2/3}\, f_\beta\left(\sqrt{2}\, N^{2/3}\, \left(\lambda_{\max} - \sqrt{2}\,\right)\right)$$

Top Eigenvalue of a random matrix $\lambda_{\max}$





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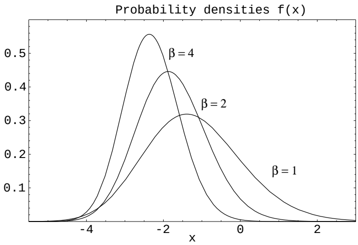
$$f_\beta(x) \rightarrow \text{Painlevé-II}$$

Tracy-Widom Distribution for $\lambda_{\max}$



- Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE) & $\beta = 4$ (GSE)

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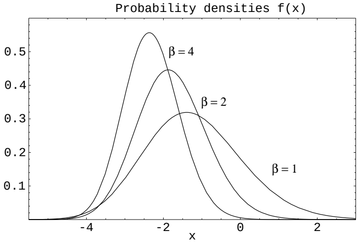

- Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE) & $\beta = 4$ (GSE)
- Asymptotics: $f_\beta(x) \sim \exp\left[-\frac{\beta}{24}|x|^3\right] \quad \text{as} \quad x \to -\infty$

$$\sim \exp\left[-\frac{2\beta}{3}\, x^{3/2}\right] \quad \text{as} \quad x \to \infty$$

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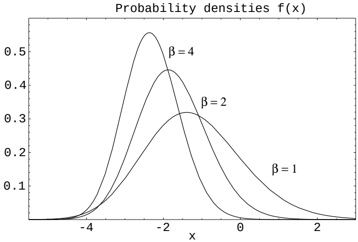

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Typical fluctuations (small) $\Rightarrow$ Tracy-Widom distribution $\rightarrow$ ubiquitous

directed polymer, random permutation, growth models–KPZ equation, sequence alignment, large N gauge theory, liquid crystals, spin glasses,...

Tracy-Widom distribution: Experiments

PRL **104**, 230601 (2010) PHYSICAL REVIEW LETTERS week ending 11 JUNE 2010

Universal Fluctuations of Growing Interfaces: Evidence in Turbulent Liquid Crystals

Kazumasa A. Takeuchi* and Masaki Sano
Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

RAPID COMMUNICATIONS

PHYSICAL REVIEW E **85**, 020101(R) (2012)

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

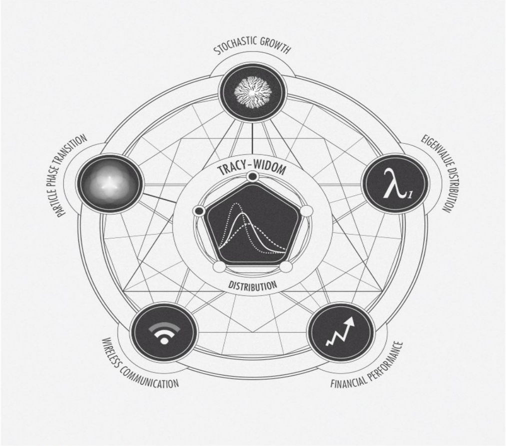
Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson*
Weizmann Institute of Science, Department of Physics of Complex Systems, Rehovot 76100, Israel

PHYSICAL REVIEW B **87**, 184509 (2013)

Universal scaling of the order-parameter distribution in strongly disordered superconductors

G. Lemarié,[1,2] A. Kamlapure,[3] D. Bucheli,[2] L. Benfatto,[2] J. Lorenzana,[2] G. Seibold,[4] S. C. Ganguli,[3] P. Raychaudhuri,[3] and C. Castellani[2]
[1]*Laboratoire de Physique Théorique UMR-5152, CNRS and Université de Toulouse, F-31062 France*
[2]*ISC-CNR and Department of Physics, Sapienza University of Rome, P.le A. Moro 2, 00185 Rome, Italy*
[3]*Tata Institute of Fundamental Research, Homi Bhabha Rd., Colaba, Mumbai 400005, India*
[4]*Institut Für Physik, BTU Cottbus, P.O. Box 101344, 03013 Cottbus, Germany*

Ubiquity of Tracy-Widom distribution

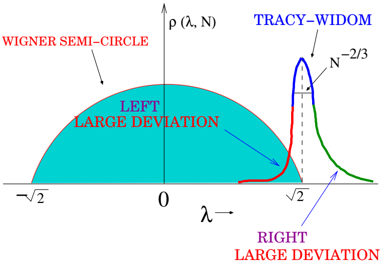


Olena Shmahalo/Quanta Magazine

"Equivalence Principle", M. Buchanan, Nature Phys. 10, 543 (2014)

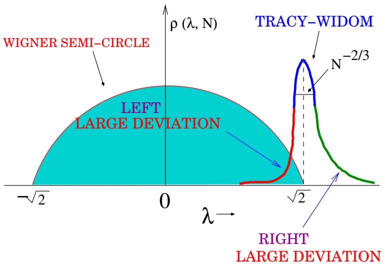
"At the far ends of a new universal law", N. Wolchover, Quanta Magazine (October, 2014)

Probability of Large Deviations of $\lambda_{\max}$:



- Tracy-Widom law $\mathrm{Prob}[\lambda_{\max} \le w, N] \to F_\beta\left(\sqrt{2}\, N^{2/3}\,(w-\sqrt{2})\right)$ describes the prob. of typical (small) fluctuations of $\sim O(N^{-2/3})$ around the mean $\sqrt{2}$, i.e., when $|\lambda_{\max} - \sqrt{2}| \sim N^{-2/3}$

Probability of Large Deviations of $\lambda_{\max}$:



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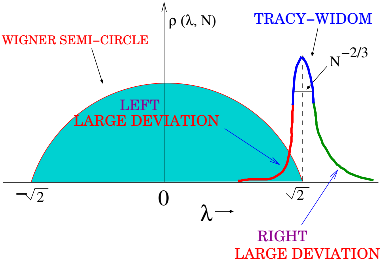

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In particular, $P_N = \mathrm{Prob.}[\lambda_{\max} \le 0, N] \Longrightarrow$ left large deviation tail

Why is Tracy-Widom ubiquitous?

A natural question: Why is Tracy-Widom distribution so ubiquitous?

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Critical Phenomena : universality $\Longleftrightarrow$ phase transition

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microscopic details become irrelevant near a critical point

Questions:

Where to look for a phase transition?

Where is the critical point ?

What are the two phases ?

Large deviations and 3-rd order phase transition

typical fluctuations of size $\sim N^{-2/3} \rightarrow$ Tracy-Widom distributed

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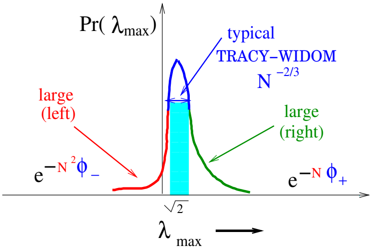

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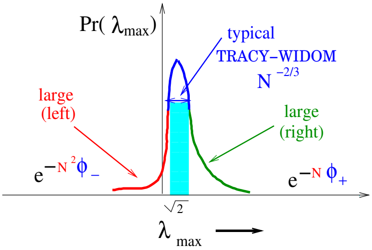

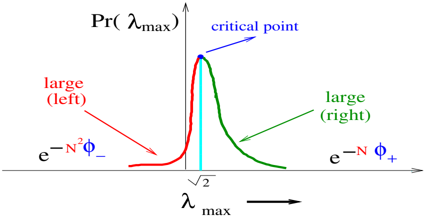

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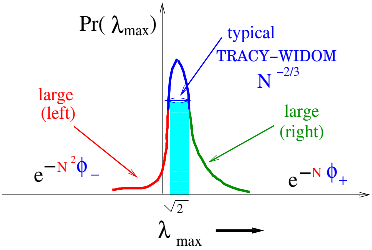

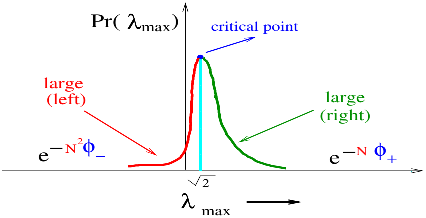

- Large deviation principle: Ben-Arous & Guionnet 1997

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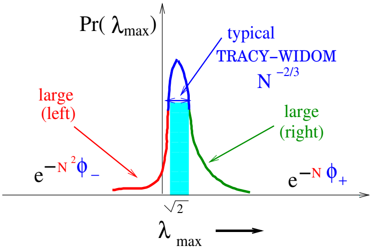

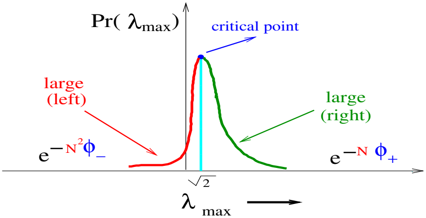

- Large deviation principle: Ben-Arous & Guionnet 1997
- nonanalytic behavior of the large deviation functions at the critical point $\sqrt{2} \Longrightarrow$ 3-**rd order phase transition**

  $\Longrightarrow$ Review: S.M. & G. Schehr, J. Stat. Mech. P01012 (2014)

IV. Large deviations of $\lambda_{\max}$ via Coulomb Gas

Distribution of $\lambda_{\max}$ via Coulomb gas

$$\mathrm{Prob}[\lambda_{\max} \leq w,\ N] = \mathrm{Prob}[\lambda_1 \leq w, \lambda_2 \leq w, \ldots, \lambda_N \leq w] = \frac{Z_N(w)}{Z_N(\infty)}$$

$$\boxed{Z_N(w) = \int_{-\infty}^{w} \cdots \int_{-\infty}^{w} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{N\sum_{i=1}^{N}\lambda_i^2 - \sum_{j\neq k}\log|\lambda_j - \lambda_k|\right\}\right]}$$

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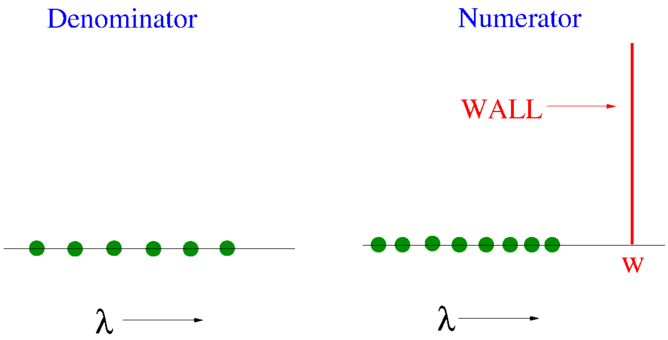

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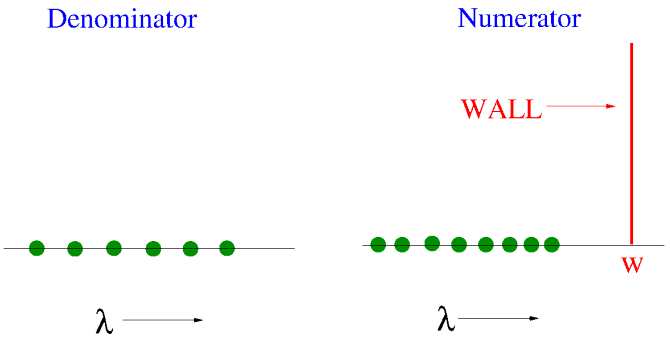
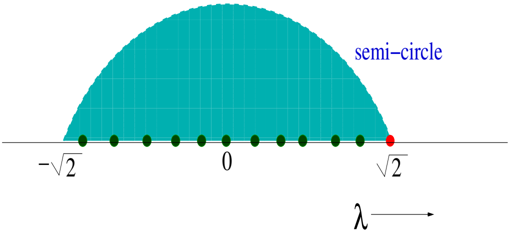


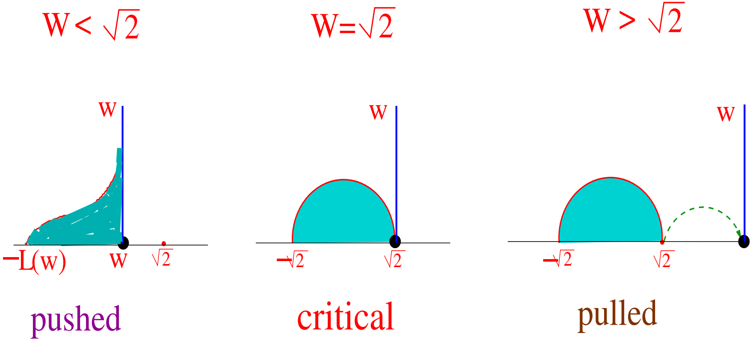
Denominator: Saddle point $\rightarrow$ Wigner semi-circle



$$\boxed{\rho_\infty(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}}$$

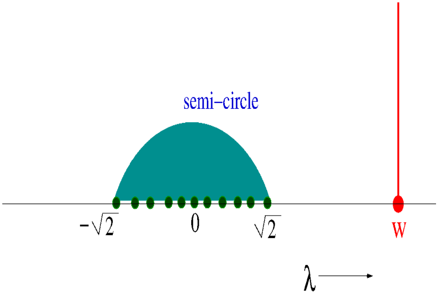
Phase transition

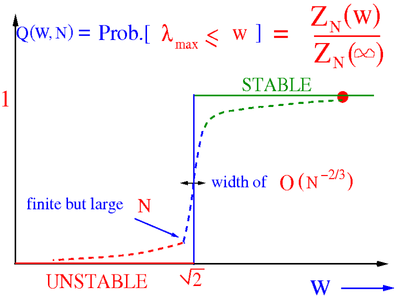
$$\lambda_{max} = W$$



$W = \sqrt{2}$ ⟶ CRITICAL POINT

Numerator: wigner semi-circle for $w > \sqrt{2}$





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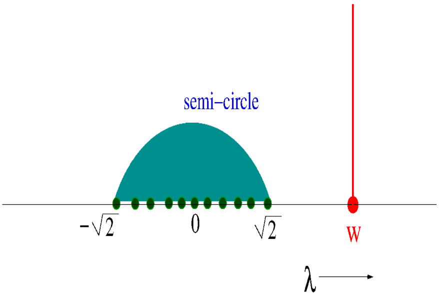

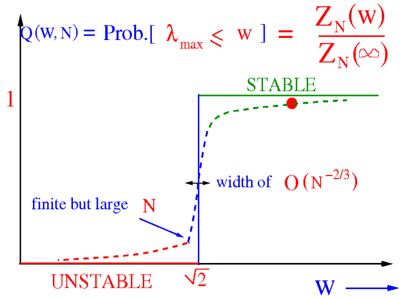

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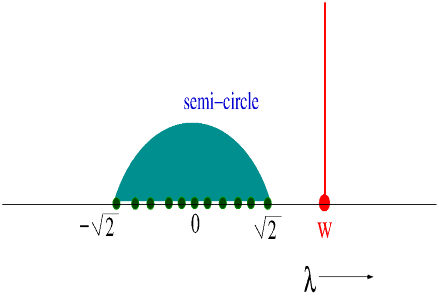

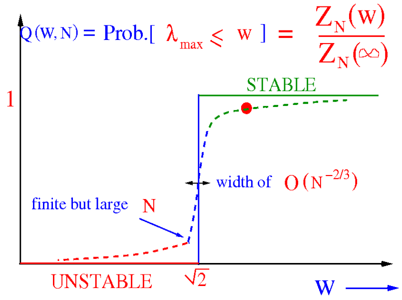

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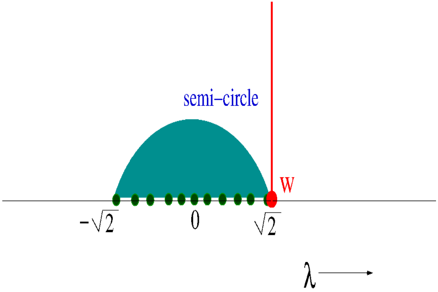
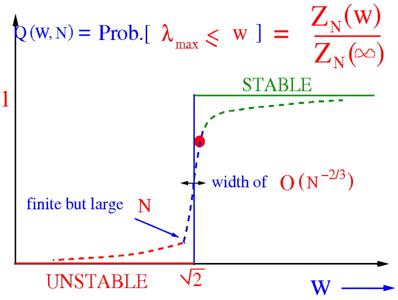
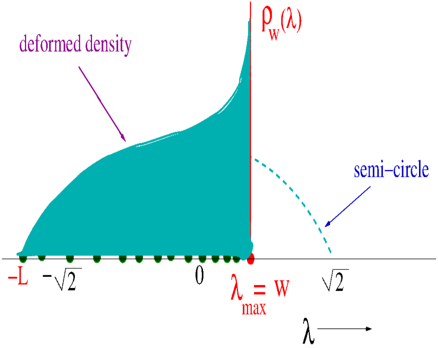
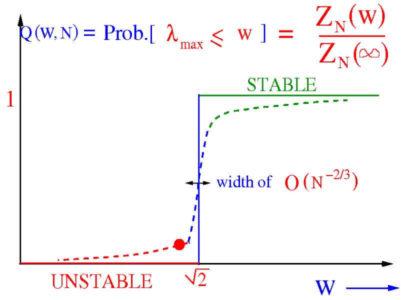


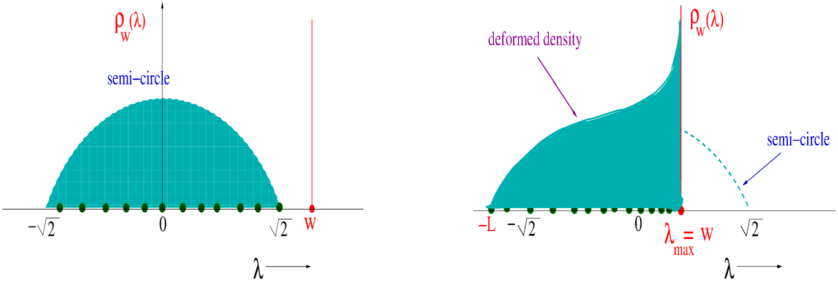


Numerator: Left large deviation $(w < \sqrt{2})$





Saddle point density



- For $w \geq \sqrt{2}$, the density is given by Wigner semicircle law:

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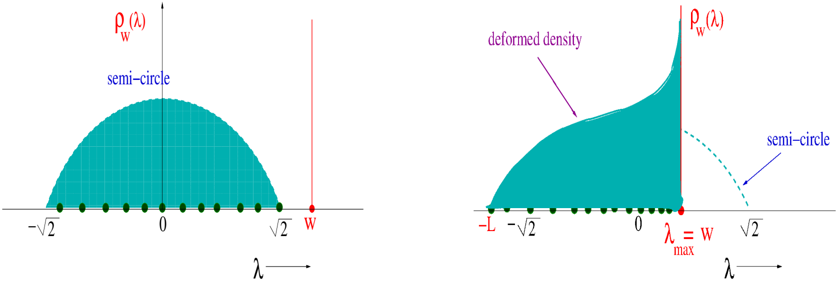

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- For $w < \sqrt{2}$, the deformed density: $\rho_w(\lambda) = \frac{\sqrt{\lambda+L(w)}}{2\pi\sqrt{w-\lambda}}\ [w + L(w) - 2\lambda]$

where $L(w) = [2\sqrt{w^2+6} - w]/3$

D.S. Dean and S.M., PRL, 97, 160201 (2006); PRE, 77, 041108 (2008)

Left Large Deviation Function

$$\mathrm{Prob}[\lambda_{\max} \le w, N] = \frac{Z_N(w)}{Z_N(\infty)} \sim \exp\left[-\beta N^2 \left\{E[\rho_w(\lambda)] - E[\rho_\infty(\lambda)]\right\}\right]$$

$$\sim \exp\left[-\beta\, N^2\, \Phi_-(w)\right]$$

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$$\Phi_-(w) = \frac{1}{108}\Big[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} + 27\left(\ln(18) - 2\ln(w + \sqrt{6+w^2})\right)\Big] \quad \text{for} \quad w < \sqrt{2}$$

(Dean & S.M., PRL, 97, 160201 (2006))

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- In particular, $P_N = \mathrm{Prob.}[\lambda_{\max} \le 0] \sim \exp[-\beta\, N^2\, \theta]$

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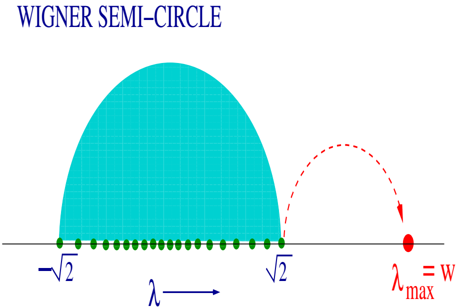
$$\boxed{\theta = \phi_-(0) = \frac{1}{4}\ln(3)}$$

- Note also that $\Phi_-(w) \approx \frac{1}{6\sqrt{2}}(\sqrt{2} - w)^3$ as $w \to \sqrt{2}$ from below

Right Large Deviation Function: $w > \sqrt{2}$



$\mathrm{Prob.}[\lambda_{\max} = w, N] \sim \exp[-N\,\Delta E(w)]$

$\Delta E(w) \Rightarrow$ energy cost in pulling a charge out of the Wigner sea

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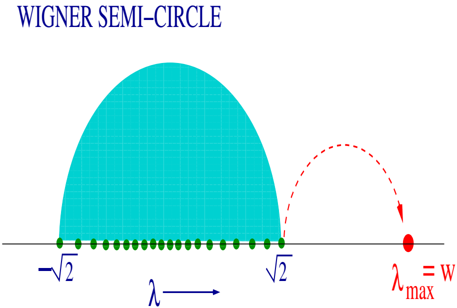

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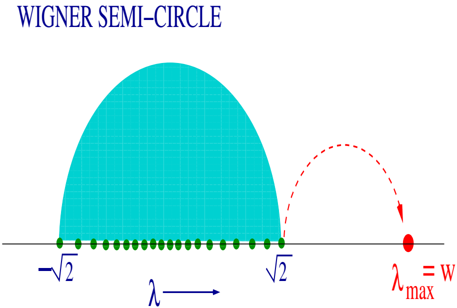

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Right large deviation: $\mathrm{Prob.}[\lambda_{\mathsf{max}} = w, N] \sim \exp[-N\,\Phi_+(w)]$

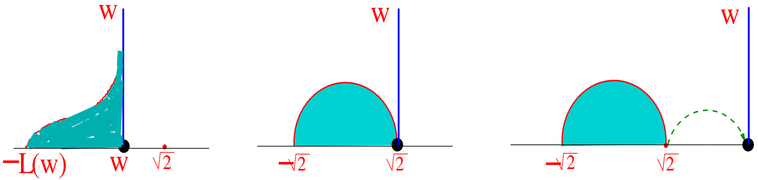
$$\boxed{\Phi_+(w) = \Delta E(w) = \frac{1}{2} w\sqrt{w^2-2} + \ln\left[\frac{w-\sqrt{w^2-2}}{\sqrt{2}}\right]} \quad (w > \sqrt{2})$$

[S.M. & Vergassola, PRL, 102, 160201 (2009)]

Phase transition

charge density that dominates $P(\lambda_{max} = W, N)$

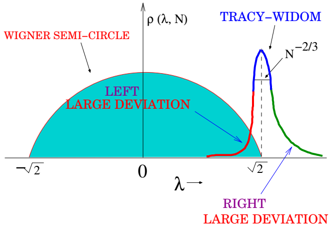
$W < \sqrt{2}$ | $W = \sqrt{2}$ | $W > \sqrt{2}$



pushed (UNSTABLE) | critical | pulled (STABLE)

$W = \sqrt{2} \longrightarrow$ CRITICAL POINT

Large Deviation Tails of $\lambda_{\max}$



Prob. density of the top eigenvalue: $\mathrm{Prob.}\,[\lambda_{\max} = w, N]$ behaves as:

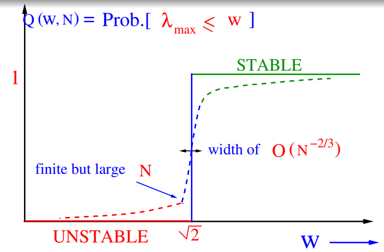
$$\sim \quad \exp\left[-\beta N^2 \Phi_-(w)\right] \qquad \text{for} \quad \sqrt{2} - w \sim O(1)$$

$$\sim \quad N^{2/3} f_\beta \left[\sqrt{2}\, N^{2/3}\, \left(w - \sqrt{2}\right)\right] \qquad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3})$$

$$\sim \quad \exp\left[-\beta N \Phi_+(w)\right] \qquad \text{for} \quad w - \sqrt{2} \sim O(1)$$

V. Summary and Implications

For Large but Finite N: Summary of Results



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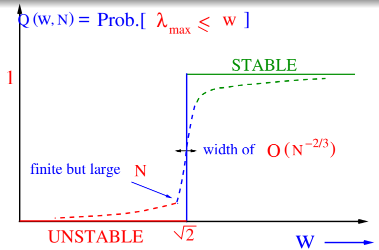
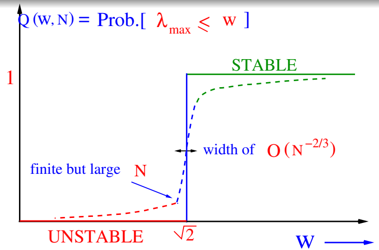


$$
\begin{aligned}
Q(w,N) &\sim \exp\left[-N^2\Phi_-(w)+\ldots\right] && \text{for} \quad \sqrt{2}-w \sim O(1) \\
&\sim F_1\left[\sqrt{2}\,N^{2/3}\,\left(w-\sqrt{2}\right)\right] && \text{for} \quad |w-\sqrt{2}| \sim O(N^{-2/3}) \\
&\sim 1-\exp\left[-N\Phi_+(w)+\ldots\right] && \text{for} \quad w-\sqrt{2} \sim O(1)
\end{aligned}
$$

For Large but Finite N: Summary of Results



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&\sim 1-\exp\left[-N\Phi_+(w)+\ldots\right] && \text{for} \quad w-\sqrt{2}\sim O(1)
\end{aligned}
$$

Crossover function: $F_1(z) \rightarrow$ Tracy-Widom (1994)

Exact rate functions:

$\Phi_-(w) \rightarrow$ Dean & S.M. 2006

$\Phi_+(w) \rightarrow$ S.M. & Vergassola 2009; see also Ben Arous, Dembo, Guionnet 2001

Higher order corrections: (Borot, Eynard, S.M., & Nadal 2011, Nadal & S.M., 2011)

Large Deviation Functions

These large deviation functions $\Phi_{\pm}(w)$ have been found useful in a large variety of problems:

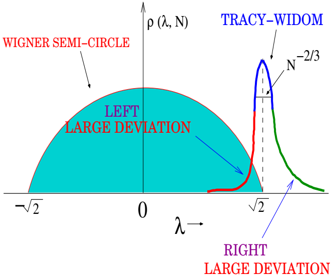
[Fyodorov 2004, Fyodorov & Williams 2007, Bray & Dean 2007, Auffinger, Ben Arous & Cerny 2010, Fydorov & Nadal 2012.... —— stationary points on random Gaussian surfaces and spin glass landscapes]

[Cavagna, Garrahan, Giardina 2000, Parisi & Rizzo 2008,... —— Glassy systems]

[Susskind 2003, Douglas et. al. 2004, Aazami & Easther 2006, Marsh et. al. 2011, ...—— String theory & Cosmology]

[Beltrani 2007, Dedieu & Malajovich, 2007, Houdre 2011, Chiani 2012...—— Random Polynomials, Random Words (Young diagrams)]

3-rd order phase transition

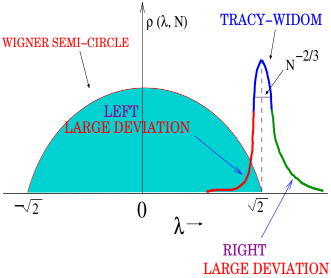


Cumulative distribution:

$$\mathrm{Prob.}[\lambda_{\max} \leq w, N] \sim e^{-\beta\, N^2\, \Phi_-(w)}$$

$\Phi_-(w) \to$ energy cost in pushing the gas of Coulomb charges to the left of $\sqrt{2}$

3-rd order phase transition



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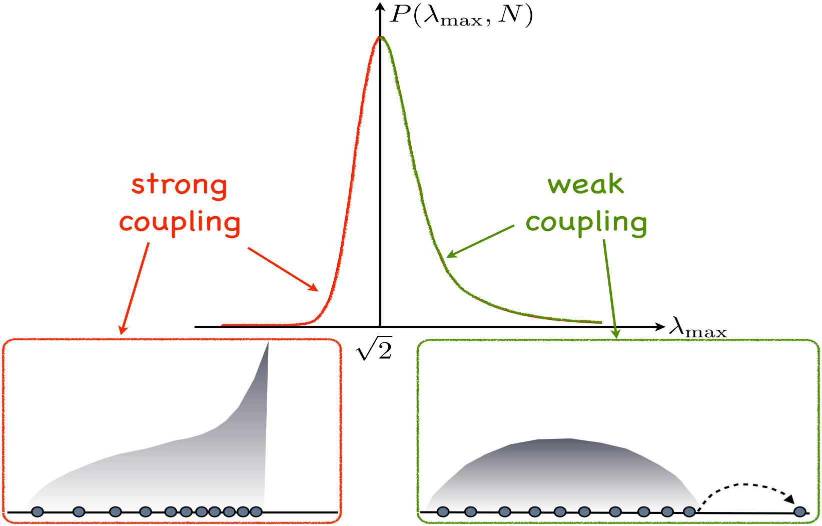
3-rd order phase transition:

$$\lim_{N\to\infty} -\frac{1}{\beta\, N^2} \ln\left[P(\lambda_{\max} \leq w, N)\right] = \begin{cases} \Phi_-(w) \sim (\sqrt{2} - w)^3 & \text{as } w \to \sqrt{2}^- \\ 0 & \text{as } w \to \sqrt{2}^+ \end{cases}$$

$\longrightarrow$ analogue of the free energy difference

3-rd derivative $\rightarrow$ discontinuous

Transition between Strong and Weak phases



PHYSICAL REVIEW D VOLUME 21, NUMBER 2 15 JANUARY 1980

Possible third-order phase transition in the large-N lattice gauge theory

David J. Gross
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

Edward Witten
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 10 July 1979)

The large-N limit of the two-dimensional U(N) (Wilson) lattice gauge theory is explicitly evaluated for all fixed $\lambda = g^2 N$ by steepest-descent methods. The λ dependence is discussed and a third-order phase transition, at $\lambda = 2$, is discovered. The possible existence of such a weak- to strong-coupling third-order phase transition in the large-N four-dimensional lattice gauge theory is suggested, and its meaning and implications are discussed.

Volume 93B, number 4 PHYSICS LETTERS 30 June 1980

$N = \infty$ PHASE TRANSITION IN A CLASS OF EXACTLY SOLUBLE MODEL LATTICE GAUGE THEORIES ★

Spenta R. WADIA
The Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

Received 27 March 1980

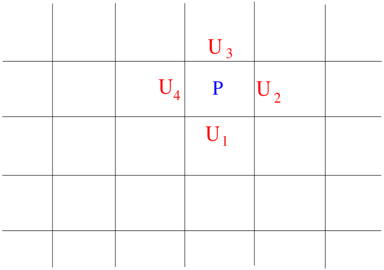
A nice review of large-N gauge theory: M. Marino, arXiv:1206.6272

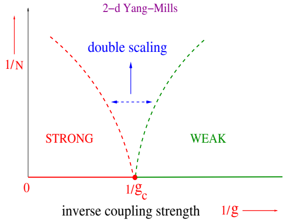
Gross-Witten-Wadia model (1980)

$$Z = \int [DU] \, \exp\left[S(U)\right]$$

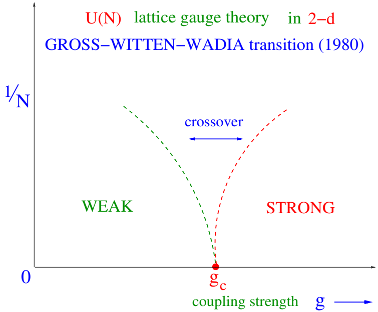
$$S(U) = \frac{N}{g^2} \sum_P \mathrm{Tr}\left(\prod_P U + h.c\right)$$

$g \to$ coupling strength

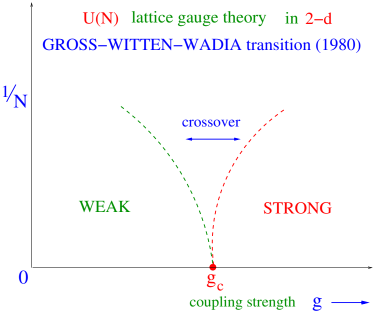


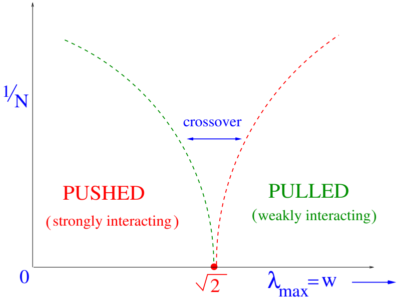


Large N Phase Transition: Phase Diagram



Large N Phase Transition: Phase Diagram





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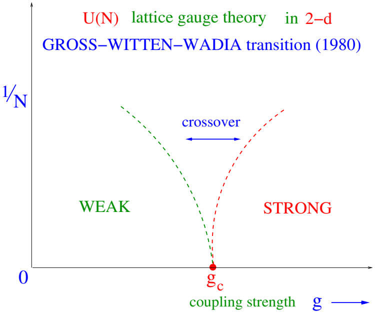
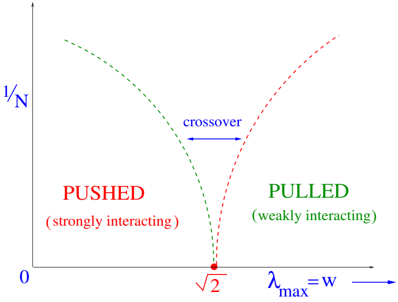
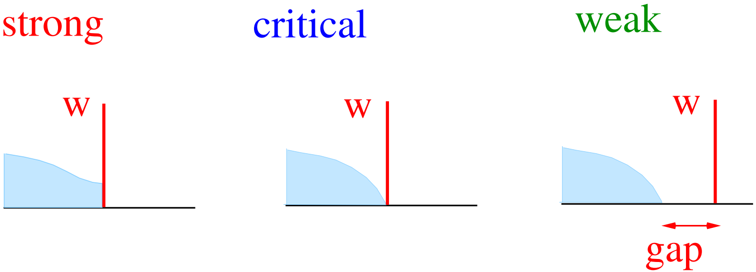




PUSHED phase $\equiv$ Strong coupling phase of Yang-Mills gauge theory
PULLED phase $\equiv$ Weak coupling phase of Yang-Mills gauge theory

Tracy-Widom $\Rightarrow$ crossover function in the double scaling regime (for finite but large N)

Basic mechanism for 3-rd order transition



Gap between the soft edge (square-root singularity) of the Coulomb droplet and the hard wall vanishes as a control parameter g goes through a critical value g_c:

$$\text{gap} \longrightarrow 0 \text{ as } g \to g_c$$

3-rd order phase transition $\Longleftrightarrow$ universal Tracy-Widom crossover

3-rd order transition $\to$ ubiquitous

- $\lambda_{\max}$ for other matrix ensembles: Wishart: $W = X^{\dagger}X \to (N \times N)$
  $\to$ covariance matrix

  Typical: Tracy-Widom [Johansson 2000, Johnstone 2001]

  Large deviations: Exact rate functions

  [Vivo, S.M., Bohigas 2007, S.M. & Vergassola 2009]

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  [Vivo, S.M., Bohigas 2007, S.M. & Vergassola 2009]

- large N gauge theory in 2-d [Gross, Witten, Wadia '80, Douglas & Kazakov '93]
- Distribution of MIMO capacity [Kazakopoulos et. al. 2010]
- Complexity in spin glass models [Auffinger, Ben Arous & Cerny 2010, Fyodorov & Nadal 2013]
- Random tilings models [Colomo & Pronko 2013]

3-rd order transition → ubiquitous

We have found 3-rd order phase transitions in a wide variety of systems:

- Conductance and Shot Noise in Mesoscopic Cavities
- Entanglement entropy of a random pure state in a bipartite system
- Maximum displacement in Vicious walker problem
- Distribution of Wigner time-delay ...

More recently:

- Cold atoms: free fermions in a 1-d harmonic trap at $T = 0$
- Height distribution in $(1+1)$-d KPZ growth models
- 1d Plasma: position of the rightmost charge
- Ginibre ensemble of RMT

Bohigas, Comtet, Dean, Forrester, Le Doussal, Nadal, Schehr, Texier, Vergassola, Vivo+...S.M. ('08-'17)

Main Conjecture:

Wherever Tracy-Widom distribution occurs, there is an underlying 3-rd order phase transition

TW $\longrightarrow$ finite-size crossover function at a 3-rd order critical point

Review: S.M. & G. Schehr, J. Stat. Mech. P01012 (2014)

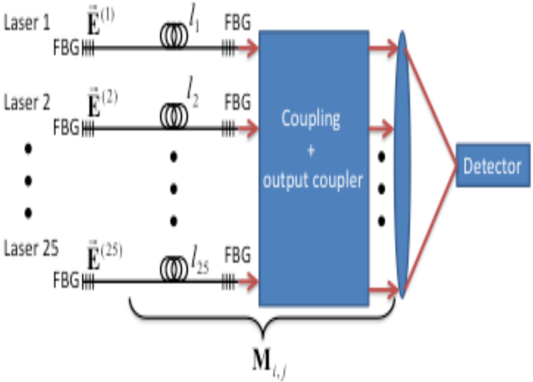
Experimental Verification with Coupled Lasers

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson*
Weizmann Institute of Science, Dept. of Physics of Complex Systems, Rehovot 76100, Israel
(Dated: May 30, 2011)

We determined the probability distribution of the combined output power from twenty five coupled fiber lasers and show that it agrees well with the Tracy-Widom, Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions of the largest eigenvalue of Wishart random matrices with no fitting parameters. This was achieved with 500,000 measurements of the combined output power from the fiber lasers, that continuously changes with variations of the fiber lasers lengths. We show experimentally that for small deviations of the combined output power over its mean value the Tracy-Widom distribution is correct, while for large deviations the Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions are correct.

Experimental Verification with Coupled Lasers

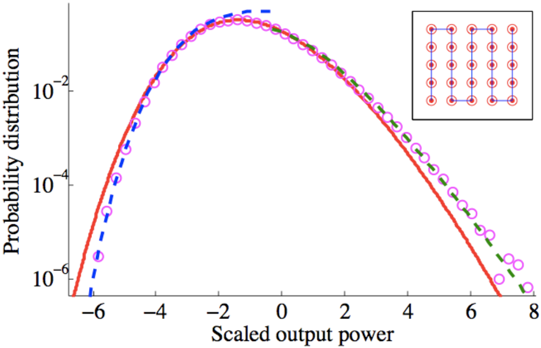


combined output power from fiber lasers $\propto \lambda_{\max}$

$\lambda_{\max} \to$ top eigenvalue of the Wishart matrix $W = X^t X$

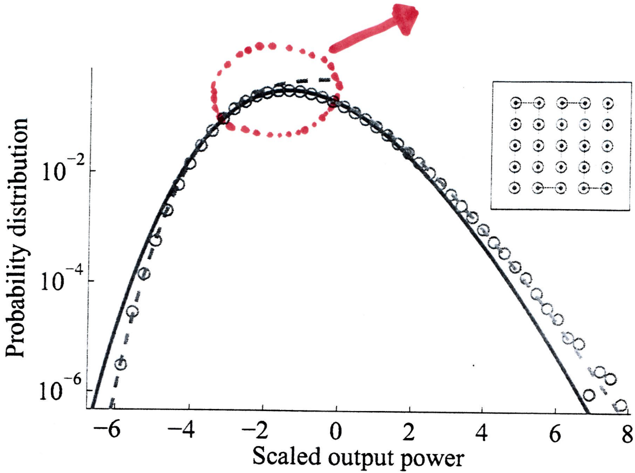
where $X \to$ real symmetric Gaussian matrix $(\beta = 1)$

Experimental Verification with Coupled Lasers



(a)

Experimental Verification with coupled lasers

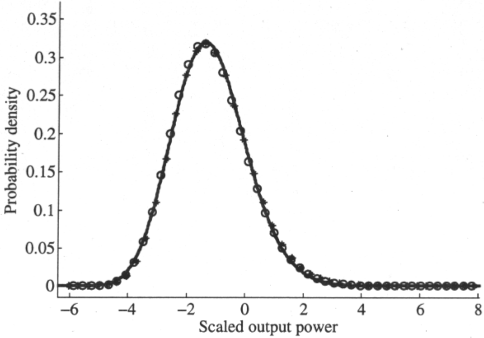


(a)

Tracy-Widom density with $\beta = 1$

Fridman et. al. arXiv: 1012.1282

2



Collaborators

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Postdocs: A. Kundu (ICTS, Bangalore), D. Villamaina (CFM, Paris), J. Grela (LPTMS)

Collaborators:

O. Bohigas, A. Comtet, G. Schehr, C. Texier (LPTMS, Orsay, France), R. Allez (Berlin, Germany), G. Borot (Geneva, Switzerland), J.-P. Bouchaud & M. Potters (CFM, Paris, France), P. Calabrese (Sissa, Trieste, Italy), K. Damle & V. Tripathi (Tata Institute, Bombay, India), A. Dhar (ICTS, Bangalore), D.S. Dean (Bordeaux, France), B. Eynard (Saclay, France), P. J. Forrester (Melbourne, Australia), A. Lakshminarayan (IIT Madras, India), P. Le Doussal (ENS, Paris, France), S. Sabhapandit (RRI, Bangalore), A. Scardichio (ICTP, Trieste, Italy), S. Tomsovic (Washington State Univ., USA), M. Vergassola (UCSD, USA), P. Vivo (King's College, UK)

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Review

Review: S.M. & G. Schehr, arXiv: 1311.0580

J. Stat. Mech. P01012 (2014)