Current fluctuations and dynamical phase transitions

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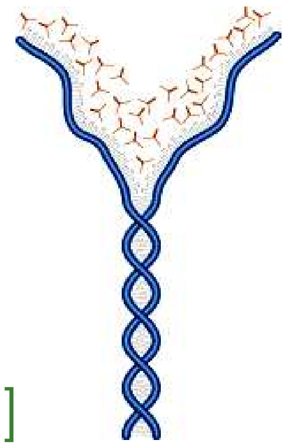
Outline: Current fluctuations and dynamical phase transitions

- Introduction:
  - Current fluctuations in *non-equilibrium* statistical physics
  - Large deviation framework

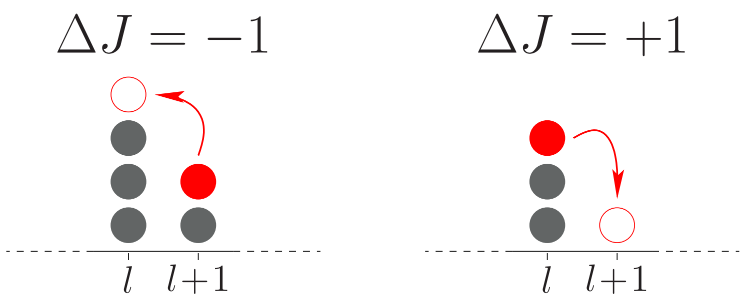


- Recent results:
  1. **Reset processes**
     [RJH & H. Touchette, J. Phys. A: Math. Theor. 50, 10LT01 (2017)]
     ...with a connection to an *equilibrium* model of DNA denaturation
  2. **Hidden states**
     [M. Cavallaro, R. J. Mondragón & RJH, Phys. Rev. E 92, 022137 (2015)]
     ...and temporal correlations in the zero-range process

- Perspective on dynamical phase transitions

Introduction: current fluctuations, large deviations

- Interested in *distributions* of currents in stochastic particle systems ("microscopic" picture)
- E.g., consider integrated particle current across bond



- *Non-equilibrium* (broken detailed balance) characterized by non-zero mean currents
- In Markov processes distribution of J_t/t generically obeys *large deviation principle*:

$$\text{Prob}(J_t/t = j) \sim e^{-I(j)t}$$

  – *Rate function* $I(j)$ quantifies asymptotic probability of rare fluctuations

$$I(j) = -\lim_{t\to\infty} \frac{1}{t} \ln \text{Prob}(J_t/t = j)$$

Introduction: current fluctuations, large deviations

- Alternatively look at long-time behaviour of generating function

$$G(k,t) = \langle e^{kJ_t} \rangle \sim e^{\lambda(k)t}$$

  - Exponent is *scaled cumulant generating function* (SCGF)

$$\lambda(k) = \lim_{t\to\infty} \frac{1}{t} \ln G(k,t)$$

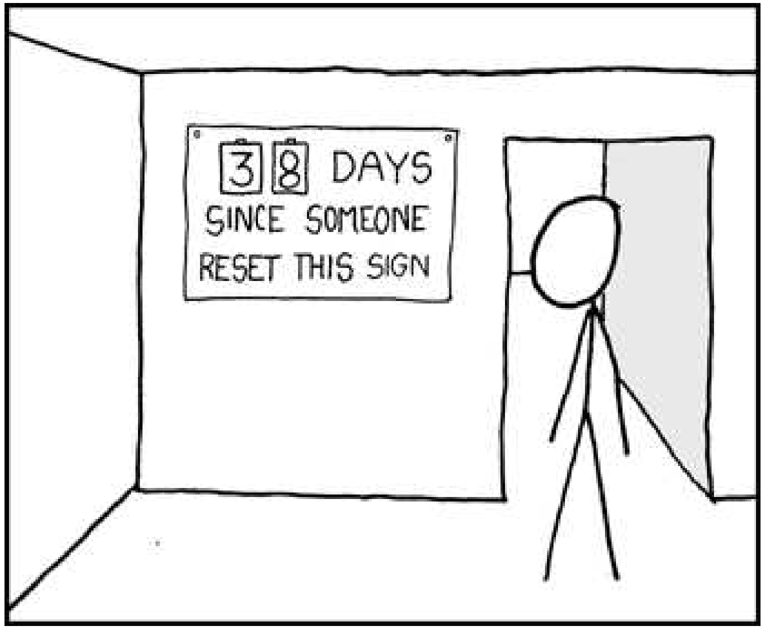
- Insight into structure of non-equilibrium statistical mechanics:
  - If $\lambda(k)$ is differentiable then $I(j)$ is its Legendre-Fenchel (LF) transform

$$I(j) = \max_k \{kj - \lambda(k)\}$$

  - If $\lambda(k)$ is non-differentiable then LF transform gives only convex hull
  - In general, non-analytic points in $\lambda(k)$ correspond to *dynamical phase transitions*...

- *What type of models show dynamical phase transitions?*

1. Reset processes



[http://xkcd.com/363/]

- Interest in processes involving stochastic resets to a fixed state [Evans, Majumdar, Eule, Metzger, Meylahn, Sabhapandit etc.]
  - Random search reinitialized to its starting position
  - Population catastrophes, buffer clearing, attachment of molecular motors

- *What about current fluctuations in particle systems with reset...?*

Framework

- Markov chain X_i evolving in *discrete* time
  (possibly with weakly time-dependent transition probabilities)
- Assume original process has current generating function

$$G_0(k, n) = \langle e^{kJ_n} \rangle_0 \sim e^{\lambda_0(k)n},$$

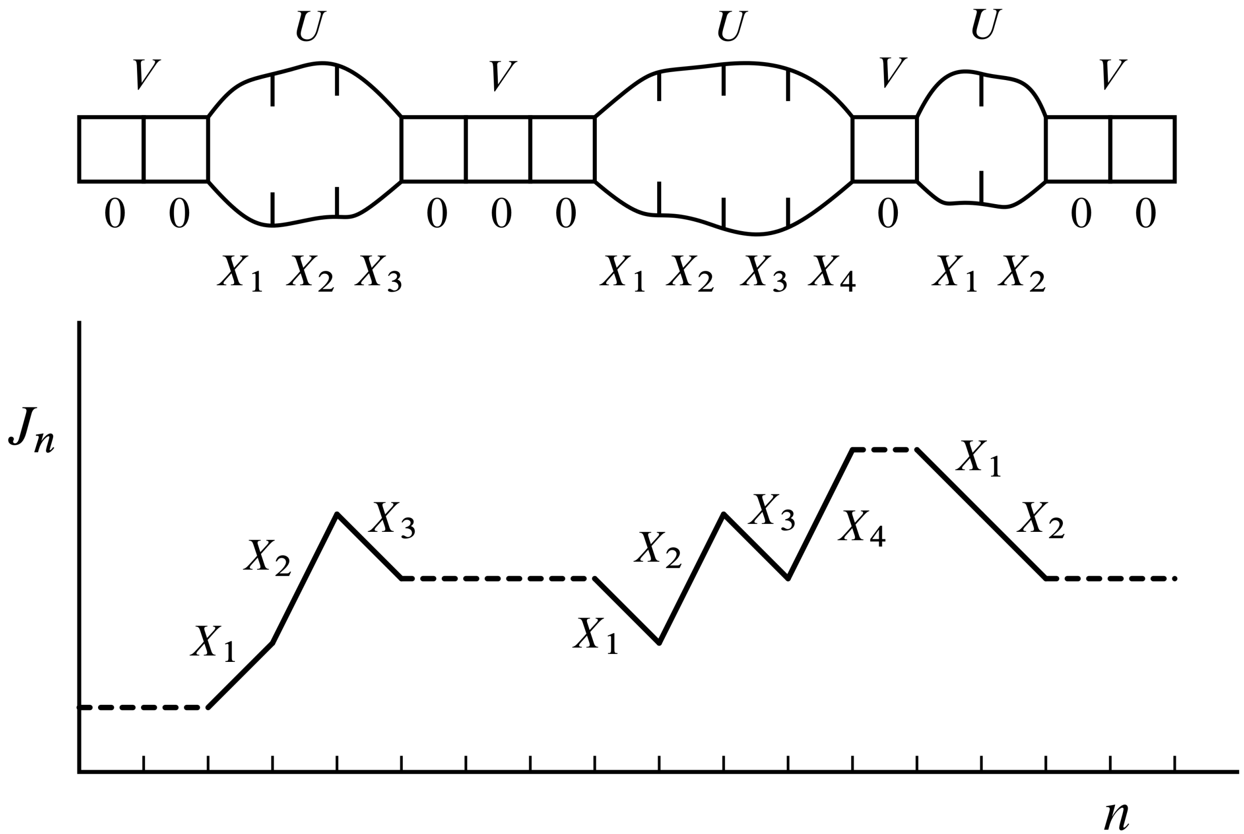
  and rate function

$$I_0(j) = \max_k \{kj - \lambda_0(k)\}$$

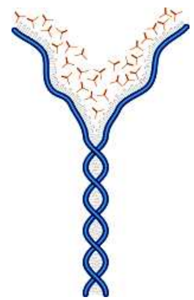
- Now add reset s.t. at each time step
  – Probability r: reset occurs, no current
  – Probability $1 - r$: process evolves, current incremented
- What is the generating function $G_r(k, n)$ with reset?
  – Do finite-time contributions to $G_0(k, n)$ play a role?
  – *Is there a phase transition to a regime where current fluctuations are optimally realised by trajectories involving no reset?*

Mapping to DNA denaturation

- Generating function maps to partition function in PS model [Poland & Scheraga '66]

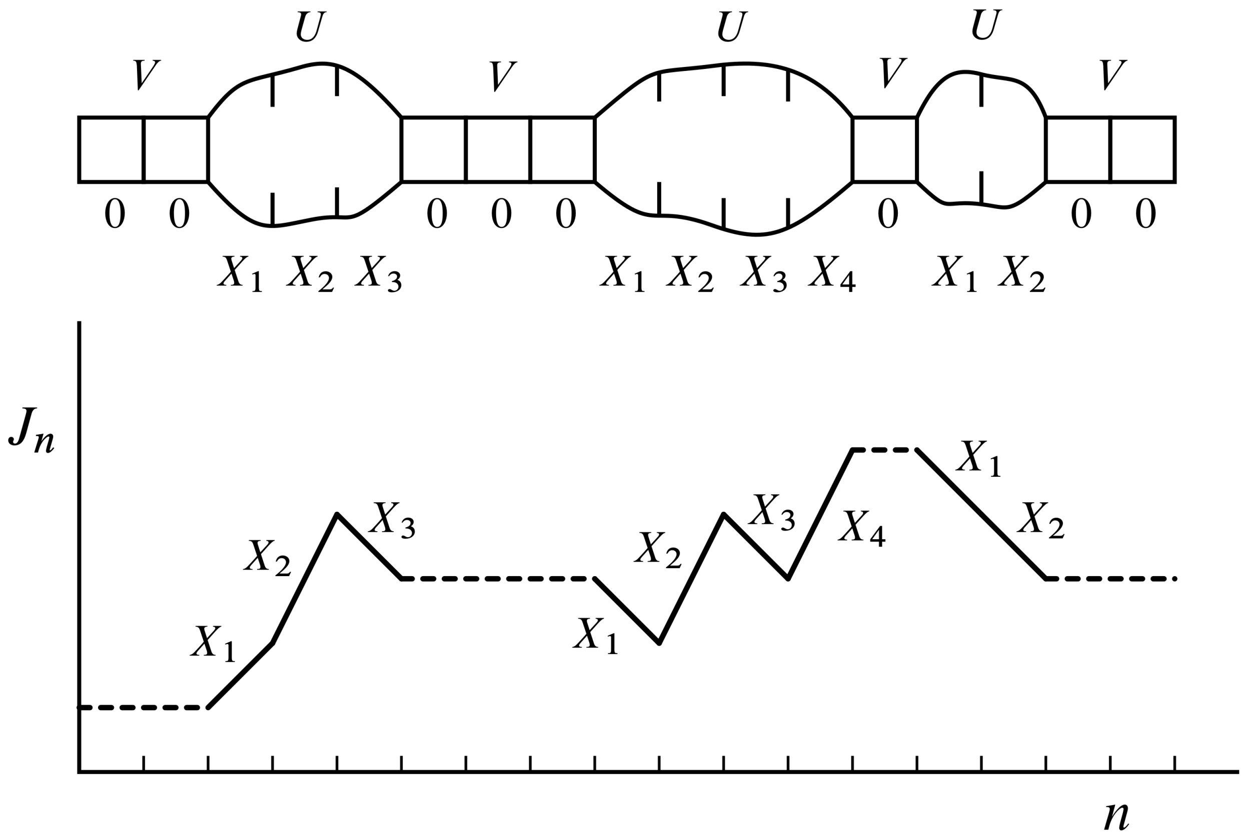


  – PS model: phase transitions as function of temperature,
  order parameter – fraction of bound monomers
  – Our model: phase transitions as function of conjugate parameter k,
  order parameter – fraction of steps with reset



Mapping to DNA denaturation

- Generating function maps to partition function in PS model [Poland & Scheraga '66]



– Current generating function for "loop" of n consecutive steps without reset...

$$U(k,n) = (1-r)^n G_0(k,n)$$

– ...and that for a period of n consecutive reset steps

$$V(k,n) = r^n e^0 = r^n$$

Calculation of generating function

- As in PS model, convenient to consider discrete-Laplace transforms:

$$\tilde{G}_r(k,z) = \sum_{n=1}^{\infty} G_r(k,n) z^{-n},$$

$$\tilde{U}(k,z) = \sum_{n=1}^{\infty} U(k,n) z^{-n},$$

$$\tilde{V}(k,z) = \sum_{n=1}^{\infty} V(k,n) z^{-n} = \frac{r}{z-r}$$

  where total number of steps now fluctuates

- Alternating segments of consecutive no-reset steps and consecutive reset steps; geometric sum of $\tilde{U}\tilde{V}$ terms:

$$\tilde{G}_r(k,z) = \frac{\tilde{U}(k,z) + \tilde{V}(k,z) + 2\tilde{U}(k,z)\tilde{V}(k,z)}{1 - \tilde{U}(k,z)\tilde{V}(k,z)}$$

- SCGF $\lambda_r(k)$ determined by largest real z at which $\tilde{G}_r$ diverges...

Long-time behaviour

$$\tilde{G}_r(k,z) = \frac{\tilde{U}(k,z) + \tilde{V}(k,z) + 2\tilde{U}(k,z)\tilde{V}(k,z)}{1 - \tilde{U}(k,z)\tilde{V}(k,z)}$$

- In absence of phase transition

$$\lambda_r(k) = \ln z^*(k)$$

where z^* is largest real solution of

$$\tilde{U}(k,z)\tilde{V}(k,z) = 1$$

- But if $z^*(k)$ reaches convergence boundary point $z_c(k)$ of $\tilde{U}(k,z)$, crossover to

$$\lambda_r(k) = \ln z_c(k) = \lambda_0(k) + \ln(1-r)$$

Phase transition at k_c to regime where current fluctuations optimally realised by trajectories with no reset events

Nature of phase transition

- Type of phase transition determined by behaviour of $\tilde{U}(k,z)$ in neighbourhood of z_c – depends on *subleading* (power-law) terms in the long-time limit of $U(k,n)$:

$$U(k,n) \sim \frac{(1-r)^n e^{n\lambda_0(k)}}{n^{c(k)}}$$

1. For $1 < c(k_c) \leq 2$:
   - Derivative $\partial\tilde{U}(k,z)/\partial z$ diverges at k_c
   - Continuous dynamical phase transition
   - Average length of a segment without reset is infinite at transition point

2. For $c(k_c) > 2$:
   - Derivative $\partial\tilde{U}(k,z)/\partial z$ converges at k_c
   - First-order dynamical phase transition with "cusp" in $\lambda_r(k)$
   - Average length of a segment without reset is *finite* at the transition point

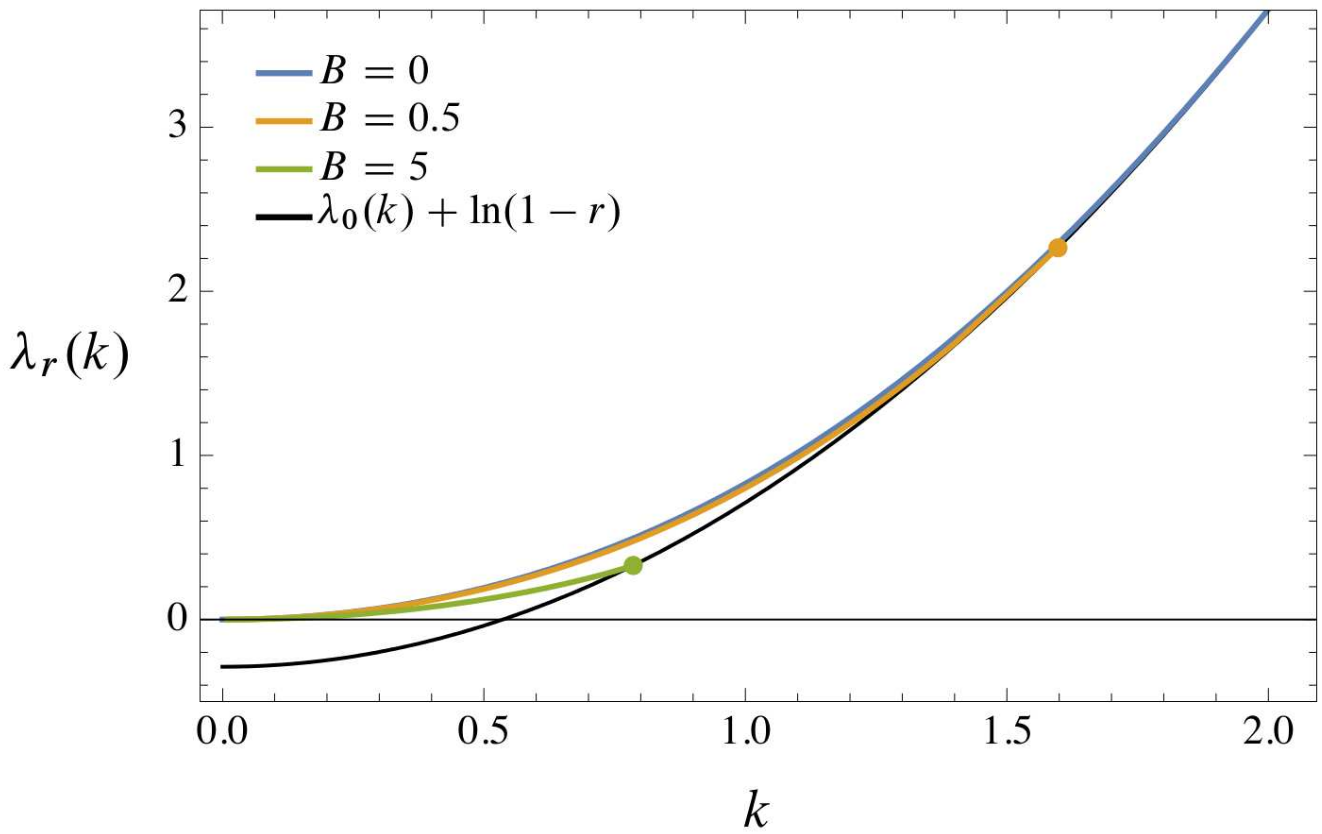
- Same classification as PS model although with added subtlety that c is function of k

Example 1: Gaussian random walk with varying variance

- ith step after reset drawn from Gaussian with mean 0 and variance $2[1 - B/(i + d)]$
- Generating function for n consecutive steps without reset

$$U(k, n) = (1 - r)^n \exp\left[nk^2 - Bk^2(H_{n+d} - H_d)\right] \sim \frac{(1 - r)^n e^{nk^2}}{n^{Bk^2}}$$

- $d = 10$, SCGF $\lambda_r(j)$:



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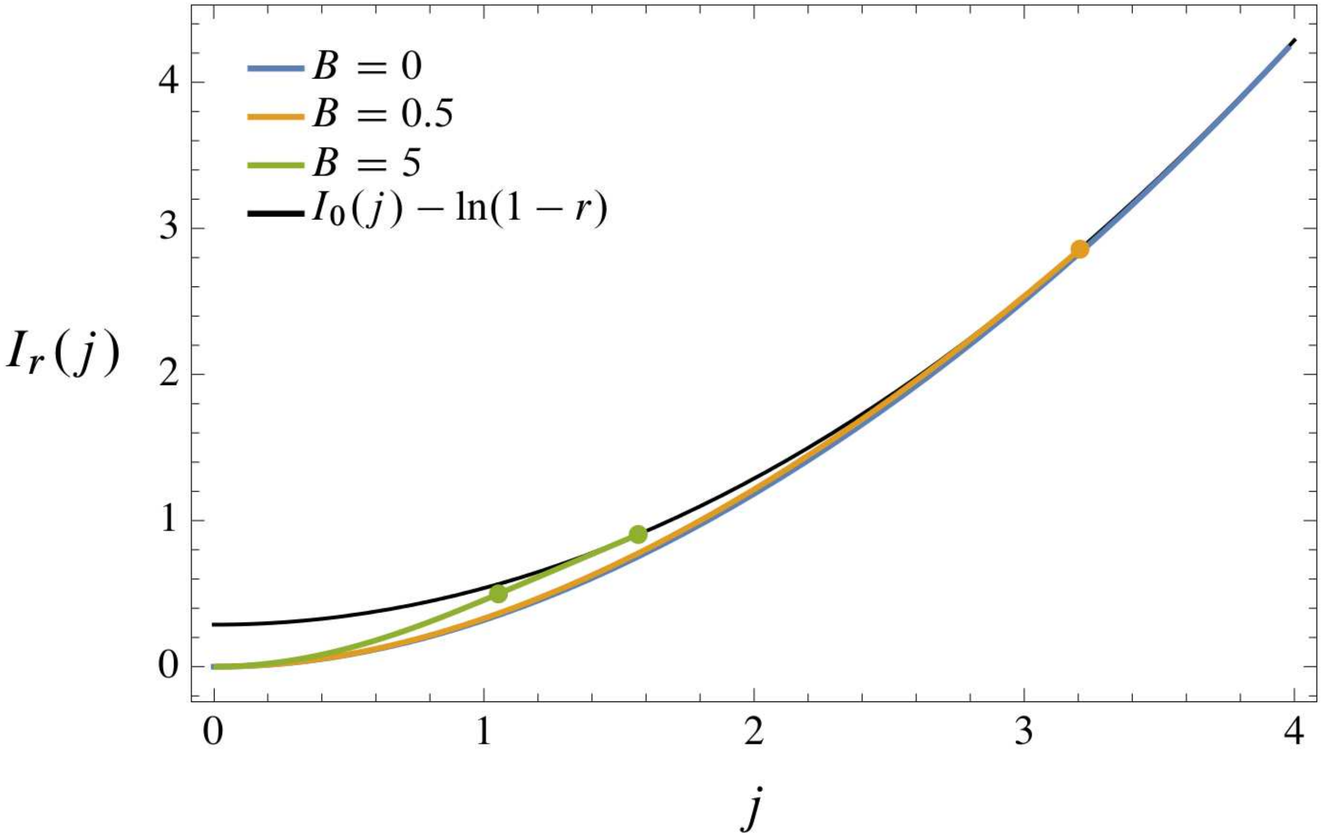
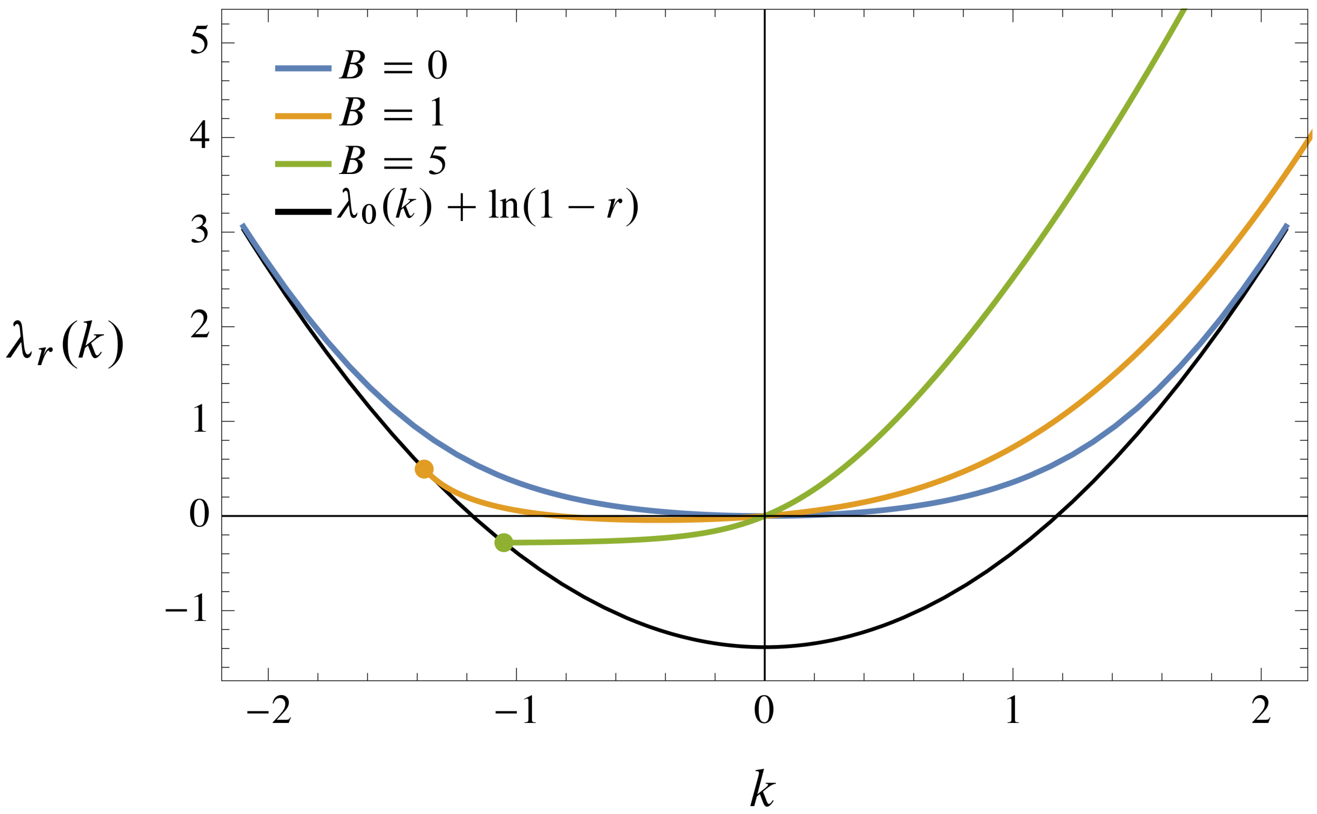
- $d = 10$, rate function $I_r(j)$:



Example 2: Gaussian random walk with decaying mean

- ith step after reset now Gaussian with mean B/i and variance 2:

$$U(k,n) \sim \frac{(1-r)^n e^{nk^2}}{n^{-Bk}}$$



- Addition of reset breaks the symmetry, bringing a non-zero (positive) mean current
- Similar analysis for random walk in discrete space

Extensions/Variations

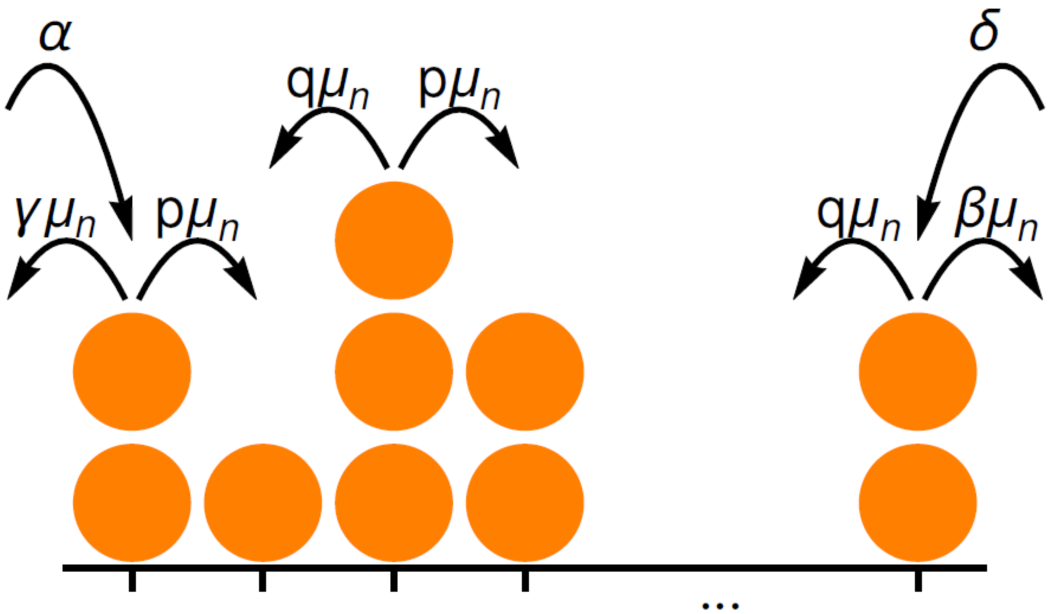
- Can calculate joint rate function for current J_n and number R_n of resets
  - Value of R_n/n minimizing this rate function for a given J_n/n corresponds to expected fraction of resets
  - At dynamical phase transition point, expected fraction is $\left(1 - r \left.\frac{\partial \tilde{U}}{\partial z}\right|_{z=z^*(k)}\right)^{-1}$
  - Could investigate statistics of "efficiency" ratio J_n/R_n... [cf. Verley et al. '14]

- Similar analysis is possible for time-homogenous dynamics with non-constant reset probabilities
  - Reset probabilities depending on time since the last reset [cf. Pal et al. '16]
    (subtlety: different statistics for final unterminated no-reset "loop")
  - Reset probabilities drawn from a given distribution and then constant until a reset actually occurs, spirit of "superstatistics" [cf. Beck et al. '04]

- **Power-law corrections in generating functions for no-reset segments <u>or</u> reset segments can lead to dynamical phase transitions**

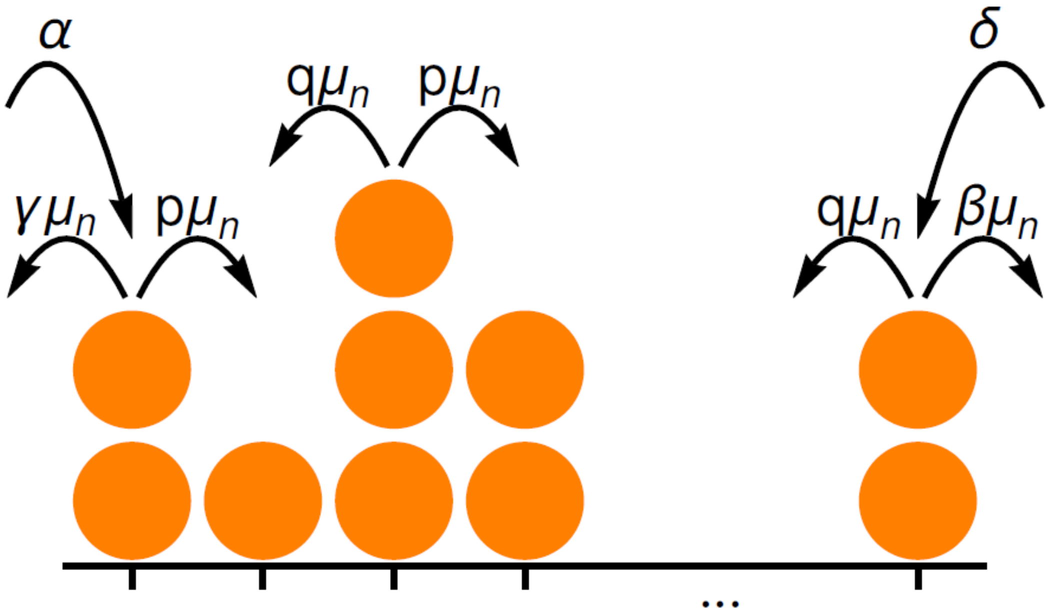
2. **Hidden states** in the zero-range process

- Zero-range process (ZRP) is a paradigmatic interacting particle model (discrete space, continuous time)
- We study current fluctuations in L-site open-boundary ZRP [Levine et al. '05]:



- Hopping rates μ_n depend only on occupation of departure site
- Driving due to asymmetric jump rates and boundaries
- If μ_n bounded, strong boundary driving can lead to boundary condensation... ...and dynamical phase transitions for currents even with a well-defined steady state

Markovian ZRP

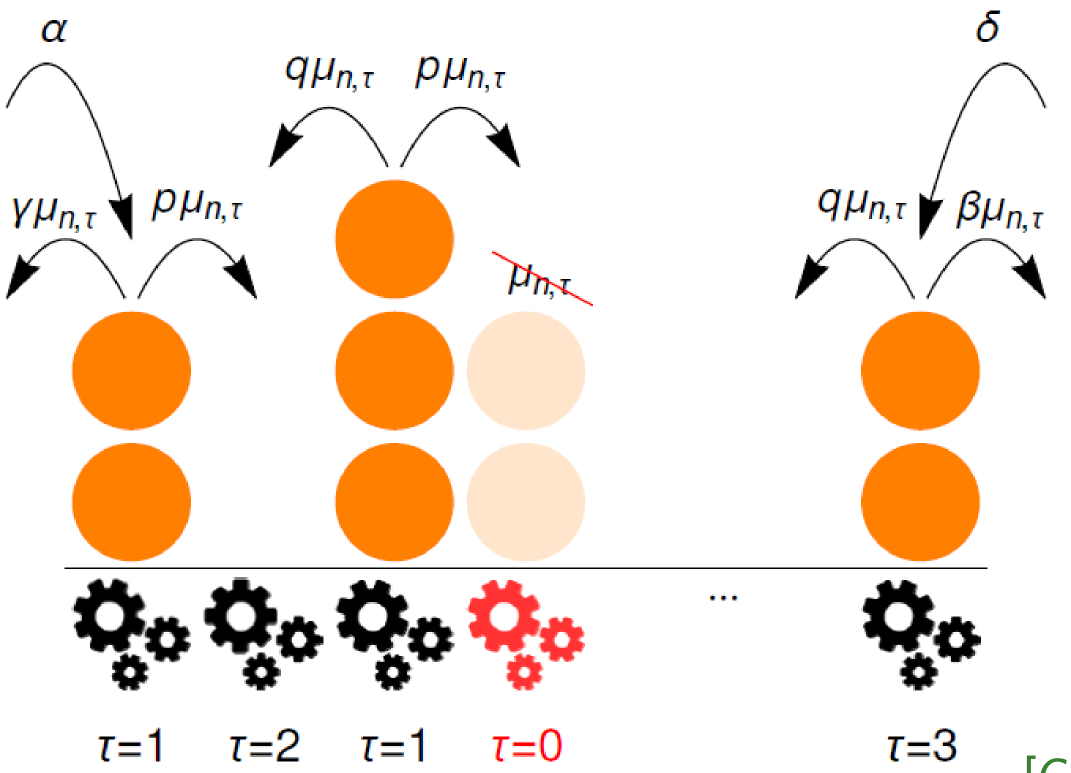


- If $\mu_n \to \infty$ as $n \to \infty$, no dynamical phase transitions in Markovian case
- $\lambda(k)$ obtained as principal eigenvalue of a tilted generator; $I(j)$ via LF transform:

$$I(j) = \frac{(p-q)[\alpha\beta(p/q)^{L-1} + \gamma\delta]}{\gamma(p-q-\beta) + \beta(p-q+\gamma)(p/q)^{L-1}} - \sqrt{j^2 + \frac{4\alpha\beta\gamma\delta(p/q)^{L-1}(p-q)^2}{[\gamma(p-q-\beta)+\beta(p-q+\gamma)(p/q)^{L-1}]^2}}$$
$$- j\ln\left[\frac{2\alpha\beta(p/q)^{L-1}(p-q)}{\gamma(p-q-\beta)+\beta(p-q+\gamma)(p/q)^{L-1}}\right] + j\ln\left[j + \sqrt{j^2 + \frac{4\alpha\beta\gamma\delta(p/q)^{L-1}(p-q)^2}{[\gamma(p-q-\beta)+\beta(p-q+\gamma)(p/q)^{L-1}]^2}}\right].$$

[RJH, Rákos and Schütz, '05]

Non-Markovian "on-off" ZRP



[Cavallaro et al., '15]

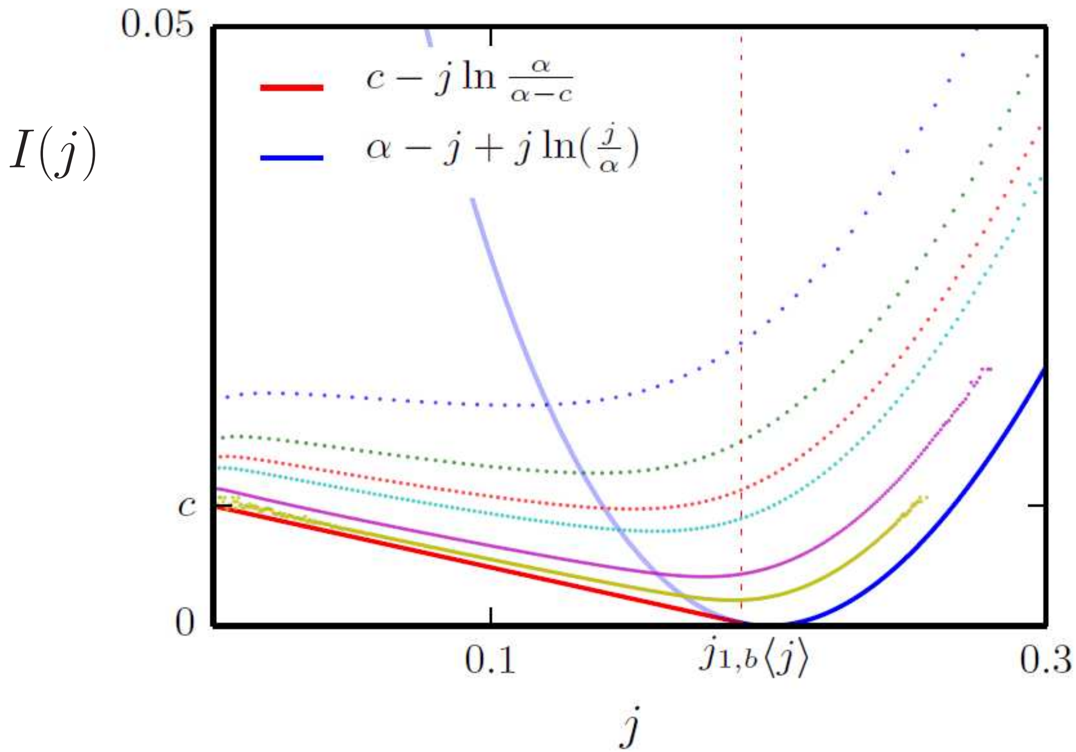
- Each site has an additional hidden variable τ; particles only hop during an on phase
- The site turns off at particle arrival and turns on with rate c
- This mechanism favours congestion of particles on a site
- Open boundary version of model introduced in [Hirschberg et al., '09]

Memory-induced dynamical phase transition

- On-off ZRP has complicated structure of current fluctuations...
- ...but some analytical progress is possible working in extended state space

[Cavallaro et al., '15]

- **Evidence for first-order dynamical phase transition, even for μ_n unbounded**
- Example: output current in totally asymmetric single-site case $(\mu_n = n)$



Perspective: Current fluctuations and dynamical phase transitions

- Current fluctuations have rich structure, sensitive to the underlying dynamics...

- Temporal correlations can lead to dynamical phase transitions
  - Sub-exponential terms in compound process "amplified" by reset
  - Interplay of hidden variables with infinite state space

Nothing new under the sun?

- Insight into non-equilibrium problems from old equilibrium models
  - Long-time limit plays rôle of thermodynamic limit
  - Need long-range correlations or two diverging dimensions for phase transition?

[cf. Tsobgni Nyawo & Touchette '16]

- Large deviation theory provides framework for establishing rigorous conditions for dynamical phase transitions...

References

- *Phase transitions in large deviations of reset processes*,
  R. J. Harris and H. Touchette, J. Phys. A: Math. Theor. 50, 10LT01 (2017).

- *Current fluctuations in the zero-range process with open boundaries*,
  R. J. Harris, A. Rákos and G. M. Schütz, J. Stat. Mech., P08003 (2005).

- *Temporally correlated zero-range process with open boundaries: steady state and fluctuations*,
  M. Cavallaro, R. J. Mondragón and R. J. Harris, Phys. Rev. E 92, 022137 (2015).