Infinite-covariant density for fat-tailed systems

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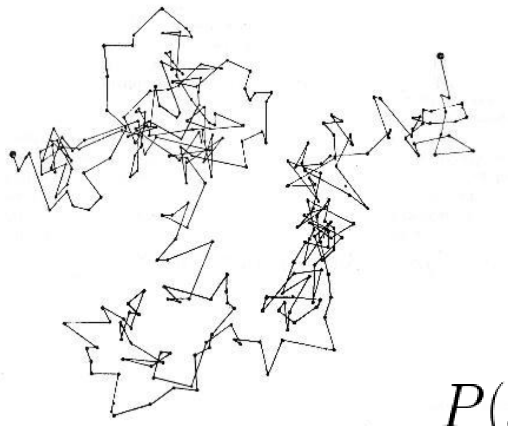
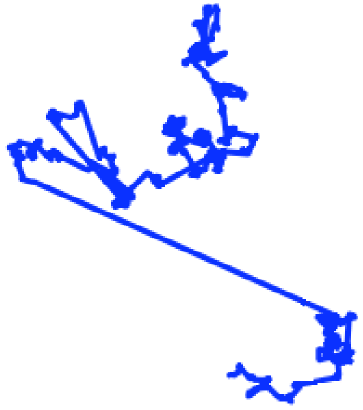
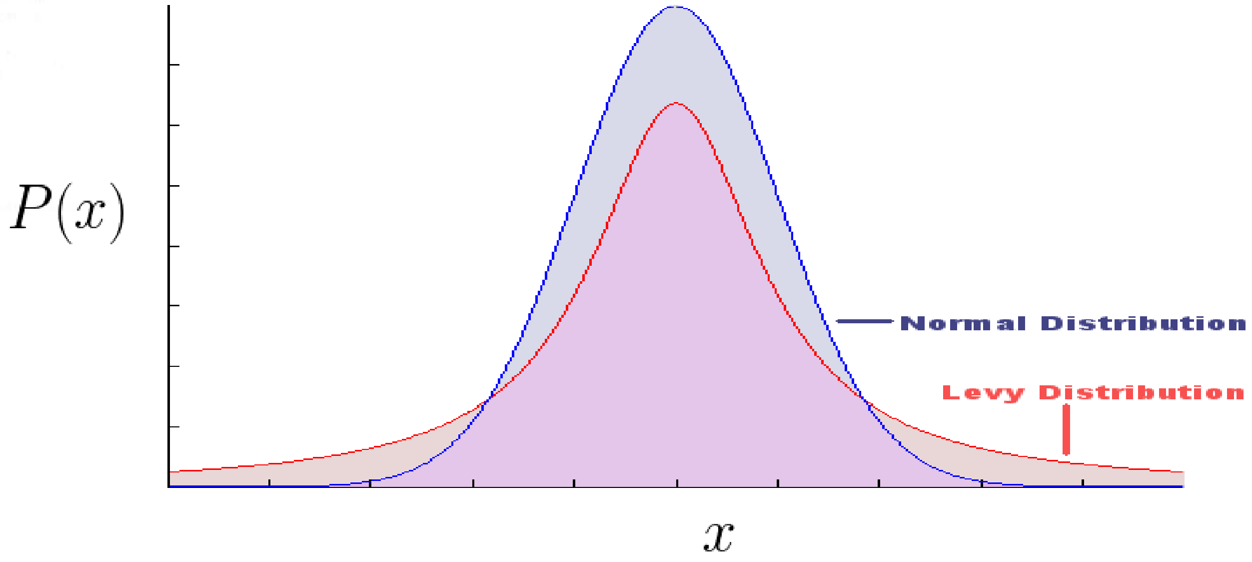
Large Deviation Theory in Statistical Physics: Recent Advances and Future Challenges
14 Aug, 2017 - 13 Oct, 2017

Brownian motion



Lévy flight

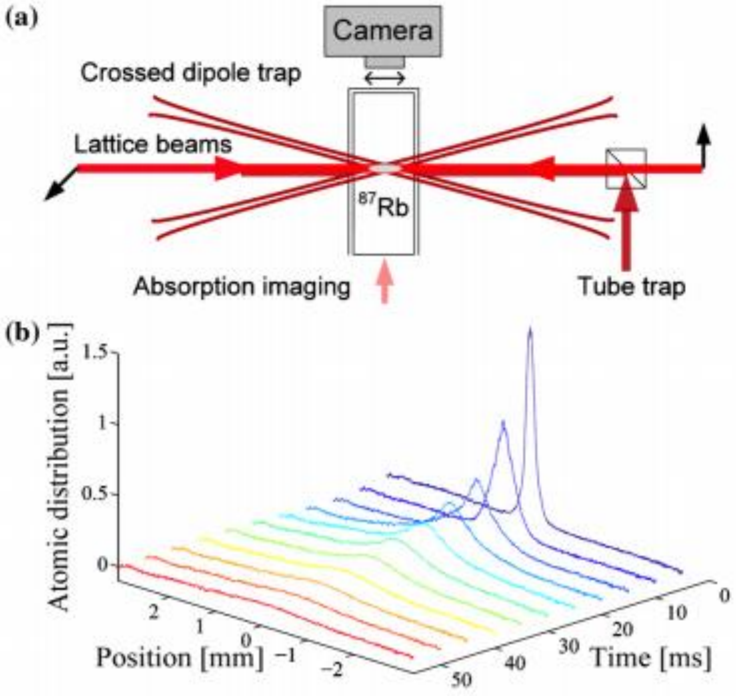




The infinite-covariant density may describe the large-fluctuations at the tails

Observation of Anomalous Diffusion and Fractional Self-Similarity in One Dimension

Yoav Sagi, Miri Brook, Ido Almog, and Nir Davidson

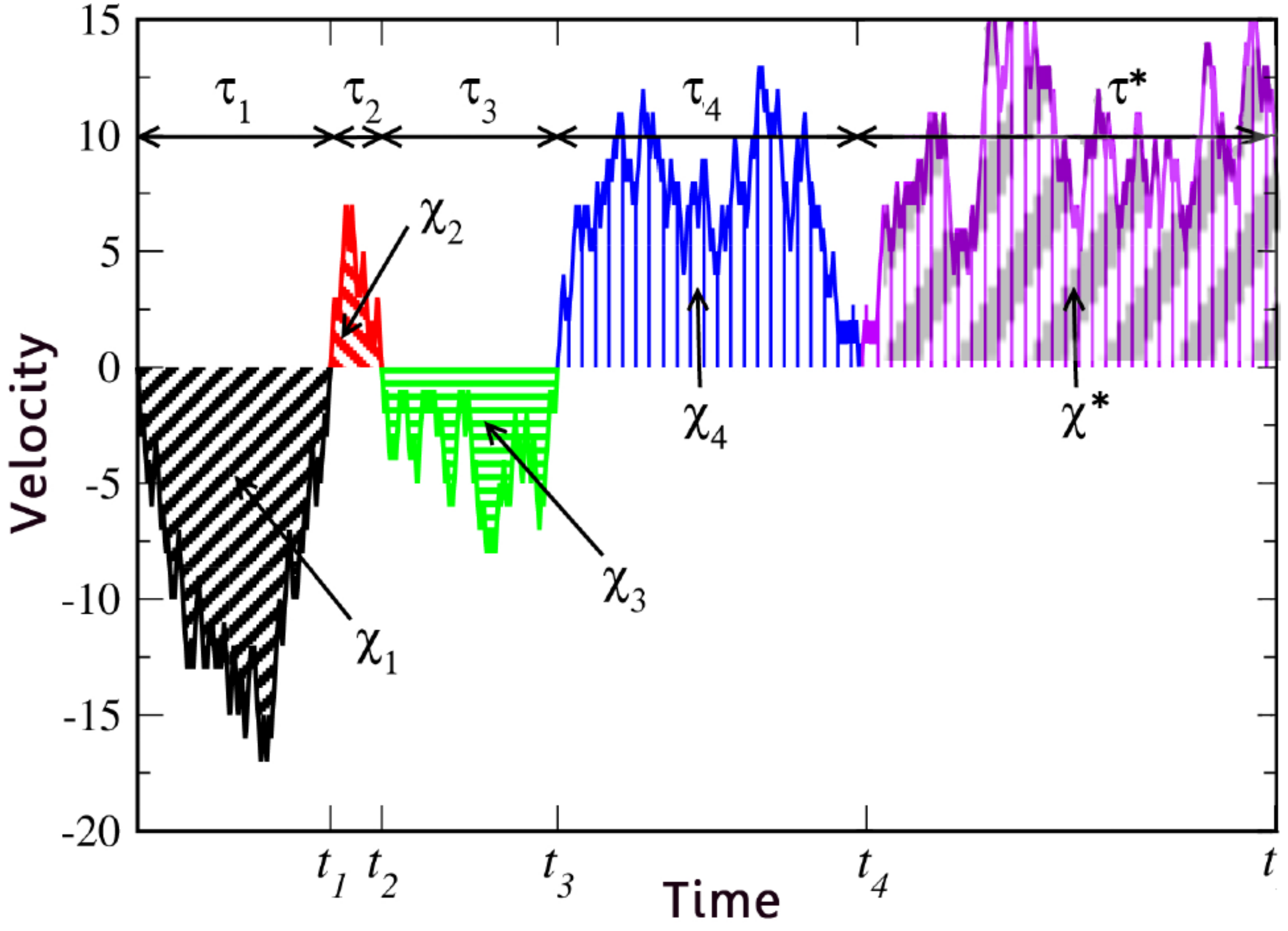


$$\frac{\partial P_t(x,p)}{\partial t} + p\frac{\partial P_t(x,p)}{\partial x} = \left[D\frac{\partial^2}{\partial p^2} - \frac{\partial}{\partial p}F(p)\right] P_t(x,p)$$

$$\frac{dp}{dt} = F(p) + \sqrt{2\ D}\ \Gamma(t) \qquad \dot{x}(t) = p(t)/m$$

$$F(p) = -\frac{p}{1+p^2} \quad \Rightarrow \quad \propto -1/p$$

At large p



$$t = \sum_{i=1}^{N} \tau_i + \tau^*$$

$$x(t) = \sum_{i=1}^{N} \chi_i + \chi^*$$

$$\chi_i = \int_0^{\tau_i} v\left(t'\right) dt'$$

The Moment-Generating function:

$$\hat{P}_t(k) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \langle (x(t))^n \rangle \, k^n$$

Inverse Fourier-transform should yield the probability density:

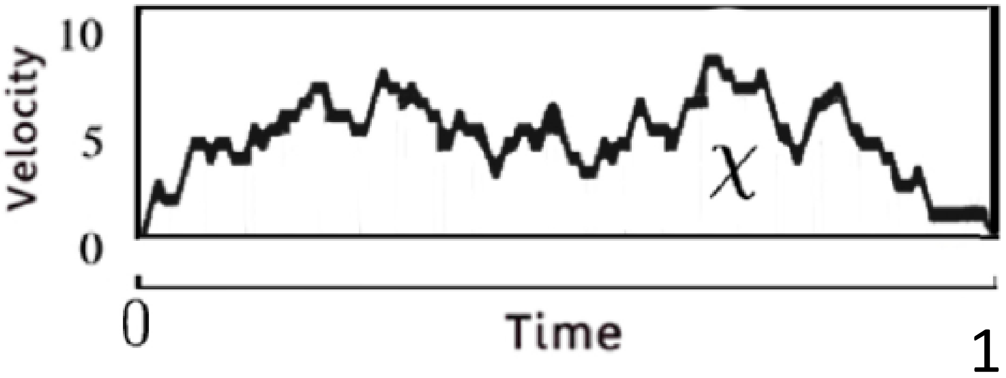
$$P_t(x) = FT^{-1}[\hat{P}_t(k)]_{k \to x}$$

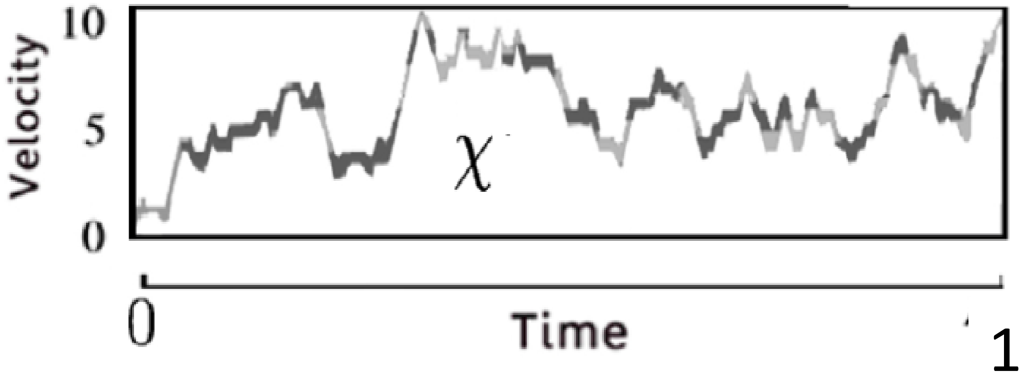
$$\nu = \frac{1+D}{3D}$$

$$\nu > 2/3$$

The moments at the long time, large x, limit:

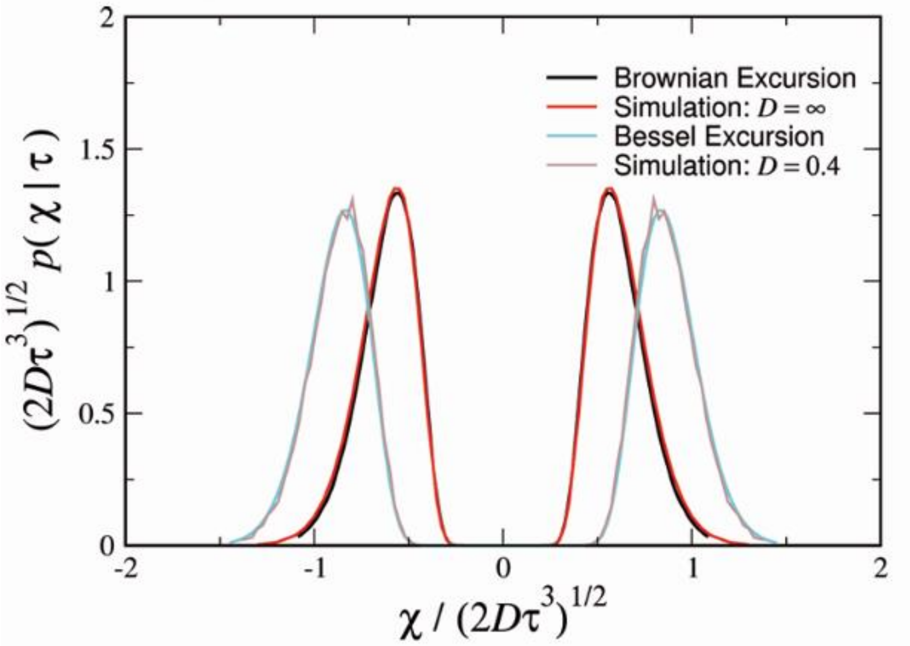
$$\left\langle x^{2m}(t) \right\rangle \sim ct^{3m-3\nu/2+1} \left[\frac{\langle \chi^{2m} \rangle_E}{|(3m-3\nu/2)(3m-3\nu/2+1)|} \quad \text{(+ meander part)} \right]$$





Excursions:

$$\Phi(\chi,\tau) = \frac{g(\tau)}{\tau^{3/2}} B_E\left(\chi/\tau^{3/2}\right)$$



$$\hat{P}(k,u) = \frac{\hat{\Psi}(k,u)}{1+\hat{\Phi}(k,u)}$$

$$\hat{P}(k,u) = \int_0^\infty dt\,\exp(-ut)\int_{-\infty}^{\infty} dx P(x,t)\exp(ikx)$$

The ``Generating function” of the asymptotic moments:

$$P_t^A(k) = 1+ct^{-3\nu/2+1}\sum_{m=1}^{\infty}\frac{(-1)^m(kt^{3/2})^{2m}}{(2m)!}\left[\int_{-\infty}^{\infty}\chi^{2m}B_E(\chi)\mathrm{d}\chi\left(\frac{1}{3m-3\nu/2}-\frac{1}{3m-3\nu/2+1}\right)\right]$$

(+ meander part)

Taylor expansion:

$$\sum_{m=1}^{\infty} \frac{(-1)^m (k v_{3/2} t^{3/2})^{2m}}{(2m)!(3m - 3\nu/2)} = \int_0^1 \frac{\cos\left(\omega^{3/2} y\right) - 1}{\omega^{3\nu/2+1}} \mathrm{d}\omega$$

Yields

$$\hat{P}_t^A(k) = 1 + ct^{1-3\nu/2} \int_{-\infty}^{\infty} \mathrm{d}\chi \int_0^1 \mathrm{d}\omega \left[\cos\left(\omega^{3/2} k \chi t^{3/2}\right) - 1\right] \frac{(1-\omega) B_E(\chi)}{\omega^{3\nu/2+1}}$$

(**Recall!** $P_t(x) = FT^{-1}[\hat{P}_t(k)]_{k \to x}$)

- Use the long time asymptotic moments to get the ``asymptotic generating function’’

- Perform the inverse Fourier-transform, to get the long-time, large x limit shape of the probability density

After Inverse-Fourier transform:

$$P_t^A(x) \sim$$

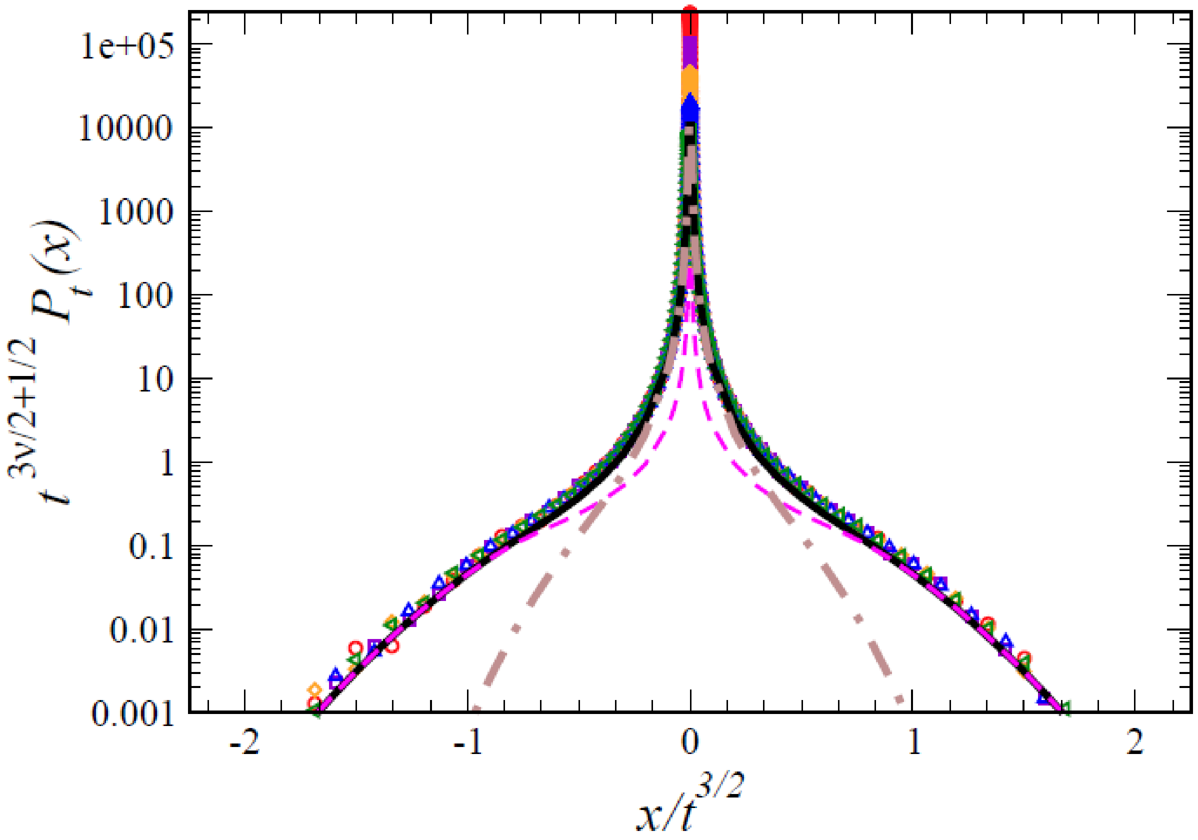
$$\frac{ct^{-3\nu/2+1}}{2} \int_{-\infty}^{\infty} \mathrm{d}\chi \int_0^1 \mathrm{d}\omega \left[\delta\left(x - \omega^{3/2}\chi t^{3/2}\right) + \delta\left(x + \omega^{3/2}\chi t^{3/2}\right)\right] \left[\frac{B_E(\chi)}{\omega^{3\nu/2+1}} + 2\frac{B_M(\chi) - (3\nu/2)B_E(\chi)}{3\nu\omega^{3\nu/2}}\right]$$

The Infinite-Covariant Density:

$$\mathcal{I}(z) = \frac{2c}{3}\frac{1}{|z|^{\nu+1}}\left[\int_{|z|}^{\infty} B_E(\chi)|\chi|^{\nu}\mathrm{d}\chi + |z|^{2/3}\int_{|z|}^{\infty}\left(\frac{2}{3\nu}B_M(\chi) - B_E(\chi)\right)|\chi|^{\nu-2/3}\mathrm{d}\chi\right]$$

$\nu > 2/3$ $\qquad$ $\mathcal{I}(z) = \lim_{t\to\infty} t^{3\nu/2+1/2}P_t(x)$ $\qquad$ $z = x/t^{3/2}$

Not dependent directly on time… And not normalized!



$$\mathcal{I}(z) = \lim_{t\to\infty} t^{3\nu/2+1/2} P_t(x)$$

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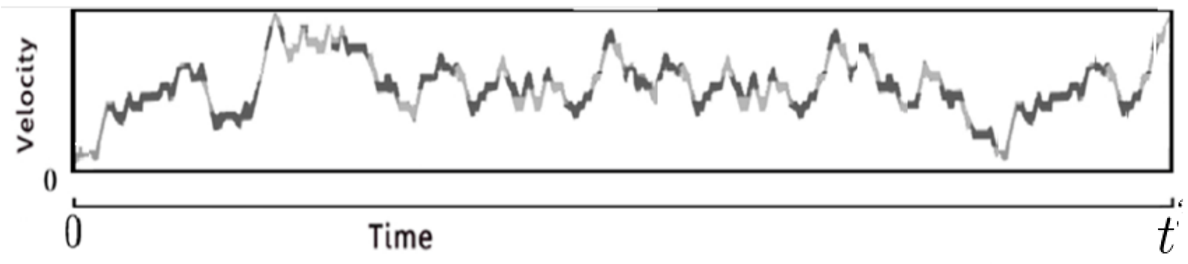


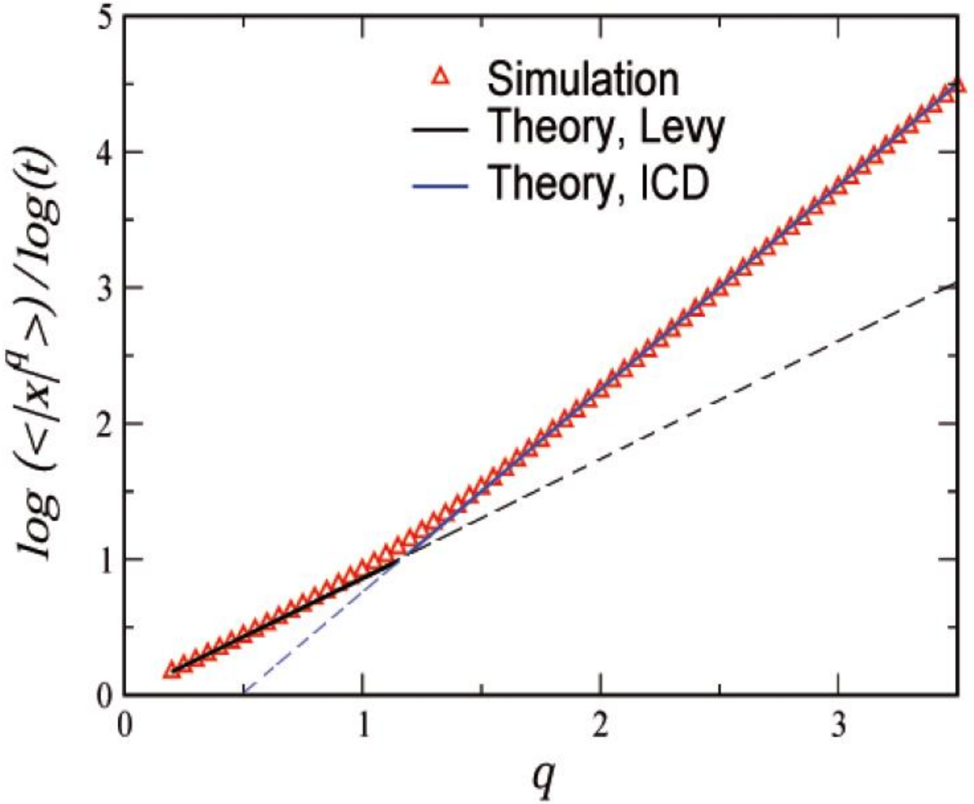
Dominance of rare events

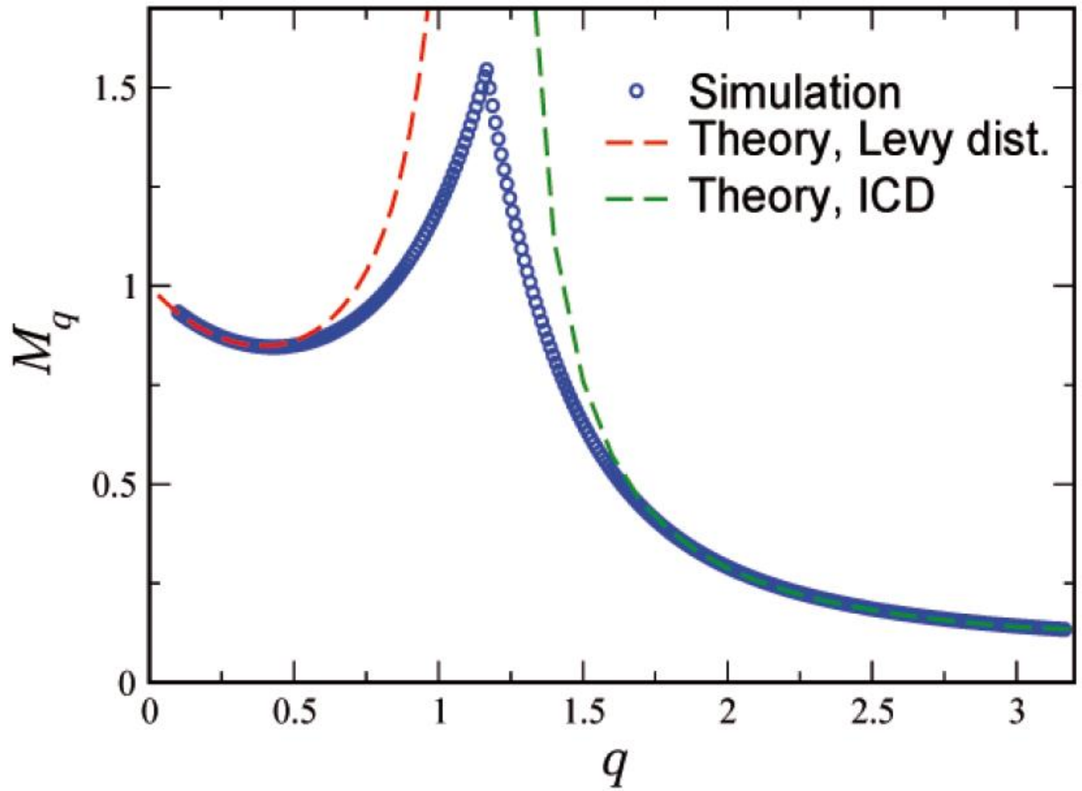
$z \ll 1:$

$$\mathcal{I}(z) \approx \frac{c}{3}\langle|\chi|^\nu\rangle_E |z|^{-\nu-1}$$

$z \gg 1:$

$$\mathcal{I}(z) \approx \frac{4c}{9\nu}|z|^{-\nu-1/3}\int_z^\infty |\chi|^{\nu-2/3} B_M(\chi)\mathrm{d}\chi.$$





$$\langle |x|^q \rangle = M_q t^{\zeta(q)} \qquad \zeta(q) = \begin{cases} q/\nu & ,q < \nu \\ 3q/2 - 3\nu/2 + 1 & ,q > \nu \end{cases}$$

Here

$t=10^5$,

$\nu=1.1667$

Gladly discuss in detail after the talk…

Large deviations theory --> thin tailed systems

Infinite density --> fat-tailed

Large deviations theory --> exponential shape (*rate function)

Infinite density --> power-law regime,

$$\mathcal{I}(\tilde{z}) = \lim_{t\to\infty} t^{\alpha} P_t(x), \qquad \tilde{z} = x/t^{\beta}, \qquad \alpha > \beta \geq 0$$

- Moment-generating function + asymptotic moments
- Time-independent, non-normalizeable
- Bi-scaling

- Phys. Rev. Lett. **118**, 260601 (2017),
- Phys. Rev. X **4**, 021036 (2014),
- EPJ B (soon, hopefully…)

The discussion brings the light!
(Thanks for listening)