The importance of large deviations in non-equilibrium systems

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Chandrasekhar Lecture Series

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Outline

The importance of large deviations in non-equilibrium systems

Large deviations

Density fluctuations in non-equilibrium systems

Current fluctuations

Diffusive systems

Concluding remarks

Reviews

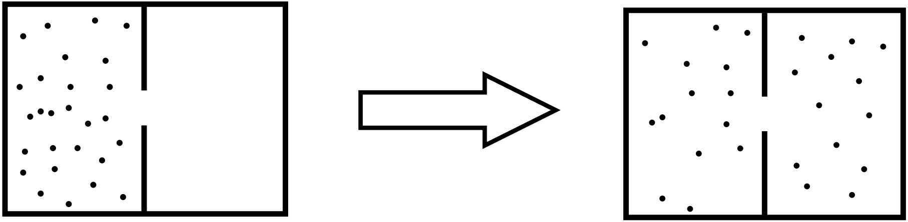
Den Hollander
(2008) *Large deviations*
(Vol. 14). American Mathematical Soc.

Touchette
(2009) *The large deviation approach to statistical mechanics*
Physics Reports

Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim
(2015) *Macroscopic fluctuation theory*
Reviews of Modern Physics

NON EQUILIBRIUM SYSTEMS

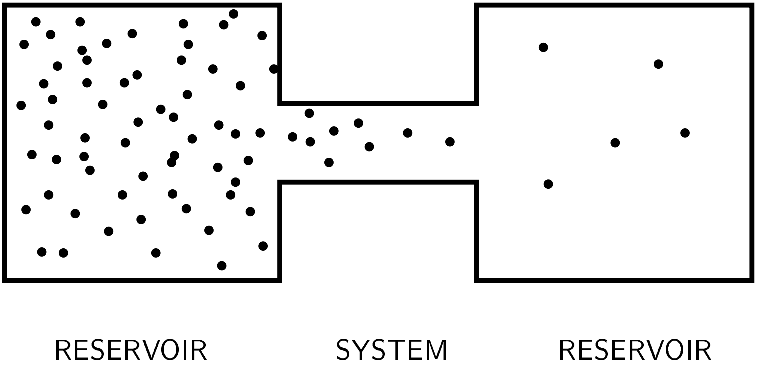
Non equilibrium systems



NON EQUILIBRIUM

EQUILIBRIUM

Non equilibrium steady states

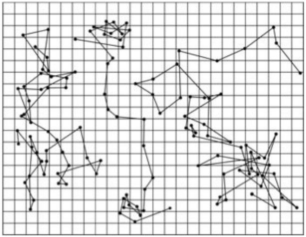


LARGE DEVIATION FUNCTIONS

Random walk - Brownian motion

$$S = x_1 + x_2 + \cdots x_N$$

Albert Einstein



Jean Perrin

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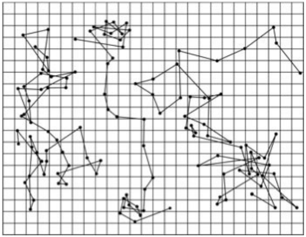

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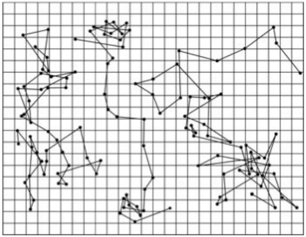

Jean Perrin

1. Law of large numbers

$$S \sim \langle S \rangle = N \langle x \rangle$$

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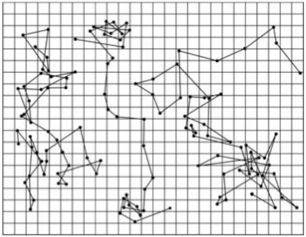

Jean Perrin

1. Law of large numbers

$$S \sim \langle S \rangle = N \langle x \rangle$$

2. Central limit theorem

$$S - \langle S \rangle = \sqrt{N}\,\eta \quad ; \quad P(\eta) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{\eta^2}{2\sigma}\right]$$

Random walk - Brownian motion

$$S = x_1 + x_2 + \cdots x_N$$

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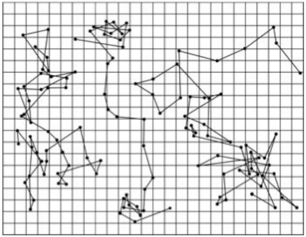

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3. Large deviation function

$$\mathrm{Pro}\,(S = N\ r) \sim \exp[N\ g(r)]$$

Large deviation funtion

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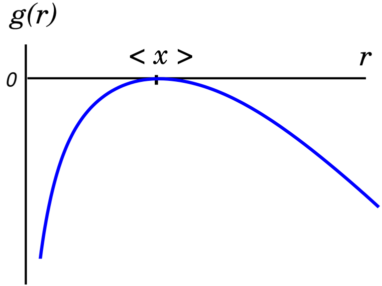
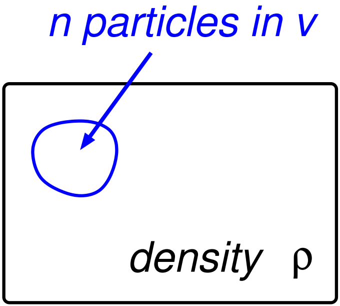


The large deviation function gives all the cumulants of the sum

$$g(r) = -\frac{(r - \langle x \rangle)^2}{2\langle x^2 \rangle_c} + \frac{(r - \langle x \rangle)^3\ \langle x^3 \rangle_c}{6\langle x^2 \rangle_c^3} + \cdots$$

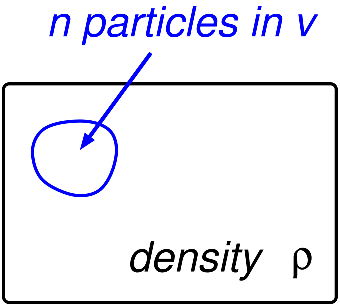
DENSITY FLUCTUATIONS

Density fluctuations



Time to see a 1% fluctuation of the density $\frac{n}{v}$ in a sphere of radius a?

Density fluctuations



Time to see a 1% fluctuation of the density $\frac{n}{v}$ in a sphere of radius a?

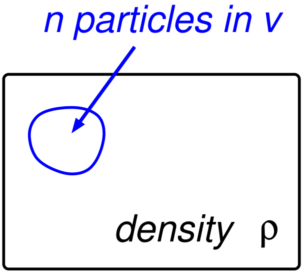
Chandrasekhar 1943:

Radius a	Time
10^{-5} cm	10^{-11} second
2.5×10^{-5} cm	1 second
$3. \times 10^{-5}$ cm	10^{6} seconds
5.10^{-5} cm	10^{68} seconds
1 cm	$10^{10^{14}}$ seconds

Large deviations of the density

1. Law of large number

$$\langle n \rangle = v \ \rho$$



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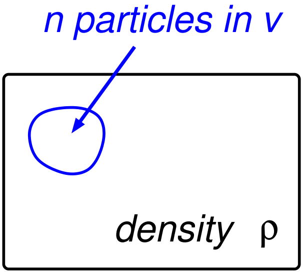

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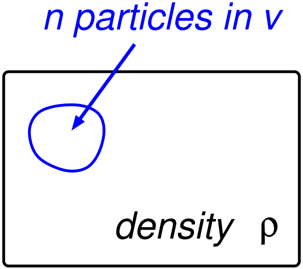
$$\langle n \rangle = v \ \rho$$

2. Fluctuation dissipation

$$\langle n^2 \rangle - \langle n \rangle^2 = v \ kT \ \kappa(\rho)$$ where $\kappa(\rho)$ is the compressibility

$$\boxed{\text{variance} = v \ k \text{ response coefficient}}$$

Large deviations of the density



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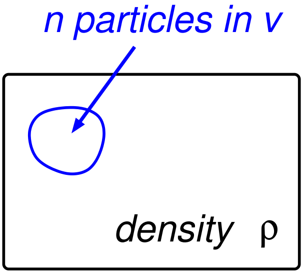
3. Large deviations

Large deviations of the density



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variance $= v\ k$ response coefficient

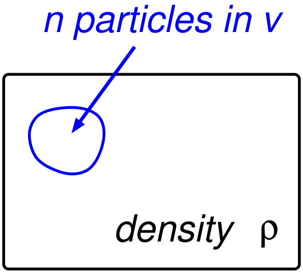
3. Large deviations

$$\mathrm{Pro}(n = v\ r) \sim \exp[v\ g(r)]$$

$$g(r) = -\frac{f(r) - f(\rho) - (r-\rho)f'(\rho)}{kT}$$

where $f(\rho)$ is the free energy per unit volume at density ρ

Large deviations of the density



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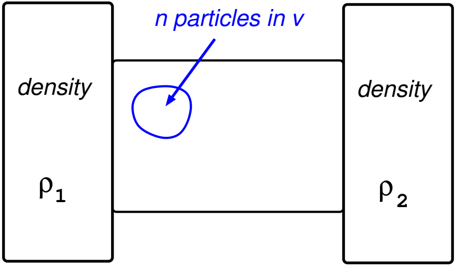
3. Large deviations

$$\text{Pro}(n = v \ r) \sim \exp[v \ g(r)] \qquad ; \qquad g(r) \equiv \text{free energy}$$

$$\text{Pro}(n = v \ r) \simeq \frac{Z_v(n) \ Z_{V-v}(N - N)}{Z_V(N)}$$

Non equilibrium steady state



$$\mathrm{Pro}(n = v\ r) \sim \exp[v\ g(r)]$$

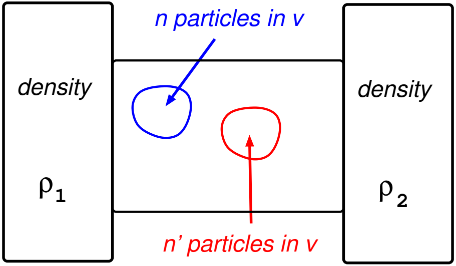
1. $\rho_1 = \rho_2$ (equilibrium)

$$g(r) \equiv \text{free energy}$$

2. $\rho_1 \neq \rho_2$

$$g(r) \equiv ?$$

Non locality



$$\mathrm{Pro}(n = v\ r\ ,\ n' = v\ r') \sim \exp[v\ G(r,r')]$$

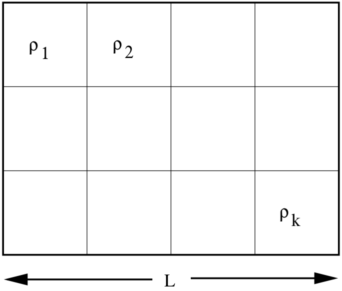
1. $\rho_1 = \rho_2$ (equilibrium)

$$G(r,r') = g(r) + g(r')$$

2. $\rho_1 \neq \rho_2$

$G(r,r')$ is not additive $\Rightarrow$ Long range correlations

Large deviation functional



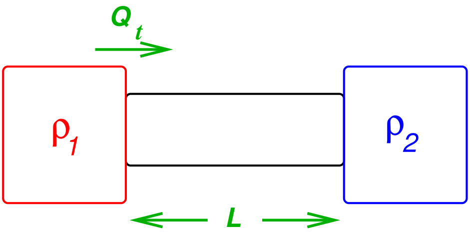
$$\mathrm{Pro}(\rho_1, \ldots \rho_k) \sim \exp\left[-L^d \mathcal{F}(\rho_1, \ldots \rho_k)\right]$$

Large number k of boxes $\vec{r} = L\vec{x}$

$$\boxed{\mathrm{Pro}(\{\rho(\vec{x})\}) \sim \exp\left[-L^d \mathcal{F}(\{\rho(\vec{x})\})\right]}$$

CURRENT FLUCTUATIONS

Current fluctuations and large deviations



1. Law of large number : Second law

2. variance : fluctuation dissipation relation

3. large deviations : fluctuation theorem

Second law

Clausius: "Heat generally cannot flow spontaneously from a material at lower temperature to a material at higher temperature"

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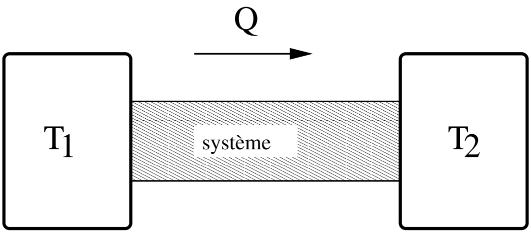
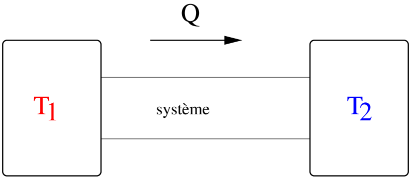


Energy flows from the hot to the cold

$$\boxed{Q > 0 \text{ if } T_1 > T_2}$$

Currents: fluctuation dissipation relation



Q= energy transferred during time t

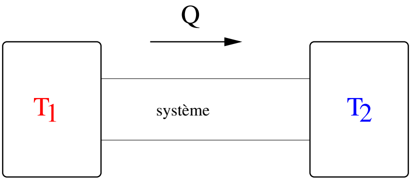
- Current fluctuation at equilibrium ($T_1 = T_2 = T$)

$$\langle Q^2 \rangle = k \times (2DT^2) \times t$$

- Response to a small temperature difference

$$\langle Q \rangle = D \times (T_1 - T_2) \times t$$

Fluctuation theorem (large deviations)

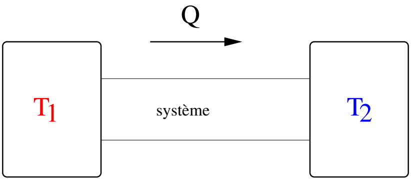


Evans Cohen Morris 1994
Gallavotti Cohen 1995
⋮
Lebowitz Spohn 1999

In the long time limit

$$\frac{\mathrm{Pro}(Q)}{\mathrm{Pro}(-Q)} \sim \exp\left[Q\left(\frac{1}{kT_2} - \frac{1}{kT_1}\right)\right]$$

Fluctuation theorem (large deviations)



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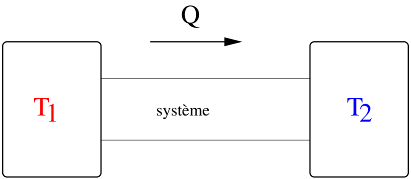
$$\lim_{t\to\infty} \frac{1}{t} \log \frac{\text{Pro}(Q = jt)}{\text{Pro}(Q = -jt)} = j\left(\frac{1}{kT_2} - \frac{1}{kT_1}\right)$$

Fluctuation theorem (large deviations)



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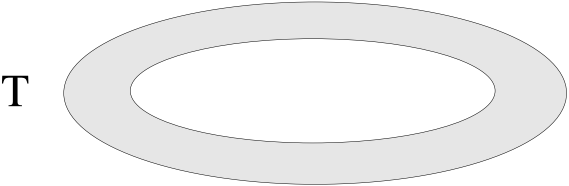
- Probabilistic character of second law
- Universal relation

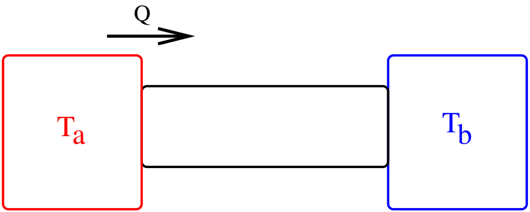
Current fluctuations in non-equilibrium steady states

Current fluctuations in non-equilibrium steady states

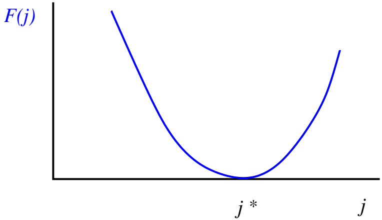
Current fluctuations on a ring



Current fluctuations and large deviations



Q_t Energy transferred during time t

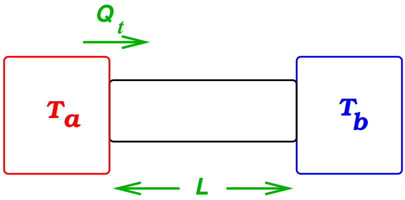


$$\text{Pro}\left(\frac{Q_t}{t} = j\right) \sim \exp[-t \; F(j)]$$

Expansion of $F(j)$ near j^* gives all cumulants of Q_t

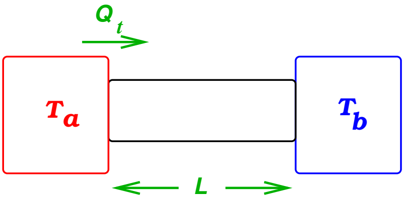
FOURIER's and FICK's LAWS

AVERAGE CURRENT OF **HEAT**



$$\frac{\langle Q_t \rangle}{t} = F(T_a, T_b, L, \text{contacts})$$

AVERAGE CURRENT OF **HEAT**



$$\frac{\langle Q_t \rangle}{t} = F(T_a, T_b, L, \text{contacts})$$

FOURIER's law:

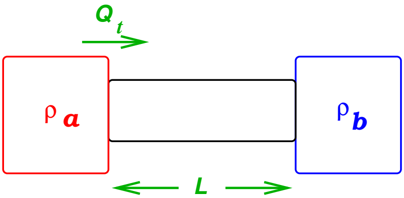
large L

$$\frac{\langle Q_t \rangle}{t} \simeq \frac{G(T_a, T_b)}{L}$$

large L and $T_a - T_b$ small

$$\frac{\langle Q_t \rangle}{t} \simeq D(T) \frac{T_a - T_b}{L} \simeq -D(T) \nabla T$$

AVERAGE CURRENT OF **PARTICLES**



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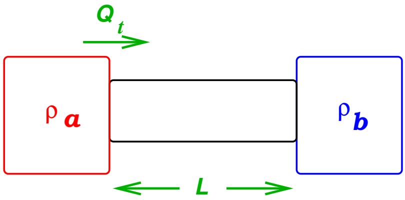


$$\frac{\langle Q_t \rangle}{t} = F(\rho_a, \rho_b, L, \text{contacts})$$

FICK's law:

large L

$$\frac{\langle Q_t \rangle}{t} \simeq \frac{G(\rho_a, \rho_b)}{L}$$

large L and $\rho_a - \rho_b$ small

$$\frac{\langle Q_t \rangle}{t} \simeq D(\rho) \frac{\rho_a - \rho_b}{L} \simeq -D(\rho) \nabla \rho$$

ONE DIMENSIONAL MECHANICAL SYSTEMS

Ideal gas

No Fourier's law

Harmonic chain

$$E = \sum \frac{p_i^2}{2m} + \sum_i g(x_{i+1} - x_i)^2$$

$$\frac{\langle Q_t \rangle}{t} \simeq G(T_a, T_b)$$

ONE DIMENSIONAL MECHANICAL SYSTEMS

Ideal gas

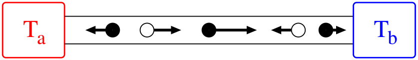
No Fourier's law

Harmonic chain

$$E = \sum \frac{p_i^2}{2m} + \sum_i g(x_{i+1} - x_i)^2 \qquad \frac{\langle Q_t \rangle}{t} \simeq G(T_a, T_b)$$

Hard particle gas

Anomalous Fourier's law



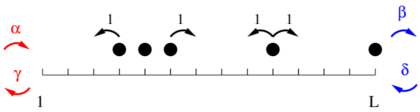
$$\frac{\langle Q_t \rangle}{t} \simeq \frac{G(T_a, T_b)}{L^{1-\alpha}}$$

$.3 \leq \alpha \leq .5$ Delfini Lepri Livi Politi Livi, Spohn, Grassberger, Dhar ...

Anharmonic chain

DIFFUSIVE SYSTEMS: average current

Symmetric exclusion



Fick's law

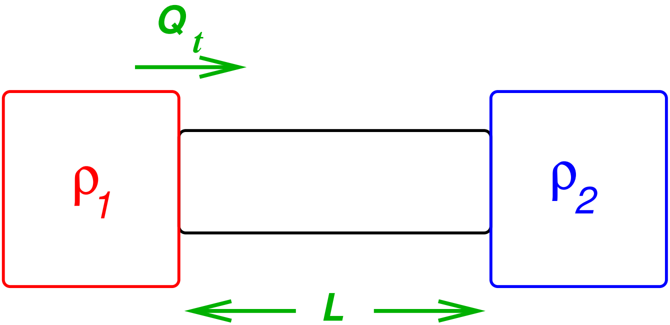
$$\frac{\langle Q_t \rangle}{t} \simeq \frac{G(\rho_a, \rho_b)}{L}$$

General lattice gas

Random walkers

KMP model

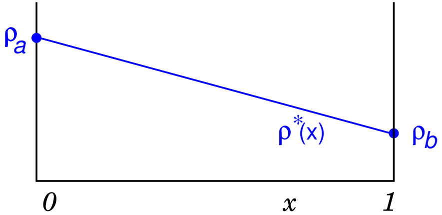
DIFFUSIVE SYSTEMS



For large L:

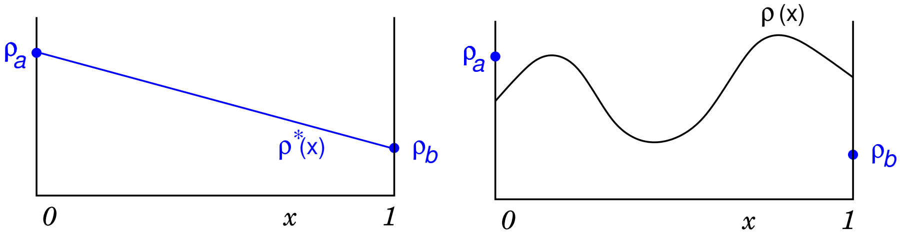
$$\frac{\langle Q_t \rangle}{t} \simeq \frac{F(\rho_1, \rho_2)}{L} \qquad \text{Fick's law}$$

Density profiles



Steady state

Density profiles



Steady state — Fluctuation

- Equilibrium $\rho_a = \rho_b$

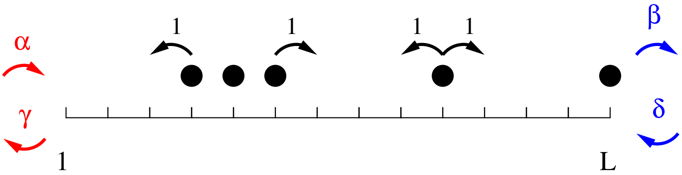
$$P(\mathcal{C}) = \frac{e^{-\beta E(\mathcal{C})}}{Z}$$

- Non-equilibrium $\rho_a \neq \rho_b$

$$P(\mathcal{C}) = ???$$

Exclusion processes

SSEP (Symmetric simple exclusion process)



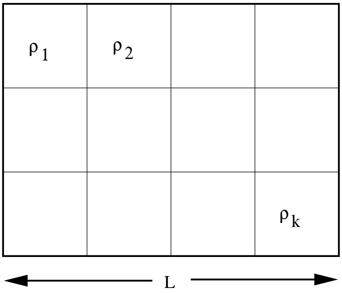
$$\rho_a = \frac{\alpha}{\alpha+\gamma} \,, \qquad \rho_b = \frac{\delta}{\beta+\delta}$$

DENSITY FLUCTUATIONS

IN

DIFFUSIVE SYSTEMS

Large deviation functional



$$\mathrm{Pro}(\rho_1, ...\rho_k) \sim \exp\left[-L^d \mathcal{F}(\rho_1, ...\rho_k)\right]$$

Large number k of boxes $\vec{r} = L\vec{x}$

$$\boxed{\mathrm{Pro}(\{\rho(\vec{x})\}) \sim \exp\left[-L^d \mathcal{F}(\{\rho(\vec{x})\})\right]}$$

Large deviation functional for the SSEP in $d = 1$

$$\text{Pro}(\{\rho(x)\}) \sim \exp[-L\, \mathcal{F}(\{\rho(x)\})]$$

Equilibrium $\rho_a = \rho_b = F$

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left[(1-\rho(x)) \log \frac{1-\rho(x)}{1-F} + \rho(x) \log \frac{\rho(x)}{F} \right]$$

Large deviation functional for the SSEP in $d = 1$

$$\text{Pro}(\{\rho(x)\}) \sim \exp[-L\ \mathcal{F}(\{\rho(x)\})]$$

Equilibrium $\rho_a = \rho_b = F$

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx\ \left[(1-\rho(x))\log\frac{1-\rho(x)}{1-F} + \rho(x)\log\frac{\rho(x)}{F}\ \right]$$

Non-equilibrium $(\rho_a \neq \rho_b)$

D Lebowitz Speer 2001-2002
Bertini De Sole Gabrielli Jona-Lasinio Landim 2002

$$\mathcal{F} = \sup_{F(x)} \int dx \left[(1-\rho(x))\log\frac{1-\rho(x)}{1-F(x)} + \rho(x)\log\frac{\rho(x)}{F(x)} + \log\frac{F'(x)}{\rho_b-\rho_a}\right]$$

with $F(x)$ monotone, $F(0) = \rho_a$ and $F(1) = \rho_b$

Non-equilibrium $\rho_a \neq \rho_b$

$$\mathcal{F} = \sup_{F(x)} \int dx \left[(1-\rho(x)) \log \frac{1-\rho(x)}{1-F(x)} + \rho(x) \log \frac{\rho(x)}{F(x)} + \log \frac{F'(x)}{\rho_b - \rho_a} \right]$$

$$\Downarrow$$

$\mathcal{F}$ is non-local:

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \ (1-\rho(x)) \log \frac{1-\rho(x)}{1-\rho^*(x)} + \rho(x) \log \frac{\rho(x)}{\rho^*(x)}$$
$$+ \frac{(\rho_a - \rho_b)^2}{(\rho_a - \rho_a^2)^2} \int_0^1 dx \int_x^1 dy \ x(1-y) \ (\rho(x) - \rho^*(x)) \ (\rho(y) - \rho^*(y))$$
$$+ O(\rho_a - \rho_b)^3$$

where $\rho^*(x) = (1-x)\ \rho_a + x\ \rho_b$

Long-range correlations Spohn 82

$$\langle \rho(x)\rho(y) \rangle - \langle \rho(x) \rangle \langle \rho(y) \rangle \simeq \frac{1}{L} G(x,y) = -\frac{(\rho_a - \rho_b)^2}{L} x(1-y)$$

Large deviation functional

$$\boxed{\mathrm{Pro}(\{\rho(x)\}) \sim \exp[-\mathrm{Action}] = \exp\left[-L\ \mathcal{F}(\{\rho(x)\})\right]}$$

Equilibrium

1. $\mathcal{F}$ local
2. $\mathcal{F} = \int dx\ g(\rho(x))$; $g(\rho) \equiv$ free energy per unit volume
3. No phase transition in one dimension (short range interactions)

Large deviation functional

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Equilibrium

1. $\mathcal{F}$ local
2. $\mathcal{F} = \int dx\ g(\rho(x))$; $g(\rho) \equiv$ free energy per unit volume
3. No phase transition in one dimension (short range interactions)

Non-equilibrium

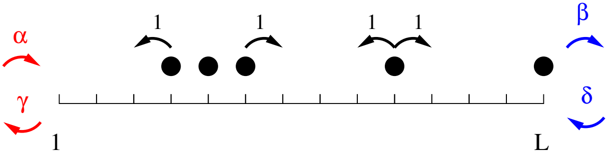
1. $\mathcal{F}$ non local
2. Weak long range correlations (SSEP for $x < y$)

$$\langle(\rho(x) - \rho^*(x))(\rho(y) - \rho^*(y))\rangle \simeq -\frac{(\rho_a - \rho_b)^2}{L} x(1-y)$$

Matrix ansatz

Fadeev 1980, ...,
D Evans Hakim Pasquier 1993

Steady state of the SSEP



$$P(\tau_1,\ldots,\tau_L) = \frac{\langle W|X_1\ldots X_L|V\rangle}{\langle W|(D+E)^L|V\rangle}$$

where $X_i = \begin{cases} D & \text{if site } i \text{ occupied} \\ E & \text{if site } i \text{ empty} \end{cases}$

Matrix ansatz

Fadeev 1980, ...,
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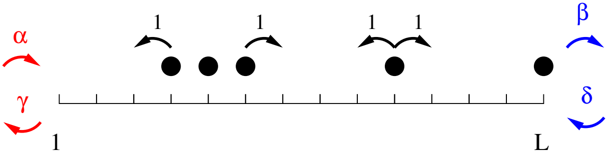
Steady state of the SSEP



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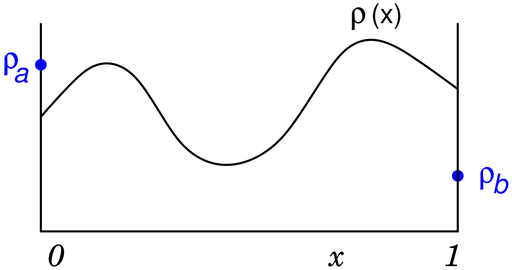
where $X_i = \begin{cases} D & \text{if site } i \text{ occupied} \\ E & \text{if site } i \text{ empty} \end{cases}$

$$\langle W|(\alpha E - \gamma D) = \langle W|$$
$$DE - ED = D + E$$
$$(\beta D - \delta E)|V\rangle = |V\rangle$$

Macroscopic fluctuation theory

$$\text{Pro}(\{\rho(x)\}) \sim \exp[-L\ \mathcal{F}(\{\rho(x)\})]$$



Bertini De Sole Gabrielli
Jona-Lasinio Landim 2001-2002

Macroscopic fluctuation theory

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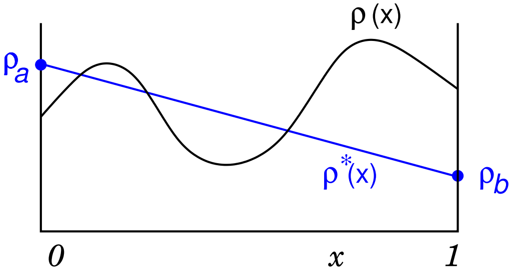

Bertini De Sole Gabrielli
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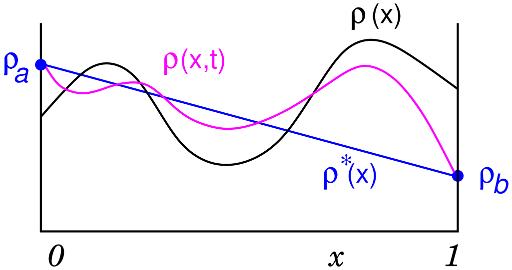
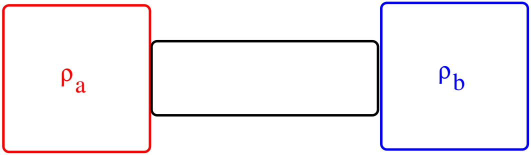


Bertini De Sole Gabrielli
Jona-Lasinio Landim 2001-2002

$$\mathcal{F}(\{\rho(x)\}) = \min_{\rho(x,t)} \int_{-\infty}^{0} dt \int_0^1 dx \frac{[j(x,t) + \rho'(x,t) D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}$$

with $\rho(x,0) = \rho(x)$; $\rho(x,-\infty) = \rho^*(x)$

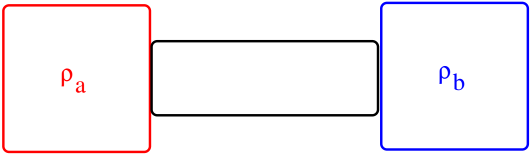
Large diffusive systems



A diffusive system

- For $\rho_a - \rho_b$ small: $$\frac{\langle Q_t \rangle}{t} = \frac{(\rho_a - \rho_b) D(\rho)}{L}$$ Fick's law
- $\rho_a = \rho_b = \rho$: $$\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$$

Large diffusive systems



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- $\rho_a = \rho_b = \rho$: $\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$

SSEP $D(\rho) = 1$; $\sigma(\rho) = 2\rho(1-\rho)$

Random walkers $D(\rho) = 1$; $\sigma(\rho) = 2\rho$

Macroscopic fluctuation theory for diffusive systems

Kipnis Olla Varadhan 89
Spohn 91
⋮
Bertini De Sole Gabrielli Jona-Lasinio Landim 2001 →

Onsager Machlup theory for non equilibrium

$$\frac{d\rho}{dt} = -\frac{dj}{dx} \quad \text{(conservation law)}$$

Evolution of a profile $\rho(x,t)$ for $0 \leq t \leq T$

$$\mathrm{Pro}(\{\rho(x,t), j(x,t)\})$$
$$\exp\left[-L \int_0^{T/L^2} dt \int_0^1 dx \; \frac{[j(x,t) + \rho'(x,t) D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}\right]$$

$$\Updownarrow$$

$$j(x,t) = -\rho'(x,t) D(\rho(x,t)) + \frac{1}{\sqrt{L}}\, \eta(x,t)$$

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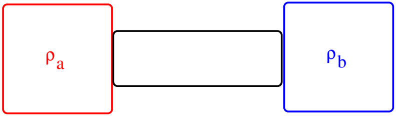
$$\Updownarrow$$

$$j(x,t) = -\rho'(x,t) D(\rho(x,t)) + \frac{1}{\sqrt{L}} \, \eta(x,t)$$

with the white noise $\langle \eta(x,t)\eta(x',t')\rangle = 2\sigma(\rho(x,t))\delta(x-x')\delta(t-t')$

Time reversal symmetry of fluctuations at equilibrium



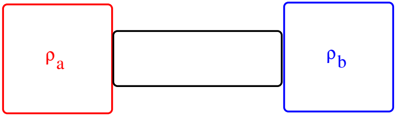
Given a fluctuation at $t = 0$

$$\rho(x, 0) = \rho_{\text{steady state}}(x) + \phi(x, 0)$$

How does it relax?

$t > 0$ $$\rho(x, t) = \rho_{\text{steady state}}(x) + \phi(x, t)$$

Time reversal symmetry of fluctuations at equilibrium



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How does it relax?

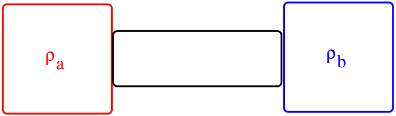
$t > 0$ $$\rho(x, t) = \rho_{\text{steady state}}(x) + \phi(x, t)$$

How is it produced?

$t < 0$ $$\rho(x, t) = \rho_{\text{steady state}}(x) + \phi(x, t)$$

Onsager Machlup 1953 Equilibrium $\Rightarrow$ $\boxed{\phi(x, t) = \phi(x, -t)}$

Time reversal asymmetry of fluctuations out of equilibrium



Given a fluctuation at $t = 0$

$$\rho(x,0) = \rho_{\text{steady state}}(x) + \phi(x,0)$$

How does it relax?

$t > 0$ $\quad \rho(x,t) = \rho_{\text{steady state}}(x) + \phi(x,t)$

How is it produced?

$t < 0$ $\quad \rho(x,t) = \rho_{\text{steady state}}(x) + \phi(x,t)$

Non equilibrium steady state $\Rightarrow$ $\boxed{\phi(x,t) \neq \phi(x,-t)}$

SSEP

How does a fluctuation $\rho(x,0)$ relax?

$$\frac{d\rho(x,t)}{dt} = \frac{d^2\rho(x,t)}{dx^2}$$

How is a fluctuation $\rho(x,0)$ produced?

$$\frac{d\rho(x,s)}{ds} = \frac{d^2\rho(x,s)}{dx^2} - 2\frac{(\rho_a - \rho_b)}{\rho_a(1-\rho_a)}(1-2\rho(x,s))\frac{d\rho(x,s)}{dx} + O\left((\rho_a - \rho_b)^2\right)$$

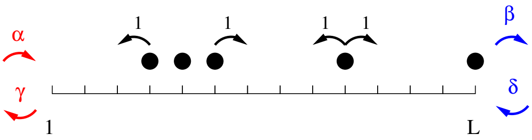
CURRENT FLUCTUATIONS

IN

DIFFUSIVE SYSTEMS

SSEP (Symmetric simple exclusion process)

D. Douçot Roche 2004



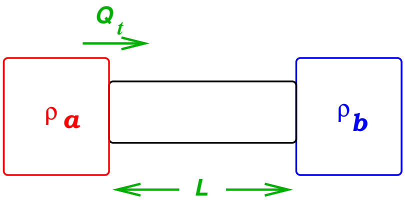
$$\rho_a = \frac{\alpha}{\alpha+\gamma} \;, \qquad\qquad \rho_b = \frac{\delta}{\beta+\delta}$$

$$\lim_{t\to\infty} \frac{\langle Q(t)\rangle}{t} \simeq \frac{1}{L}[\rho_a - \rho_b] \qquad \text{Fick's law}$$

$$\lim_{t\to\infty} \frac{\langle Q^2(t)\rangle_c}{t} \simeq \frac{1}{L}\left[\rho_a + \rho_b - \frac{2(\rho_a^2 + \rho_a\rho_b + \rho_b^2)}{3}\right]$$

$$\lim_{t\to\infty} \frac{\langle Q^3(t)\rangle_c}{t} \simeq \frac{1}{L}(\rho_a - \rho_b)\left[1 - 2(\rho_a+\rho_b) + \frac{16\rho_a^2 + 28\rho_a\rho_b + 16\rho_b^2}{15}\right]$$

DIFFUSIVE SYSTEMS: higher cumulants



Bodineau D. 2004

- For $\rho_a - \rho_b$ small: $$\frac{\langle Q_t \rangle}{t} = \frac{D(\rho)(\rho_a - \rho_b)}{L}$$
- $\rho_a = \rho_b = \rho$: $$\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$$

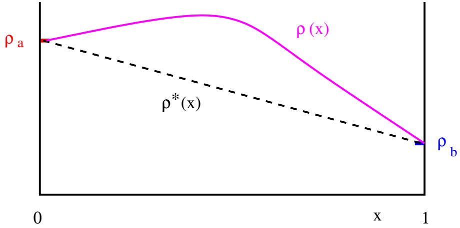
Then one can calculate

All cumulants of Q_t for arbitrary ρ_a and ρ_b

Variational problem

Bodineau D. 2004

$$F_L(j, \rho_a, \rho_b) = \frac{1}{L} \min_{\rho(x)} \int_0^1 dx \, \frac{[j\ L\ + \rho'(x)\ D(\rho(x))]^2}{2\sigma(\rho(x))}$$



Satisfies the fluctuation theorem

$$F(j) - F(-j) = j \int_{\rho_a}^{\rho_b} \frac{D(\rho)}{\sigma(\rho)} d\rho$$

Gallavotti Cohen 1995
Evans Searls 1994
.... Kurchan 1998
Lebowitz Spohn 1999

TRUE VARIATIONAL PRINCIPLE

Bertini De Sole Gabrielli
Jona-Lasinio Landim 2005

$$F(j) = \frac{1}{L} \lim_{T\to\infty} \min_{\rho(x,t),j(x,t)} \frac{1}{T} \int_0^T dt \int_0^1 dx \frac{[j(x,t) + \rho'(x,t) D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}$$

with $\frac{d\rho}{dt} = -\frac{dj}{dx}$ (conservation), $\rho_t(0) = \rho_a$, $\rho_t(1) = \rho_b$ and

$$j\ T = \int_0^T j_t(x) dt$$

- Sufficient condition for the optimal profile to be time independent
- Dynamical phase transition

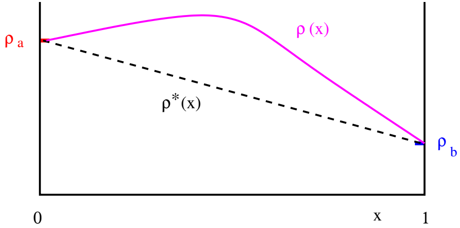
Bodineau D. 2005-2007

the optimal $\rho_t(x)$ starts to become time dependent

Variational problem

Bodineau D. 2004

$$F_L(j, \rho_a, \rho_b) = \frac{1}{L} \min_{\rho(x)} \int_0^1 dx \, \frac{[j \; L \; + \rho'(x) \; D(\rho(x))]^2}{2\sigma(\rho(x))}$$



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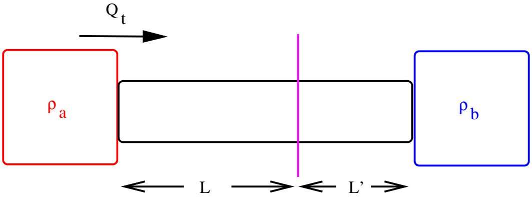
Evans Searls 1994
.... Kurchan 1998
Lebowitz Spohn 1999

Additivity principle

$$\mathrm{Pro}\left(\frac{Q_t}{t} = j, \rho_a, \rho_b\right) \sim \exp[-t\ F_{L+L'}(j, \rho_a, \rho_b)]$$

Bodineau D. 2004



Additivity

$$P_{L+L'}(Q, \rho_a, \rho_b) \sim \max_{\rho} [P_L(Q, \rho_a, \rho)\ P_{L'}(Q, \rho, \rho_b)]$$

$$F_{L+L'}(j, \rho_a, \rho_b) = \min_{\rho} [F_L(j, \rho_a, \rho)\ +\ F_{L'}(j, \rho, \rho_b)]$$

Diffusive systems: all the cumulants

Bodineau D. 2004

$$\frac{\langle Q_t \rangle_c}{t} = \frac{1}{L} \mathcal{I}_1$$

$$\frac{\langle Q_t^2 \rangle_c}{t} = \frac{1}{L} \frac{\mathcal{I}_2}{\mathcal{I}_1}$$

$$\frac{\langle Q_t^3 \rangle_c}{t} = \frac{1}{L} \frac{3\,(\,\mathcal{I}_3\,\mathcal{I}_1 - \mathcal{I}_2^2\,)}{\mathcal{I}_1^3}$$

$$\frac{\langle Q_t^4 \rangle_c}{t} = \frac{1}{L} \frac{3\,(\,5\,\mathcal{I}_4\,\mathcal{I}_1^2 - 14\,\mathcal{I}_1\mathcal{I}_2\,\mathcal{I}_3 + \,9\,\mathcal{I}_2^3\,)}{\mathcal{I}_1^5}$$

where

$$\mathcal{I}_n = \int_{\rho_b}^{\rho_a} D(\rho)\sigma(\rho)^{n-1} d\rho$$

For the SSEP $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$

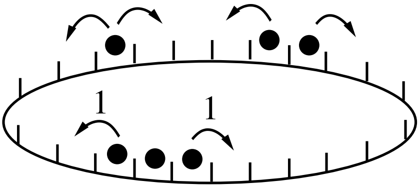
For the random walkers $D(\rho) = 1$ and $\sigma(\rho) = 2\rho$

SSEP on a ring

Appert D Lecomte Van Wijland 2008

N particles
L sites

$$\rho = \frac{N}{L}$$



Q_t flux through a bond during time t

Universal cumulants of the current

$$\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L}$$

Gaussian

$$\frac{\langle Q^4 \rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6 \rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8 \rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}$$

Universal

Universal cumulants of the current

$$\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L}$$ Gaussian

$$\frac{\langle Q^4 \rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6 \rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8 \rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}$$ Universal

Macroscopic fluctuation theory

Universal cumulants of the current

$$\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L}$$ Gaussian

$$\frac{\langle Q^4 \rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6 \rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8 \rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}$$ Universal

Macroscopic fluctuation theory

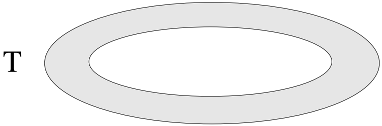
+ fluctuations around a flat profile

DIFFUSIVE SYSTEMS: summary

Open system

$$\langle Q^n \rangle_c / t \sim L^{-1}$$

Ring

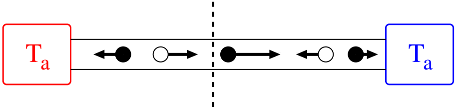


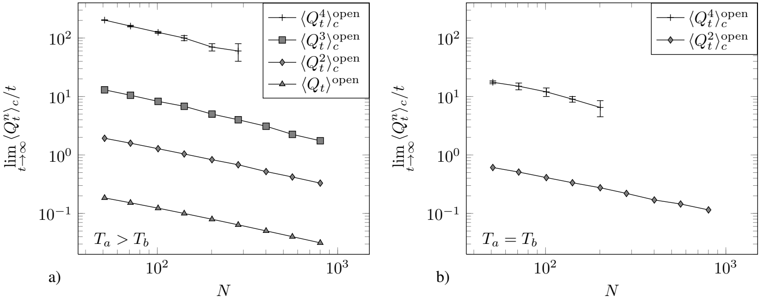
$$\langle Q^2 \rangle_c / t \sim L^{-1}$$

$$\langle Q^{2n} \rangle_c / t \sim L^{-2} \quad \text{for} \quad n \geq 2$$

NON DIFFUSIVE SYSTEMS

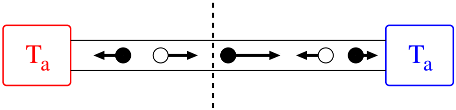
Hard particle gas: OPEN

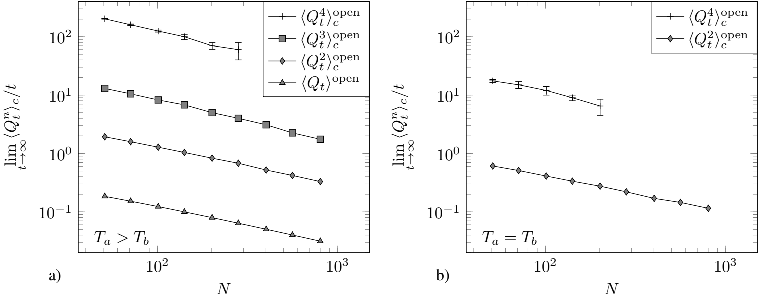




$T_a > T_b$ $T_a = T_b$

Hard particle gas: OPEN



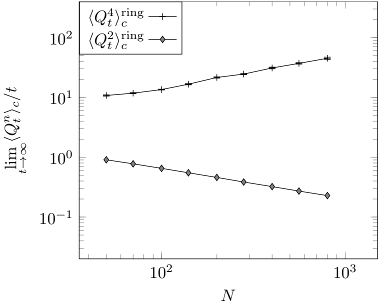


$T_a > T_b$ $T_a = T_b$

$$\frac{\langle Q_t^n \rangle}{t}_c \sim L^{-\frac{2}{3}}$$

Anomalous Fourier's law

Hard particle gas: RING



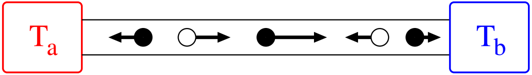
$$\frac{\langle Q_t^2 \rangle}{t} \sim L^{-\frac{1}{2}}$$

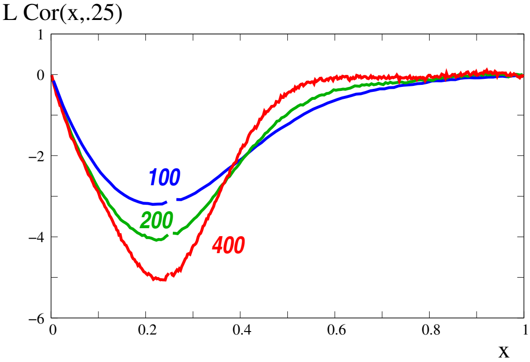
$$\frac{\langle Q_t^4 \rangle_c}{t} \sim L^{\frac{1}{2}}$$

For diffusive systems

$$\frac{\langle Q_t^2 \rangle}{t} \sim L^{-1} \quad ; \quad \frac{\langle Q_t^4 \rangle_c}{t} \sim L^{-2}$$

Correlations on a mesoscopic scale





Delfini, Lepri, Livi, Mejia-Monasterio, Politi 2010
Gerschenfeld, D., Lebowitz 2010

CONCLUDING REMARKS AND OPEN ISSUES

Lecture 1: Macroscopic approach for diffusive systems:

1. Macroscopic fluctuation theory
2. Langevin equation
3. Finite systems in d =1 and d >1
4. Infinite system
5. Open issues

Lecture 1: Macroscopic approach for diffusive systems:

1. Macroscopic fluctuation theory
2. Langevin equation
3. Finite systems in d =1 and d >1
4. Infinite system
5. Open issues

Lecture 2: Microscopic approach :

1. Representation of reservoirs
2. Markov process
3. Independent particles
4. Perturbation theory
5. Bethe ansatz for finite and infinite systems
6. Open questions

Theory for mechanical systems

Theory for mechanical systems
with conserved momentum, density, energy   Spohn

Density profiles

Correlations of the density, the momentum, the energy

Large deviations of the density

Large deviations of the current

Macroscopic fluctuation theory

Single file diffusion

Macroscopic fluctuation theory

Single file diffusion

Correlations between the density and the current

Macroscopic fluctuation theory

Single file diffusion

Correlations between the density and the current

Large deviation of the density for a general diffusive system

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Driven systems

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Phase transitions

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Large deviation of the density for a general diffusive system

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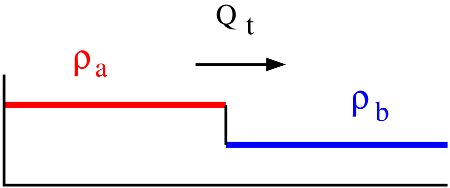
Phase transitions

Non steady state situations

Step initial condition on the infinite line



SSEP

D Gerschenfeld 2009

$$\left\langle e^{\lambda Q_t} \right\rangle \simeq \exp[\sqrt{t}\ H(\omega)]$$

with $H(\omega) = \frac{1}{\pi} \int_{\infty}^{\infty} dk \log \left[1 + \omega e^{-k^2} \right]$

and $\omega = 1 - [1 - (e^{\lambda} - 1)\rho_a][1 - (1 - e^{-\lambda})\rho_b]$

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