Breaking of ensemble equivalence in dense random graphs

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In general: a function $\boldsymbol{T} : \mathcal{G}_n \to \mathbb{R}^m$ (graph parameter)

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We choose the unique parameters θ_1^*, θ_2^* such that

$$\langle \boldsymbol{T} \rangle_{\text{can}} := \sum_{G \in \mathcal{G}_n} \boldsymbol{T}(G) \mathbb{P}_{\text{can}}(G, \theta_1^*, \theta_2^*) = \boldsymbol{T}^*.$$

E. T. Jaynes, "Information Theory and Statistical Mechanics", (1957)

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In the dense regime, if

$$s_\infty := \lim_{n \to \infty} \frac{1}{n^2} S_n(\mathbb{P}_{\text{mic}}|\mathbb{P}_{\text{can}}) = 0,$$

then the two ensembles are said to be equivalent.

H. Touchette, Equivalence and Nonequivalence of Ensembles: Thermodynamic, Macrostate, and Measure levels (2015)

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- Find a general rule of breaking of ensemble equivalence
  - Determine those constraints that yield $s_\infty > 0$
- If $s_\infty = 0$ then determine the order of S_n as $n \to \infty$

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Theorem: (S. Bhamidi,G. Bresler and A. Sly (2011)) Suppose that $\theta_1 \in \mathbb{R}$ and $\theta_2 \geq 0$. If the function

$$\phi(p;\theta_1,\theta_2) := \frac{\exp\left(\theta_1 + 6\theta_2 p^2\right)}{1 + \exp\left(\theta_1 + 6\theta_2 p^2\right)},$$

has a unique fixed point, call it p^*, then as $n \to \infty$ a graph drawn from $\mathbb{P}_{\text{can}}(\cdot)$ is indistinguishable from an Erdős-Rényi random graph with parameter p^*.

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$$\lim_{n\to\infty} \frac{1}{n^2} \log Z_n(\theta_1, \theta_2) = \sup_{0\leq u\leq 1} \left(\theta_1 u + \theta_2 u^3 - I(u)\right),$$

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Theorem: Let G_n be a graph drawn from the edge-triangle model. Assume that $\theta_2 \geq 0$.

- unique maximiser u^* then G_n is indistinguishable from the Erdös-Rényi graph with parameter u^*.
- maximiser is not unique then G_n behaves like a convex combination of Erdős-Rényi graphs.

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C. Radin and M. Yin (2013) studied the phase transition for the edge-triangle model. M. Yin (2013) has also studied more general cases.

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- $f : [0,1]^2 \to [0,1]$
- symmetric, i.e., $f(x,y) = f(y,x)$ for all $(x,y) \in [0,1]^2$

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A sequence of graphs $\{G_n\}_{\{n\geq 1\}}$ is said to converge to a function (graphon) h if for every finite simple graph H

$$\lim_{n\to\infty} t(H, G_n) = t(H, h),$$

where $t(H, G_n)$ is the homomorphism density

$$t(H, G_n) := \frac{\hom(H, G_n)}{|V(G_n)|^{|V(H)|}}.$$

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Cut-set: The set of edges which have endpoints in the two sets of the cut.

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For $F \subset W$ define

$$\mathbb{P}_{n,p}(F) := \mathbb{P}_{n,p}\left(\{G \in \mathcal{G}_n : f^G \in F\}\right),$$

where f^G is the graphon corresponding to G.

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$$I_p(\tilde{h}) := \frac{1}{2} \int\int_{[0,1]^2} h(x,y) \log \frac{h(x,y)}{p} + (1-h(x,y)) \log \frac{1-h(x,y)}{1-p} \mathrm{d}x\mathrm{d}y.$$

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We denote, for some given vector $\mathbf{T}^*$ in $\mathbb{R}^m$,

$$\mathcal{G}_n^* := \{G \in \mathcal{G}_n : \mathbf{T}(G) = \mathbf{T}^*\} \quad \text{and} \quad \mathbb{P}_{\text{mic}}(G^*) = \frac{1}{|\mathcal{G}_n^*|}.$$

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In D. Garlaschelli, F. den Hollander and A. Roccaverde (2017) cases have been identified where $s_\infty > 0$.

Contribution

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In F. den Hollander, M. Mandjes, A. Roccaverde and N. J. Starreveld (submitted) it was shown that

$$\theta_1^* \geq 0 \Leftrightarrow T_1^* \geq \frac{1}{2} \quad \text{and} \quad \theta_2^* \geq 0 \Leftrightarrow T_2^* \geq \frac{1}{8}.$$

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For the relative entropy in the limit, i.e. s_∞, we have

Theorem:[F. deb Hollander, M. Mandjes, A. Roccaverde and N. J. Starreveld.] Consider the microcanonical ensemble with constraint $\boldsymbol{T} = \boldsymbol{T}^*$, which is a continuous graph parameter, and the canonical ensemble with parameter $\boldsymbol{\theta} = \boldsymbol{\theta}^*$. Then

$$s_\infty = \sup_{\tilde{h} \in \tilde{W}} \left[\boldsymbol{\theta}^* \cdot \boldsymbol{T}(\tilde{h}) - I(\tilde{h}) \right] - \sup_{\tilde{h} \in \tilde{W}^*} \left[\boldsymbol{\theta}^* \cdot \boldsymbol{T}(\tilde{h}) - I(\tilde{h}) \right],$$

where $\tilde{W}^* = \{\tilde{h} \in \tilde{W} \colon \ \boldsymbol{T}(\tilde{h}) = \boldsymbol{T}^*\}$.

Results

Using this variational representation for the relative entropy s_∞ we managed to identify regions where breaking of ensemble equivalence (with scaling n^2) occurs

Analysis

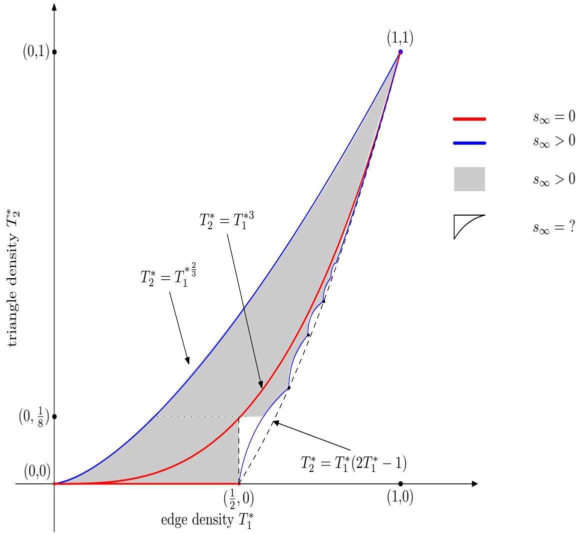

Figure: The admissible edge-triangle density region is the region on and between the blue curves (cf. C. Radin and L. Sadun (2015)).

Future work

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- Determine the relative entropy for sparse graphs.

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Consider the degree sequence as a graph parameter. Then the Erdös- Gallai Theorem yields the right topology.

A given sequence $\mathbf{d}^n = (d_1^n, \ldots, d_n^n)$ is a graphical degree sequence if and only if $\sum_{i=1}^n d_i^n$ is even and for every $r = 1, \ldots, n-1$,

$$\sum_{i=1}^{r} d_i^n \leq \frac{r(r-1)}{2} + \sum_{i=r+1}^{n} \min\{d_i^n, r\}.$$

Future work

Hence in a graphon setting, the appropriate limiting objects of graphs with a given degree sequence are functions $f : [0,1] \to [0,1]$ which satisfy, for each $x \in (0,1]$,

$$\int_0^x f(y)\mathrm{d}y < x^2 + \int_x^1 \min\{f(y), x\}\mathrm{d}y.$$

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If a given sequence $\mathbf{t}^n = (t_1^n, \ldots, t_n^n)$ is a graphical triangle degree sequence then $\sum_{i=1}^n t_i^n$ is a multiple of three and for for every $r = 1, \ldots, n-1$,

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$$\sum_{i=1}^{r} t_i^n \leq \frac{r(r-1)(r-2)}{3} + 2 \sum_{i=r+1}^{n} \min\{t_i^n, \frac{r(r-1)}{2}\} + \frac{1}{2} \sum_{i=r+1}^{n} \min\{t_i^n, r(n-r-1)\}.$$

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This is though not a sufficient condition.

Thank you!

Literature Graphons - Large deviation Principle

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(b) S. Chatterjee, "An introduction to large deviations for random graphs"

(c) C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, K. Vesztergombi, "Convergent graph sequences I: Subgraph frequencies, metric properties and testing"

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(e) S. Chatterjee, P. Diaconis and A. Sly, " Random graphs with a given degree sequence".

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(a) E.T. Jaynes, "Information theory and statistical mechanics"

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(e) H. Touchette, "General equivalence and non-equivalence of ensembles".

(f) O. Pikhurko and A. Razborov, "Asymptotic structure of graphs with the minimim number of triangles".