Branching Random Walk and Regular variation

Rajat Subhra Hazra
Joint work with Ayan Bhattacharya and Parthanil Roy

Indian Statistical Institute, Kolkata

14th August, 2017

Table of contents

What is Branching random walk?

- *Branching random walk is a natural extension of Galton-Watson process in a spatial sense.*

What is Branching random walk?

- *Branching random walk is a natural extension of Galton-Watson process in a spatial sense.*
- *Start with one particle at origin;*

What is Branching random walk?

- *Branching random walk is a natural extension of Galton-Watson process in a spatial sense.*
- *Start with one particle at origin;*
- *Its children who form the first generation are points of a point process $\mathcal{L}$ on $\mathbb{R}$.*

What is Branching random walk?

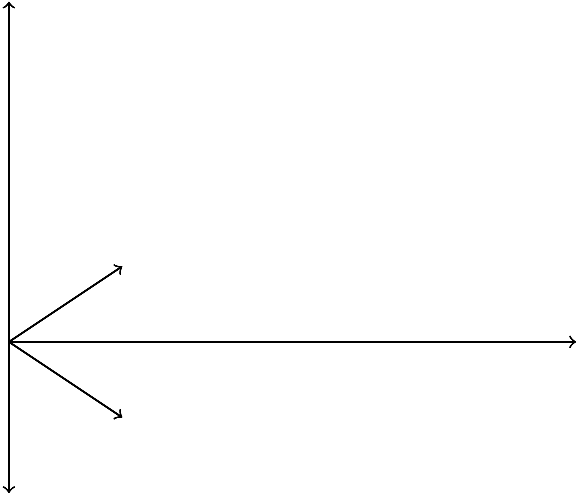
- *Branching random walk is a natural extension of Galton-Watson process in a spatial sense.*
- *Start with one particle at origin;*
- *Its children who form the first generation are points of a point process $\mathcal{L}$ on $\mathbb{R}$.*
- *Each particle produces its own children who form second generation and "positioned" (with respect to their parent) according to $\mathcal{L}$.*

What is Branching random walk?

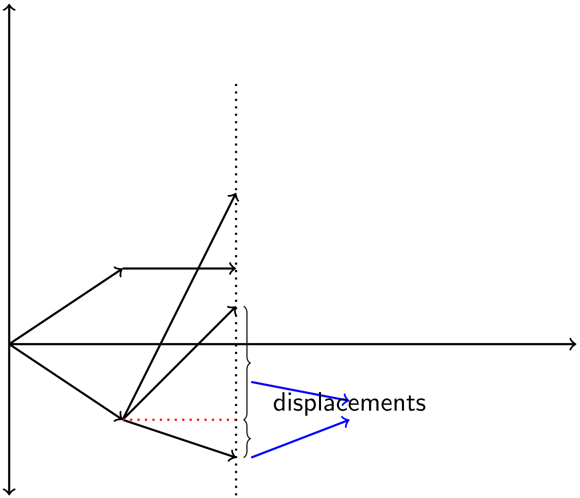
- *Branching random walk is a natural extension of Galton-Watson process in a spatial sense.*
- *Start with one particle at origin;*
- *Its children who form the first generation are points of a point process $\mathcal{L}$ on $\mathbb{R}$.*
- *Each particle produces its own children who form second generation and "positioned" (with respect to their parent) according to $\mathcal{L}$.*
- *Each individual in the n-th generation produces independently of each other and everything else.*

Growth process

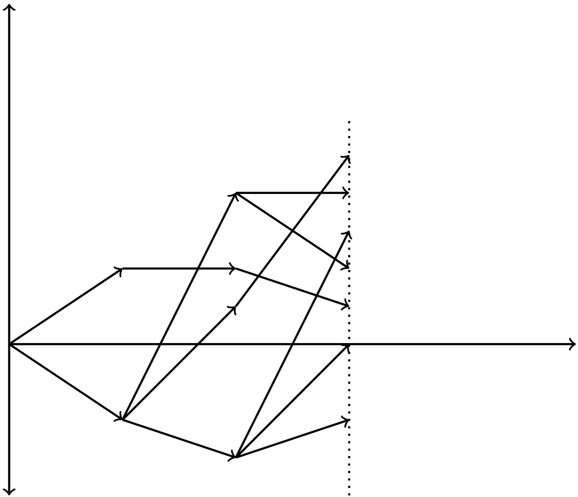
Growth process



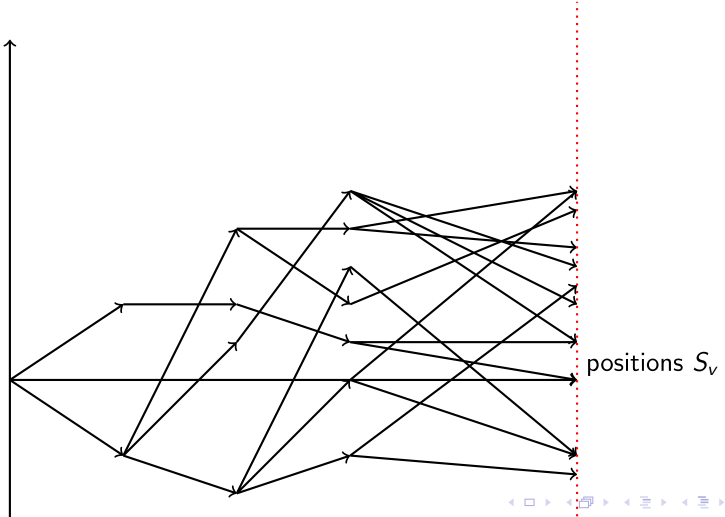
Growth process



Growth process



Growth process



Questions?

- *The underlying tree is a Galton-Watson tree.*
- *Various assumptions on displacements and positions can be assumed.*
- *Questions of interest: If S_v denotes the position of a particle v then the behaviour as $n \to \infty$ of*

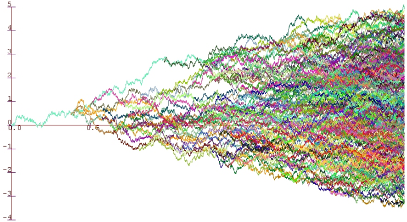
$$N_n = \sum_{|v|=n} \delta_{a_n^{-1}(S_v - b_n)}.$$

- *Position of the top most particle in the n-th generation and scaling limits.*

How did it begin? and the state of art now!

Branching Brownian motion (BBM):

- *At time* 0*, particle at* $0 \in \mathbb{R}$.
- *Particle moves by a Brownian motion for an exponential time.*
- *After the step, particle splits into two. Repeat independently.*
- $N(t) \sim e^t$ *number of particles in time* t *and positions be denoted by* $S_1(t), \cdots, S_{N(t)}(t)$.



[Picture by Matt Roberts]

Branching Brownian motion

- *Started with connections of differential equations to probability.*

Branching Brownian motion

- *Started with connections of differential equations to probability.*
- **Fisher-Kolmogorov-Petrovsky-Piscounov** *(F-KPP) equation:*

$$\partial_t u(x,t) = \frac{1}{2}\partial_x^2 u(x,t) + u - u^2 \qquad u(0,x) = \mathbb{1}_{x<0}\,.$$

Branching Brownian motion

- *Started with connections of differential equations to probability.*
- **Fisher-Kolmogorov-Petrovsky-Piscounov** *(F-KPP) equation:*

$$\partial_t u(x,t) = \frac{1}{2}\partial_x^2 u(x,t) + u - u^2 \qquad u(0,x) = \mathbb{1}_{x<0}\,.$$

- *If* $u(t,x) = \mathrm{P}(max_{1 \leq i \leq N(t)} S_i(t) > x)$ *then* **McKean (1975)** *showed that it satisfies F-KPP.*

Branching Brownian motion

- *Started with connections of differential equations to probability.*
- **Fisher-Kolmogorov-Petrovsky-Piscounov** *(F-KPP) equation:*

$$\partial_t u(x,t) = \frac{1}{2}\partial_x^2 u(x,t) + u - u^2 \qquad u(0,x) = \mathbb{1}_{x<0}\,.$$

- *If* $u(t,x) = \mathrm{P}(max_{1\le i\le N(t)} S_i(t) > x)$ *then* **McKean (1975)** *showed that it satisfies F-KPP.*
- **Bramson (1978)** *showed*

$$u(t, x + m(t)) \to w(x) \qquad m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t.$$

Branching Brownian motion

- *$w(x)$ satisfies the following equation:*

$$\frac{1}{2}\partial_x^2 w + \sqrt{2}\partial_x w + w^2 - w = 0$$

Branching Brownian motion

- *$w(x)$ satisfies the following equation:*

$$\frac{1}{2}\partial_x^2 w + \sqrt{2}\partial_x w + w^2 - w = 0$$

- *Remarkable result of* **Lalley-Sellke** *(1987) showed*

$$w(x) = \mathrm{E}\left[e^{-c\mathbf{Z}e^{-\sqrt{2}x}}\right]$$

where Z is a limit of a "derivative" martingale.

Branching Brownian motion

- *$w(x)$ satisfies the following equation:*

$$\frac{1}{2}\partial_x^2 w + \sqrt{2}\partial_x w + w^2 - w = 0$$

- *Remarkable result of* **Lalley-Sellke** *(1987) showed*

$$w(x) = \mathrm{E}\left[e^{-c\mathbf{Z}e^{-\sqrt{2}x}}\right]$$

where Z is a limit of a "derivative" martingale.

- **Arguin-Bovier-Kistler** *(2013),* **Aidekon-Brunet-Berestycki-Shi** *(2013) showed the point process*

$$L_t = \sum_{1\leq i\leq N(t)} \delta_{S_i(t)-m(t)} \to L \quad \textbf{where } L \textbf{ is superposable.}$$

Branching Random Walk

- *Strong law for topmost particle:* **Hammerseley, Kingman, Biggins** *(70's)*

Branching Random Walk

- *Strong law for topmost particle:* **Hammerseley, Kingman, Biggins** *(70's)*
- **Addario-Berry, Reed** *(2009): Order of expected maxima.*

Branching Random Walk

- *Strong law for topmost particle:* **Hammerseley, Kingman, Biggins** *(70's)*
- **Addario-Berry, Reed** *(2009): Order of expected maxima.*
- **Bramson-Zeitouni** *(2009) : Tightness for recentered maxima.*

Branching Random Walk

- *Strong law for topmost particle:* **Hammerseley, Kingman, Biggins** *(70's)*
- **Addario-Berry, Reed** *(2009): Order of expected maxima.*
- **Bramson-Zeitouni** *(2009) : Tightness for recentered maxima.*
- **Aidekon** *(2013) (weak law for minimum position, same as BBM). Relies on works of* **Biggins and Kyprianou** *(2004) on convergence of derivative martingale in boundary case.*

Branching Random Walk

- *Strong law for topmost particle:* **Hammerseley, Kingman, Biggins** *(70's)*
- **Addario-Berry, Reed** *(2009): Order of expected maxima.*
- **Bramson-Zeitouni** *(2009) : Tightness for recentered maxima.*
- **Aidekon** *(2013) (weak law for minimum position, same as BBM). Relies on works of* **Biggins and Kyprianou** *(2004) on convergence of derivative martingale in boundary case.*
- **Madaule** *(2015) : Point process convergence of the position in n-th generation (seen from the tip).*

Branching Random Walk

- *Strong law for topmost particle:* **Hammerseley, Kingman, Biggins** *(70's)*
- **Addario-Berry, Reed** *(2009): Order of expected maxima.*
- **Bramson-Zeitouni** *(2009) : Tightness for recentered maxima.*
- **Aidekon** *(2013) (weak law for minimum position, same as BBM). Relies on works of* **Biggins and Kyprianou** *(2004) on convergence of derivative martingale in boundary case.*
- **Madaule** *(2015) : Point process convergence of the position in n-th generation (seen from the tip).*
- *Non-boundary, heavy tails:* **Durrett** *(1979, 1983).*

Assumptions on Branching Mechanism

- *Underlying tree is a Galton-Watson tree.*
- Z_n *denotes the number of particles at n-th generation and* $\mu := \mathrm{E}(Z_1) \in (1, \infty)$.
- *We shall assume that* $\mathrm{P}(Z_1 = 0) = 0$ *(no leaves).*
- *Using martingale convergence theorem,*

$$\frac{Z_n}{\mu^n} \to W (\geq 0) \text{ almost surely.}$$

- *Kesten-Stigum condition :*

$$\mathrm{E}(Z_1 \log Z_1) < \infty \Leftrightarrow \mathrm{P}(W > 0) = 1.$$

Assumptions on Displacement Random Variables

- *Each particle produces an independent copy of*

$$\mathcal{L} = \sum_{i=1}^{Z_1} \delta_{X_i}$$

Assumptions on Displacement Random Variables

- *Each particle produces an independent copy of*

$$\mathcal{L} = \sum_{i=1}^{Z_1} \delta_{X_i}$$

where $Z_1 \perp (X_1, X_2, \ldots)$ *is a* $\mathbb{K} := [0, \infty)^{\infty}$*-valued random variable such that*

Assumptions on Displacement Random Variables

- *Each particle produces an independent copy of*

$$\mathcal{L} = \sum_{i=1}^{Z_1} \delta_{X_i}$$

where $Z_1 \perp (X_1, X_2, \ldots)$ *is a* $\mathbb{K} := [0, \infty)^{\infty}$*-valued random variable such that*

  - *each* $X_i \sim F \in RV_{-\alpha} (\alpha > 0)$*;*

Assumptions on Displacement Random Variables

- *Each particle produces an independent copy of*

$$\mathcal{L} = \sum_{i=1}^{Z_1} \delta_{X_i}$$

where $Z_1 \perp (X_1, X_2, \ldots)$ *is a* $\mathbb{K} := [0, \infty)^\infty$*-valued random variable such that*

  - *each* $X_i \sim F \in RV_{-\alpha}(\alpha > 0)$*;*

  - $(X_1, X_2, \ldots) \in RV_{-\alpha}(\mathbb{K} \setminus 0_\infty, \lambda)$

  - **For this talk:** $\{X_i\}_{i \geq 1}$ *be independent!*

First main result

Let us denote the random point process of the positions of the particles by

$$N_n = \sum_{|v|=n} \delta_{c_n^{-1} S_v}$$

where $c_n \approx \mu^{n/\alpha}$.

Theorem (Bhattacharya, H. and Roy (2016, 2017))

Under our assumptions, the random point configuration converges in distribution to the Cox cluster process N_* *where*

$$N_* \stackrel{d}{=} \sum_{l=1}^{\infty} T_l \delta_{W^{1/\alpha} j_l}$$

Simplified expression in binary to take away!

- *Suppose all the displacements are independent and the tree is binary.*
- *Then*

$$N_* \stackrel{d}{=} \sum_{k=1}^{\infty} 2^{G_k} \delta_{j_k}$$

- $\{j_k\}_{k\geq 1}$ *are such that* $\sum_{k=1}^{\infty} \delta_{j_k} \sim PRM(\mu_\alpha)$.
- $(G_k)_{k\geq 1}$ *is a sequence of iid Geo(1/2) random variables independent of* $(j_k)_{k\geq 1}$.

Maxima

- *Let* M_n *denotes the maximal position of the* n^{th} *generation particles.*

Maxima

- *Let M_n denotes the maximal position of the n^{th} generation particles.*

Theorem (Bhattacharya, H. and Roy (2016))

Under the assumptions, for every $x > 0$,

$$\lim_{n\to\infty} \mathrm{P}(M_n > c_n x) = \mathrm{E}\left[e^{-CWx^{-\alpha}}\right]$$

where $C > 0$ is a constant.

Maxima

- *Let M_n denotes the maximal position of the n^{th} generation particles.*

Theorem (Bhattacharya, H. and Roy (2016))

Under the assumptions, for every $x > 0$,

$$\lim_{n\to\infty} \mathrm{P}(M_n > c_n x) = \mathrm{E}\left[e^{-CWx^{-\alpha}}\right]$$

where $C > 0$ is a constant.

- *This is an extension of main result of* **Durrett(1983)**.
- *Extensions of point process result to multi-type in article by* **Bhattacharya, Maulik, Palmowski, Roy (2017)**

Stable Point process

- *(Scalar Multiplication) For every* $a \in (0, \infty)$, *define*

$$a \circ \mathcal{P} = \sum_{i=1}^{\infty} \delta_{au_i}.$$

Stable Point process

- *(Scalar Multiplication) For every* $a \in (0, \infty)$, *define*

$$a \circ \mathcal{P} = \sum_{i=1}^{\infty} \delta_{au_i}.$$

- *(Superposition) The superposition of two point measures* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *will be denoted by* $\mathcal{L}_1 + \mathcal{L}_2$.

Stable Point process

- *(Scalar Multiplication) For every* $a \in (0, \infty)$, *define*

$$a \circ \mathcal{P} = \sum_{i=1}^{\infty} \delta_{au_i}.$$

- *(Superposition) The superposition of two point measures* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *will be denoted by* $\mathcal{L}_1 + \mathcal{L}_2$.

Definition (Davydov, Molchanov and Zuyev (2008,2011))

A point process N is called a strictly α-stable ($\alpha > 0$) point process if for all $a_1, a_2 \in (0, \infty)$

Stable Point process

- *(Scalar Multiplication) For every* $a \in (0, \infty)$, *define*

$$a \circ \mathcal{P} = \sum_{i=1}^{\infty} \delta_{au_i}.$$

- *(Superposition) The superposition of two point measures* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *will be denoted by* $\mathcal{L}_1 + \mathcal{L}_2$.

Definition (Davydov, Molchanov and Zuyev (2008,2011))

A point process N is called a strictly α-stable ($\alpha > 0$) point process if for all $a_1, a_2 \in (0, \infty)$

$$a_1 \circ N_1 + a_2 \circ N_2 \stackrel{d}{=} (a_1^{\alpha} + a_2^{\alpha})^{1/\alpha} \circ N$$

Stable Point process

- *(Scalar Multiplication) For every* $a \in (0, \infty)$, *define*

$$a \circ \mathcal{P} = \sum_{i=1}^{\infty} \delta_{au_i}.$$

- *(Superposition) The superposition of two point measures* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *will be denoted by* $\mathcal{L}_1 + \mathcal{L}_2$.

Definition (Davydov, Molchanov and Zuyev (2008,2011))

A point process N is called a strictly α-stable ($\alpha > 0$) point process if for all $a_1, a_2 \in (0, \infty)$

$$a_1 \circ N_1 + a_2 \circ N_2 \stackrel{d}{=} (a_1^{\alpha} + a_2^{\alpha})^{1/\alpha} \circ N$$

where N_1 and N_2 are two independent copies of N.

Representation of Stable Point Processes

Theorem (Davydov, Molchanov, Zuyev (2008,2011))

N be a strictly α*-stable (*$\alpha > 0$*) point process if and only if*

Representation of Stable Point Processes

Theorem (Davydov, Molchanov, Zuyev (2008,2011))

N be a strictly α-stable ($\alpha > 0$) point process if and only if

$$N \stackrel{d}{=} \sum_{i=1}^{\infty} \lambda_i \circ \mathcal{P}_i$$

where

- $\{\lambda_i : i \geq 1\}$ *are such that* $\Lambda = \sum_{i=1}^{\infty} \delta_{\lambda_i} \sim PRM(\nu_\alpha)$ *where*

Representation of Stable Point Processes

Theorem (Davydov, Molchanov, Zuyev (2008,2011))

N be a strictly α-stable ($\alpha > 0$) point process if and only if

$$N \stackrel{d}{=} \sum_{i=1}^{\infty} \lambda_i \circ \mathcal{P}_i$$

where

- $\{\lambda_i : i \geq 1\}$ *are such that* $\Lambda = \sum_{i=1}^{\infty} \delta_{\lambda_i} \sim PRM(\nu_\alpha)$ *where* $\nu_\alpha((x, \infty]) = x^{-\alpha}$ *for all* $x > 0$*;*
- $\mathcal{P}_i$*s are independent copies of the point process* $\mathcal{P}$ *and also independent of* Λ*.*

Brunet-Derrida in Our Setup

Brunet-Derrida in Our Setup

BD1 *The analogue of the* **superposability** *is*

$$N_* \stackrel{d}{=} W^{1/\alpha} \circ \mathcal{Q}$$

where W is the martingale limit and $\mathcal{Q}$ is a strictly α-stable point process. (Randomly scaled strictly α-stable point process).

Brunet-Derrida in Our Setup

BD1 *The analogue of the* **superposability** *is*

$$N_* \stackrel{d}{=} W^{1/\alpha} \circ \mathcal{Q}$$

where W is the martingale limit and $\mathcal{Q}$ is a strictly α-stable point process. (Randomly scaled strictly α-stable point process).

BD2 *The analogous representation is randomly scaled scale-decorated Poisson point process (Randomly scaled DMZ representation).*

Brunet-Derrida in Our Setup

BD1 *The analogue of the* **superposability** *is*

$$N_* \stackrel{d}{=} W^{1/\alpha} \circ \mathcal{Q}$$

where W is the martingale limit and $\mathcal{Q}$ is a strictly α-stable point process. (Randomly scaled strictly α-stable point process).

BD2 *The analogous representation is randomly scaled scale-decorated Poisson point process (Randomly scaled DMZ representation).*

- *We have shown that BD1 and BD2 are equivalent*

Brunet-Derrida in Our Setup

BD1 *The analogue of the* **superposability** *is*

$$N_* \stackrel{d}{=} W^{1/\alpha} \circ \mathcal{Q}$$

where W is the martingale limit and $\mathcal{Q}$ is a strictly α-stable point process. (Randomly scaled strictly α-stable point process).

BD2 *The analogous representation is randomly scaled scale-decorated Poisson point process (Randomly scaled DMZ representation).*

- *We have shown that BD1 and BD2 are equivalent (heavy-tailed extension of* **Subag and Zeitouni (2014)***).*

Domain of Attraction Theorem

- *Recall that $\mathcal{M}$ set of point measures, is a complete, separable metric space equipped with vague metric.*

Domain of Attraction Theorem

- *Recall that $\mathcal{M}$ set of point measures, is a complete, separable metric space equipped with vague metric.*
- *One can define regular variation for measures on $\mathcal{M}$ using works of* **Hult and Lindskog(2006)**.

Domain of Attraction Theorem

- *Recall that $\mathcal{M}$ set of point measures, is a complete, separable metric space equipped with vague metric.*

- *One can define regular variation for measures on $\mathcal{M}$ using works of* **Hult and Lindskog(2006)**.

Theorem (Bhattacharya, H., Roy (2015))

Let $\mathcal{L}$ be a point process on S. Suppose $\mathcal{L}$ is $RV_{-\alpha}$, that is,

$$nP(b_n^{-1} \circ \mathcal{L} \in \cdot) \stackrel{HL}{\to} \mu_\alpha(\cdot).$$

Then

$$b_n^{-1} \circ \sum_{i=1}^{n} \mathcal{L}_i \Rightarrow \text{ Strictly } \alpha\text{-stable Point Process}$$

Further extensions and questions

- *Is there a continuum analogue of this model like Branching Brownian motion?*
- *does the solution satisfy certain analogue of F-KPP equation?*
- *What happens when the branching random variable has infinite mean?*
- *Large deviations? Critical case?*

Thank You

- **Point process convergence for branching random walks with regularly varying steps:** *Annales de l'Institut Henri Poincaré, 2017*

- **Branching random walks, stable point processes and regular variation:** *Stochastic Processes and Applications, 2017*