Unknotting Number of some knots

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Outline

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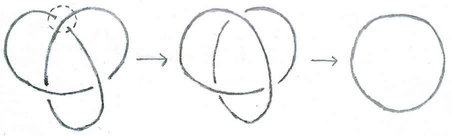
Objective

Here we find unknotting number of some knots by showing that each of them lie in an unknotting sequence of torus knot. First we find *Minimal Unknotting Crossing Data* (MUCD) for all torus knots and then by changing random sub-data from MUCD we indentify the resultant knot and its unknotting number.

Preliminaries

- **Unknotting Number:**

  The unknotting number $u(K)$ of a knot K is the minimum number of crossing changes required, taken over all knot diagrams representing K, to convert K into the trivial knot.

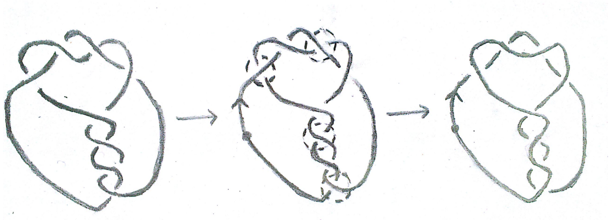


- Unknotting number for torus knot $K(p, q)$ is $(p-1)(q-1)/2$.[1]

[1] P. Kronheimer and T. Mrowka, Gauge theory for embedded surfaces, I, Topology, 32 , 773-826. 1993.

Preliminaries– Ascending and Descending diagram

- **Ascending diagram(Descending diagram):**
  Let K be a knot and $\tilde{D}$ be a based oriented knot diagram of K. Starting from the basepoint of $\tilde{D}$, proceed in the direction specified by the orientation and change each crossings that we first encountered as an over-crossing (under-crossing). The resulted diagram, denoted by $a(\tilde{D})$ ($d(\tilde{D})$) for $\tilde{D}$, is the descending diagram (ascending diagram).



- Ascending and Descending diagrams for any knot are trivial knot diagrams.

Preliminaries– Ascending and Descending diagram

- We can convert $K(p, q)$ into an ascending or descending diagram with $(p-1)(q-1)/2$ crossing changes.

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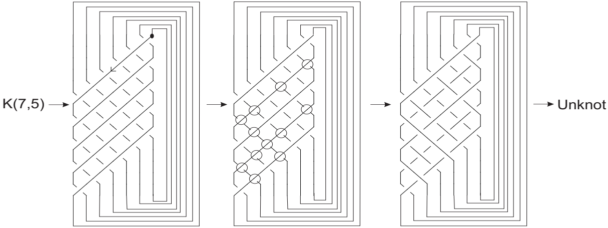
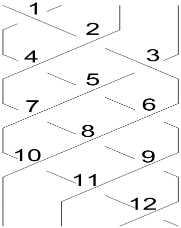


Preliminaries

- **Crossing data:** A crossing data for any n-braid β_n, denoted by $[1, 2, \dots, k]$, is a finite sequence of natural numbers enclosed in a bracket, given to the crossings starting from the first crossing from the top to the last crossing at the bottom, based on the braid representation of β_n using elementary braids $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$.



Preliminaries

- **Unknotting crossing data:**

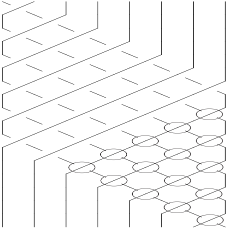
  An unknotting crossing data for an $n-$braid, β_n, is a subsequence to the crossing data of β_n such that if we make crossing change at these crossing positions then the closure of this braid is equivalent to unknot or unlink.

- **Minimal unknotting crossing data:** If the number of elements in an unknotting crossing data of a braid β_n is equal to the unknotting number of K (where K is the closure of β_n), then this unknotting crossing data is known as minimal unknotting crossing data for both the braid β_n and the knot K.

Preliminaries

- **U-Crossing data denoted by $U(B(p,q))$:**

When $p > q$

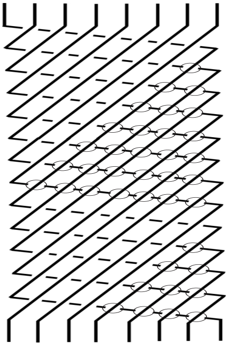


$B(8,6)$

$$U(B(8,6)) = [14, 20, 21, 26, 27, 28, 32, 33, 34, 35, 38, 39, 40, 41, 42]$$

Preliminaries

When $p < q$



$B(8, 13)$

$U(B(8, 13)) = [14, 20, 21, 26, 27, 28, 32, 33, 34, 35, 38, 39, 40, 41, 42, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 70, 76, 77, 82, 83, 84, 88, 89, 90, 91]$

Results

Theorem 1: For every n, the $(n+1)$-braid

$$\underbrace{\sigma_1\sigma_2\cdots\sigma_n}_{1}\underbrace{\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_n^{-1}}_{2}\underbrace{\sigma_1\sigma_2\cdots\sigma_{n-1}^{-1}\sigma_n^{-1}}_{3}\cdots\underbrace{\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_n^{-1}}_{n+1}$$

is a trivial braid.

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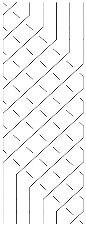

$B(7,7)$

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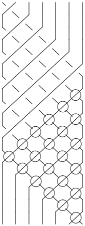

$B(7,7)$

Results

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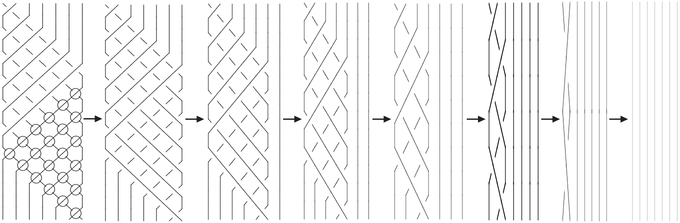


$B(7,7)$

Results

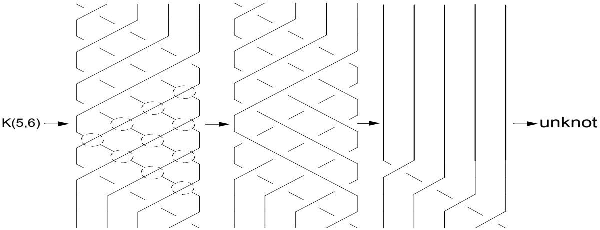
Theorem 2: Let $K(p,q)$ be a torus knot with $(p,q)=1$. If

$$q \equiv 1 \text{ or } p-1 \ (mod \ p),$$

then the $U-$crossing data for $B(p,q)$ is a minimal unknotting crossing data for $B(p,q)$ (or $K(p,q)$).

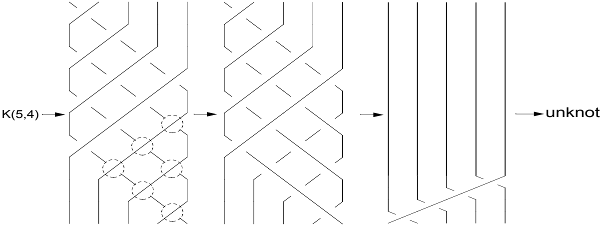
Example: when $q \equiv 1 \ (mod \ p)$

- Unknotting procedure for $K(5,6)$.



Example: when $q \equiv p - 1 \ (mod \ p)$

- Unknotting procedure for $K(5,4)$.



Results

Theorem 3: Let $K(p,q)$ be a torus knot with $(p,q)=1$. Then, the following statements are equivalent:

- Unknotting number of $K(p,q)$ is same as the number of elements in the $U-$crossing data of $B(p,q)$.
- $q \equiv 1 \text{ or } p-1 \ (mod \ p)$
- The $U-$crossing data of $K(p,q)$ is equal to minimal unknotting crossing data of $K(p,q)$.

Results

If we consider any torus knot $K(p,q)$ with $(p,q)=1$, then $q \equiv a \ mod \ p \Rightarrow q = mp + a$ for some non-negative integer m, and some $a < p$. Now, we observe that

$$B(p,q) = (\sigma_1\sigma_2\cdots\sigma_{p-1})^{mp+a}$$

$$= (\sigma_1\sigma_2\cdots\sigma_{p-1})^{mp} \cdot (\sigma_1\sigma_2\cdots\sigma_{p-1})^{a}$$

$$= (\underbrace{\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}}\cdots\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}}}_{p-factors})^{m} \cdot (\sigma_1\sigma_2\cdots\sigma_{p-1})^{a}.$$

Now, in the above braid $B(p,q)$, if we apply the crossing changes corresponding to U-crossing data for

$$(\underbrace{\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}}\cdots\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}}}_{p-factors})^{m},$$

Results

we get

$$(\eta_0\eta_1\cdots\eta_{p-1})^m\cdot(\sigma_1\sigma_2\cdots\sigma_{p-1})^a,$$

where

$\eta_0=\sigma_1\sigma_2\cdots\sigma_{p-1};\ \eta_1=\sigma_1\sigma_2\cdots\sigma_{p-1}^{-1};\ \ldots,\ \eta_{p-1}=\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{p-1}^{-1}.$

Since $\eta_0\eta_1\cdots\eta_{p-1}$ is equivalent to a trivial braid,

$$(\eta_0\eta_1\cdots\eta_{p-1})^m\cdot(\sigma_1\sigma_2\cdots\sigma_{p-1})^a\equiv(\sigma_1\sigma_2\cdots\sigma_{p-1})^a$$

Results

Our aim is to find an unknotting crossing data in $B(p,a)$, such that the number of elements in this unknotting crossing data for $B(p,a)$ is equal to the unknotting number of $K(p,q) \searrow$ number of crossings changes in $(\eta_0\eta_1\cdots\eta_{p-1})^m$

$$= \frac{(p-1)(q-1)}{2} - m\sum_{i=0}^{p-1} i = \frac{(p-1)(a-1)}{2},$$

which is the unknotting number for $K(p,a)$.

Results

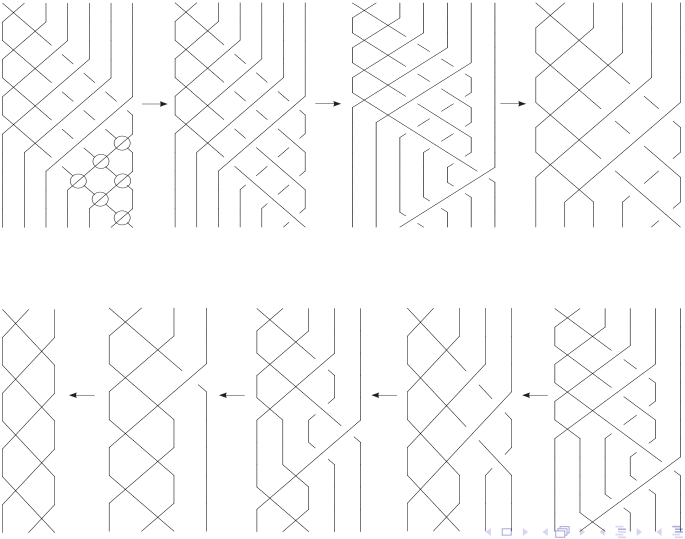
Theorem 4: For every $p-$braid, where $p > a$ for some a, we have

$$\underbrace{\eta_1\sigma_{p-a}\sigma_{p-a+1}\cdots\sigma_{p-1}}_{1}\underbrace{\eta_2\sigma_{p-a}\sigma_{p-a+1}\cdots\sigma_{p-1}^{-1}}_{2}\cdots\underbrace{\eta_a\sigma_{p-a}\sigma_{p-a+1}^{-1}\cdots\sigma_{p-1}^{-1}}_{a}$$

$$\sim_M \eta_1\eta_2\cdots\eta_a$$

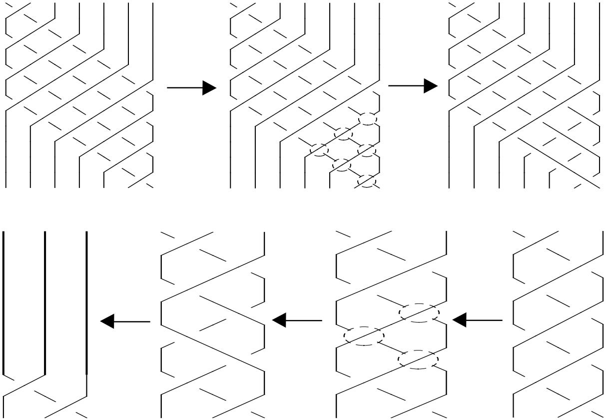
where, $\eta_i = \sigma_1^{e_{i,1}}\sigma_2^{e_{i,2}}\cdots\sigma_{p-a-1}^{e_{i,n-1}}$ $(1 \leq i \leq a)$, $e_{i,j}$ is either 1 or -1.

Results



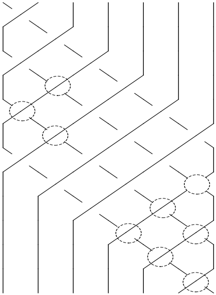
Example

- Unknotting procedure for $B(7, 4)$



Example

- Minimal Unknotting Crossing Data for $B(7,4)$



Minimal Unlinking Data For Torus Links

Theorem 5: U-crossing data for $B(p,p)$ is a minimal unknotting crossing data for $B(p,p)$(or $K(p,p)$).
Proof: It follows from Theorem 1.1, that $U(B(p,p))$ is an unknotting crossing data for $B(p,p)$. Note that, the unknotting number of $K(p,p)$ is

$$\frac{(p-1)(p-1)+(p,p)-1}{2} = \frac{(p-1)p}{2}.$$

and

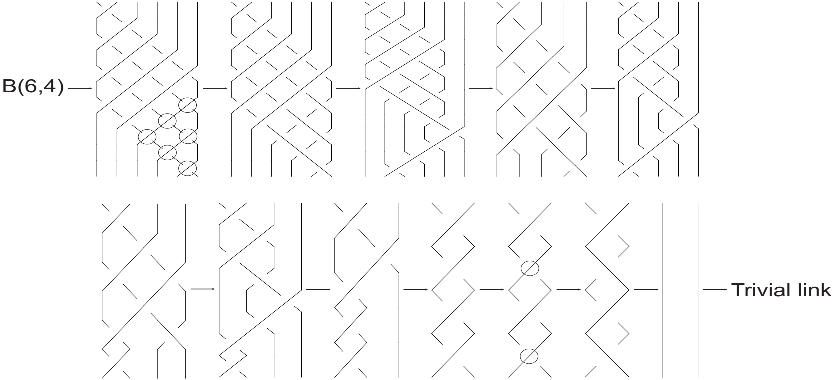
$$|U(B(p,p))| = \sum_{i=0}^{p-1} i = \frac{(p-1)p}{2},$$

are same.

Minimal Unlinking Data For Torus Links

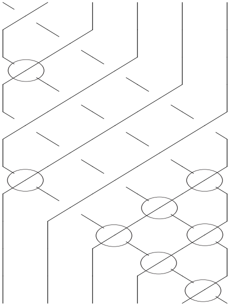
- From Theorem 4 and Theorem 5, we can find minimal unknotting crossing data for all torus links same as torus knots.

Unknotting procedure for torus link $K(6,4)$.



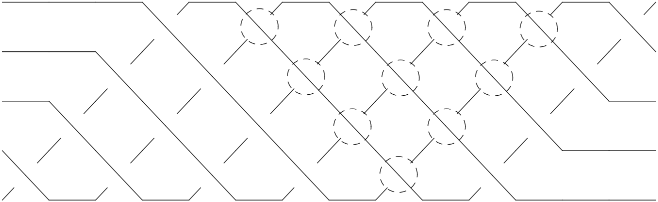
Minimal Unlinking Data For Torus Links

Minimal unknotting crossing data for torus link $K(6,4)$

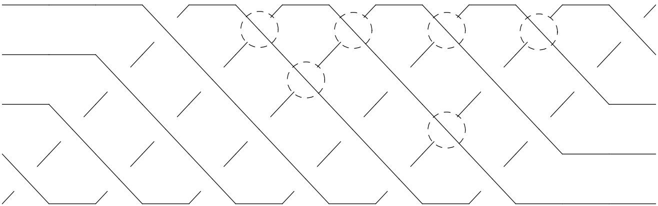


$B(6,4)$

MUCD for K(5,6)



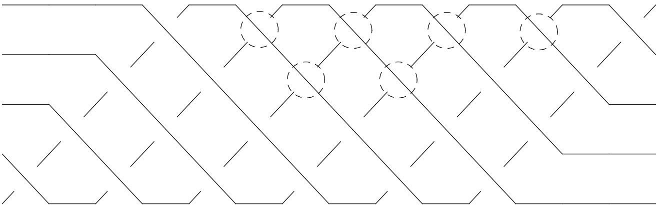
Unknotting number of different knots



If we change $\{8, 11, 12, 16, 18, 20\}$ crossings in $K(5, 6)$, we get ${}_n14_{14274}$. So

$$u({}_n14_{14274}) = 4$$

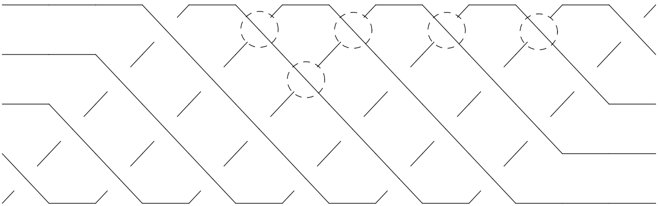
Results



If we change $\{8, 11, 12, 15, 16, 20\}$ crossings in $K(5, 6)$, we get ${}_n14_{18351}$. So

$$u({}_n14_{18351}) = 4$$

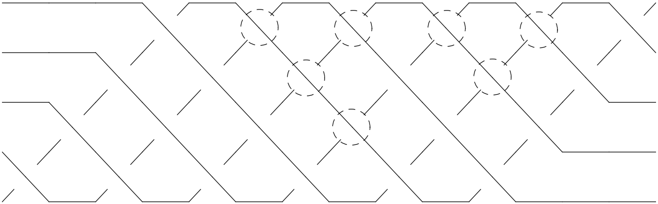
Results



If we change $\{8, 11, 12, 16, 20\}$ crossings in $K(5, 6)$, we get ${}_n14_{24498}$.
So

$$u({}_n14_{24498}) = 5$$

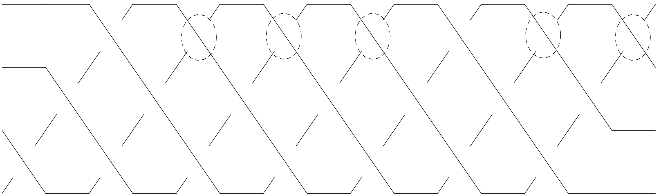
Results



If we change $\{8, 11, 12, 14, 16, 19, 20\}$ crossings in $K(5,6)$, we get 9_3. So

$$u(9_3) = 3$$

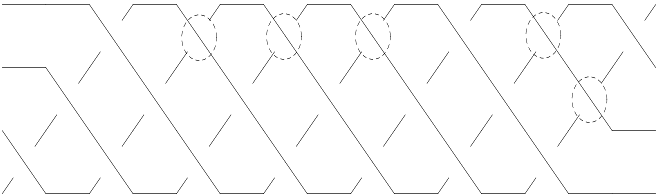
Results



If we change $\{6, 9, 12, 18, 21\}$ crossings in $K(4, 7)$, we get ${}_n12_{417}$.
So

$$u({}_n12_{417}) = 4$$

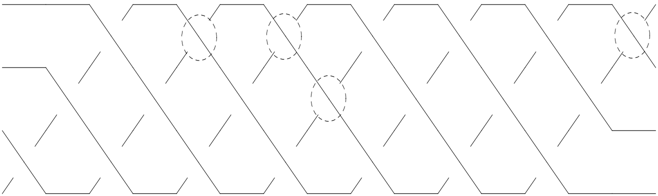
Results



If we change $\{6, 9, 12, 18, 20\}$ crossings in $K(4, 7)$, we get 10_{124}. So

$$u(10_{124}) = 4$$

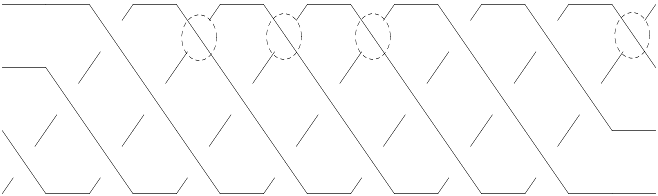
Results



If we change $\{6, 9, 11, 21\}$ crossings in $K(4, 7)$, we get ${}_n13_{604}$.
So

$$u({}_n13_{604}) = 5$$

Results



If we change $\{6, 9, 11, 21\}$ crossings in $K(4,7)$, we get ${}_n14_{17191}$.
So

$$u({}_n14_{17191}) = 5$$

Results

Table: Summary of results obtained

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
${}_n10_{21}$	4	$K(4,7)$	$6, 9, 12, 18, 20$
${}_n12_{417}$	4	$K(4,7)$	$6, 9, 12, 18, 21$
${}_n13_{604}$	5	$K(4,7)$	$6, 9, 11, 21$
${}_n14_{17191}$	5	$K(4,7)$	$6, 9, 12, 21$
${}_a9_{38}$	3	$K(5,6)$	$8, 11, 12, 14, 16, 19, 20$
${}_n14_{14274}$	4	$K(5,6)$	$8, 11, 12, 16, 18, 20$
${}_n14_{18351}$	4	$K(5,6)$	$8, 11, 12, 15, 16, 20$
${}_n14_{24498}$	5	$K(5,6)$	$8, 11, 12, 16, 20$

References

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Thank You