Lecture: Thermal emission from expanding gas

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Physical constants and astrophysical units.

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\begin{array}{ll}
G & 6.67384 \times 10^{-8}\ \mathrm{cm}^3\ \mathrm{g}^{-1}\mathrm{s}^{-2} \\
c & 2.99792458 \times 10^{10}\ \mathrm{cm\ s}^{-1} \\
h & 6.626070040 \times 10^{-27}\ \mathrm{erg\ s} \\
\hbar & 1.054571628 \times 10^{-27}\ \mathrm{erg\ s} \\
m_p & 1.6726217 \times 10^{-24}\ \mathrm{g} \\
m_u & 1.6605389 \times 10^{-24}\ \mathrm{g} \\
m_e & 9.10938291 \times 10^{-28}\ \mathrm{g} \\
e & 4.80320425 \times 10^{-10}\ \mathrm{erg}^{1/2}\mathrm{cm}^{1/2} \\
\alpha = \dfrac{e^2}{\hbar c} & \dfrac{1}{137.035999139} \\
\sigma_{\mathrm{T}} = \dfrac{8\pi e^4}{m_e^2 c^4} & 6.6524574 \times 10^{-25}\ \mathrm{cm}^2 \\
a_{\mathrm{B}} = \dfrac{\hbar}{m_e c \alpha} & 5.2917721067 \times 10^{-9}\ \mathrm{cm} \\
k_{\mathrm{B}} & 1.3806488 \times 10^{-16}\ \mathrm{erg\ K}^{-1} \\
\sigma_{\mathrm{SB}} & 5.6704 \times 10^{-5}\ \mathrm{erg\ cm}^{-2}\mathrm{s}^{-1}\mathrm{K}^{-4} \\
a & 7.5657 \times 10^{-15}\ \mathrm{erg\ cm}^{-3}\mathrm{K}^{-4} \\
G_F/(\hbar c)^3 & 1.1663787 \times 10^{-5}\ \mathrm{GeV}^{-2} \\
M_\odot & 1.9884 \times 10^{33}\ \mathrm{g} \\
GM_\odot & 1.32712440018 \times 10^{26}\ \mathrm{cm}^3\mathrm{s}^{-2} \\
R_\odot & 6.955 \times 10^{10}\ \mathrm{cm} \\
L_\odot & 3.828 \times 10^{33}\ \mathrm{erg/s} \\
\mathrm{Jy} & 10^{-23}\ \mathrm{erg\ s}^{-1}\mathrm{cm}^{-2}\mathrm{Hz}^{-1} \\
\mathrm{AU} & 1.495978707 \times 10^{13}\ \mathrm{cm} \\
\mathrm{pc} & 3.08568 \times 10^{18}\ \mathrm{cm}
\end{array} \tag{1}
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1 Thermodynamics of expanding gas

Let us consider the first law of thermodynamics of a spherical expanding gas with internal energy of E and radius of R,

$$dE = TdS - pdV = TdS - (\gamma - 1)\frac{E}{V}dV, \tag{2}$$

where we use an ideal gas equation of state with an adiabatic index of γ. The first term of the right-hand side of the equation is the energy change due to the change in entropy of the system. For instance, this term is responsible for the radioactive heating and radiative cooling. Suppose that the gas expands with a constant velocity v. The change in volume can be written as $dV/V = 3dt/t$, thereby we obtain

$$\frac{dE}{dt} = T\frac{dS}{dt} - (\gamma - 1)\frac{3E}{t}, \tag{3}$$

$$= T\frac{dS}{dt} - \frac{E}{t}, \tag{4}$$

In the last equality, we assume that the gas pressure is dominated by radiation ($\gamma = 4/3$).

For astrophysical explosions, the system often cools down due to the radiative cooling. We write this effect explicitly in the thermodynamics equation:

$$\frac{dE}{dt} = -\frac{E}{t} - L_{\rm rad} + \dot{Q}(t), \tag{5}$$

where $L_{\rm rad}$ is the radiative cooling, and $\dot{Q}$ is the heating.

For the case that the radiative cooling is very efficient, i.e., the internal energy is radiated away on much shorter time compared to one dynamical time. Then the heating balances with the radiative cooling so

$$L_{\rm rad} \approx \dot{Q}(t). \tag{6}$$

On the other hand, when the radiative cooling is not efficient enough, one needs to solve the thermodynamic equation to obtain the evolution of the internal energy. Now let us consider a situation where the gas is optically thick so that only a small fraction of photons escape from the system within one dynamical time, t. The diffusion length of photons in the gas is estimated as

$$l_{\rm diff} \sim \sqrt{Dt} \sim \sqrt{\frac{ct}{\kappa\rho}} \tag{7}$$

where $D \sim l_{\rm mfp}c$ is the diffusion coefficient, ρ is the density, and κ is the opacity. The timescale on which a photon produced at the center of the gas diffuses out from the gas, $t_{\rm rad}$, is simply given by $l_{\rm diff}(t_{\rm rad}) \sim R$, or equivalently:

$$t_{\rm rad} \sim \frac{3\kappa M}{4\pi tcv} \equiv \frac{t_{\rm diff}^2}{t}, \tag{8}$$

where M is the total mass of the gas. Therefore the radiative cooling luminosity is estimated as

$$L_{\rm rad} = \frac{E}{t_{\rm rad}}. \tag{9}$$

Now we wish to solve an equation

$$\frac{dE}{dt} = -\frac{E}{t} - \frac{E}{t_{\rm rad}} + \dot{Q}(t). \tag{10}$$

Note that this equation is no longer valid for $t_{\rm diff} \ll t$, where $t_{\rm diff}$ is the time scale on which photons created at the center diffuse out in one dynamical time.

The homogeneous form of equation (10) is

$$\frac{dE}{d\tilde{t}} = -\frac{E}{\tilde{t}} - \tilde{t}E, \tag{11}$$

where $\tilde{t} \equiv t/t_{\rm diff}$. One can easily find the general solution of equation (11):

$$E(\tilde{t}) = \frac{Ce^{-\frac{1}{2}\tilde{t}^2}}{\tilde{t}}. \tag{12}$$

The particular solution with the heating term is

$$E(\tilde{t}) = \frac{e^{-\frac{1}{2}\tilde{t}^2}}{\tilde{t}} \int_0^{\tilde{t}} \tilde{t}' e^{\frac{1}{2}\tilde{t}'^2} t_{\rm diff} \dot{Q}(t') d\tilde{t}'. \tag{13}$$

Suppose the gas is ejected at an initial radius R_0 and initial internal energy $E_0 \sim Mv^2$ at $t_0 = R_0/v$. We have

$$E(\tilde{t}_0) \approx \frac{C}{\tilde{t}_0} = E_0. \tag{14}$$

so

$$C = E_0 \frac{R_0}{vt_{\rm diff}}. \tag{15}$$

Using this coefficient, we obtain the solution with an initial condition as

$$E(t) = \frac{R_0}{R(t)} E_0 e^{-t^2/2t_{\rm diff}^2} + \frac{e^{-t^2/2t_{\rm diff}^2}}{t} \int_{t_0}^{t} dt' t' e^{\frac{1}{2}t'^2/t_{\rm diff}^2} \dot{Q}(t'). \tag{16}$$

The first term describes the adiabatic cooling $\propto R^{-1}$ and radiative cooling $e^{-t^2/2t_{\rm diff}^2}$ of the initial internal energy. Using $L_{\rm rad} = E/t_{\rm rad} = tE/t_{\rm diff}^2$, we obtain

$$L(t) = \frac{R_0}{R_{\rm diff}} \frac{E_0}{t_{\rm diff}} e^{-t^2/2t_{\rm diff}^2} + \frac{e^{-t^2/2t_{\rm diff}^2}}{t_{\rm diff}^2} \int_{t_0}^{t} dt' t' e^{\frac{1}{2}t'^2/t_{\rm diff}^2} \dot{Q}(t'). \tag{17}$$

1.1 Neutron star merger ejecta

Neutron star mergers eject a mass of $\sim 0.01M_\odot$ with a velocity of $\sim 0.2c$. When the ejecta is fully composed of heavy r-process elements, the opacity is expected to be $\kappa = 10\,{\rm cm^2/g}$, which is dominated by the bound-bound transitions of atomic electrons. The diffusion time is

$$t_{\rm diff} \approx 6\,{\rm day} \left(\frac{M}{0.01M_\odot}\right)^{1/2} \left(\frac{v}{0.2c}\right)^{-1/2} \left(\frac{\kappa}{10\,{\rm cm^2/g}}\right)^{1/2}. \tag{18}$$

The corresponding diffusion radius is $\sim 3 \cdot 10^{15}$ cm. The initial radius of the expanding ejecta is $\sim 10^7$ cm and the initial internal energy associated with the ejection is $\sim Mv^2 \sim 10^{51}$ erg. The luminosity associated with the initial energy is

$$L_{\rm init} \sim 10^{37}\,{\rm erg/s} \left(\frac{E_{\rm init}}{10^{51}{\rm erg}}\right) \left(\frac{R_{\rm init}}{10^7{\rm cm}}\right) \left(\frac{M}{0.01M_\odot}\right)^{-1} \left(\frac{\kappa}{10\,{\rm cm^2/g}}\right)^{-1}. \tag{19}$$

The radioactive heating rate is roughly $0.5 \cdot 10^{10} t_d^{-4/3}$ erg/s/g so that the peak luminosity associated with the radioactive heating is

$$L_{\rm peak} \approx 10^{40}\,{\rm erg/s} \left(\frac{M}{0.01M_\odot}\right)^{1/3} \left(\frac{v}{0.2c}\right)^{2/3} \left(\frac{\kappa}{10\,{\rm cm^2/g}}\right)^{-2/3}. \tag{20}$$

This luminosity is much larger than the luminosity associated with the initial internal energy of the ejecta unless the ejecta is shocked at somewhere large radius.

Finally, the effective temperature at the peak is given by

$$T_{\rm eff} = \left(\frac{L_{\rm peak}}{4\pi\sigma R^2}\right)^{1/4}, \tag{21}$$

$$\approx 1000\,{\rm K} \left(\frac{M}{0.01M_\odot}\right)^{-1/6} \left(\frac{v}{0.2c}\right)^{-1/12} \left(\frac{\kappa}{10\,{\rm cm^2/g}}\right)^{-5/12}. \tag{22}$$

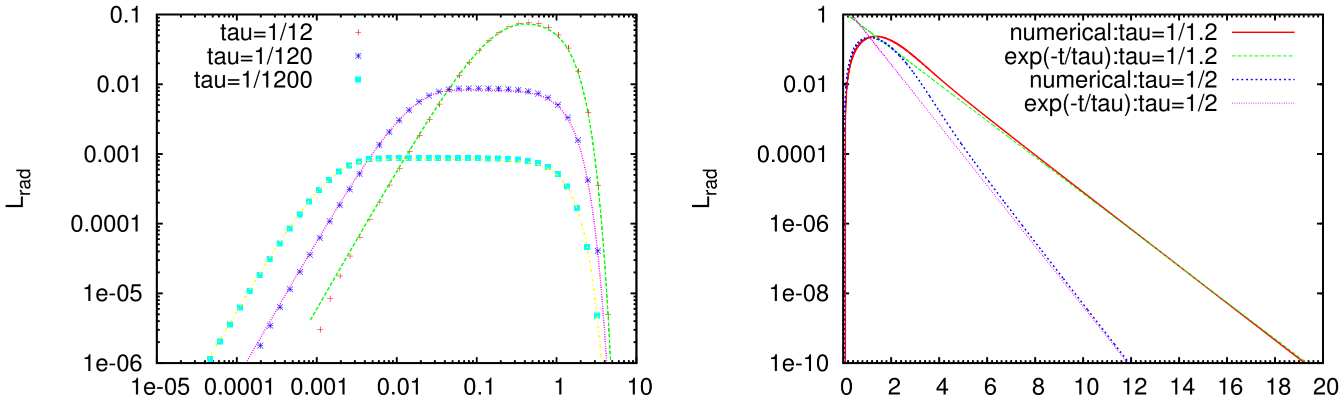

Figure 1: The bolometric luminosity for the exponential heating with $\tilde{\tau} = 1/12,\ 1/120,\ 1/1200$. The points show the results of numerical integration of Eq. (13) and the lines show the analytic expression of Eq. (25).

1.2 Exponential decay

Now the nuclear heating rate is assumed to be

$$\dot{E}_{\rm nuc}(\tilde{t}) = \frac{E_0 e^{-\frac{\tilde{t}}{\tilde{\tau}}}}{\tilde{\tau}}, \tag{23}$$

where $\tilde{\tau} \equiv \tau/t_{\rm diff}$. In the range of $\tilde{t} \ll 2/\tilde{\tau}$, $\tilde{t}^2$ is smaller than $\tilde{t}/\tilde{\tau}$ in the exponent of the integrand of Eq. (13). Neglecting $\tilde{t}^2$ term of the integrand, the integral can be solved analytically and then the solution is

$$E(\tilde{t}) \approx \frac{E_0 e^{-\frac{1}{2}\tilde{t}^2}}{\tilde{t}} \left(\tilde{\tau}^2 - \left(\tilde{\tau}\tilde{t} + \tilde{\tau}^2\right) e^{-\frac{\tilde{t}}{\tilde{\tau}}}\right). \tag{24}$$

The bolometric luminosity of the radiation from the ejecta is given by

$$L_{\rm rad}(\tilde{t}) \approx E_0 e^{-\frac{1}{2}\tilde{t}^2} \left(\tilde{\tau}^2 - \left(\tilde{\tau}\tilde{t} + \tilde{\tau}^2\right) e^{-\frac{\tilde{t}}{\tilde{\tau}}}\right). \tag{25}$$

$$L_{\rm rad}(\tilde{t}) \approx \begin{cases} E_0 \tilde{t}^2 & (\tilde{t} \ll \tilde{\tau}), \\ E_0 \tilde{\tau}^2 e^{-\frac{\tilde{t}^2}{2}} & (\tilde{t} \gg \tilde{\tau}). \end{cases} \tag{26}$$

1.3 Power-low decay

Let us now assume the nuclear heating rate to be a power-law as

$$\dot{Q}(t) = \dot{E}_{\rm nuc}(\tilde{t}) = \frac{E_0}{(a-1)\tilde{t}_0} \left(\frac{\tilde{t}}{\tilde{t}_0}\right)^{-a} \theta(\tilde{t} - \tilde{t}_0), \tag{27}$$

where $\tilde{t}_0$ corresponds to an initial time of the nuclear heating. Here we assume $1 < a < 2$. For the time $\tilde{t} \ll 1$, the integral can be done analytically as

$$E(\tilde{t}) \approx \frac{E_0 \tilde{t}_0 e^{-\frac{\tilde{t}^2}{2}}}{(2-a)\tilde{t}} \left(\frac{\tilde{t}}{\tilde{t}_0}\right)^{2-a}, \tag{28}$$

and thus, the bolometric luminosity is

$$L_{\rm rad}(\tilde{t}) \approx \frac{E_0 \tilde{t}_0 e^{-\frac{\tilde{t}^2}{2}}}{(2-a)} \left(\frac{\tilde{t}}{\tilde{t}_0}\right)^{2-a}. \tag{29}$$

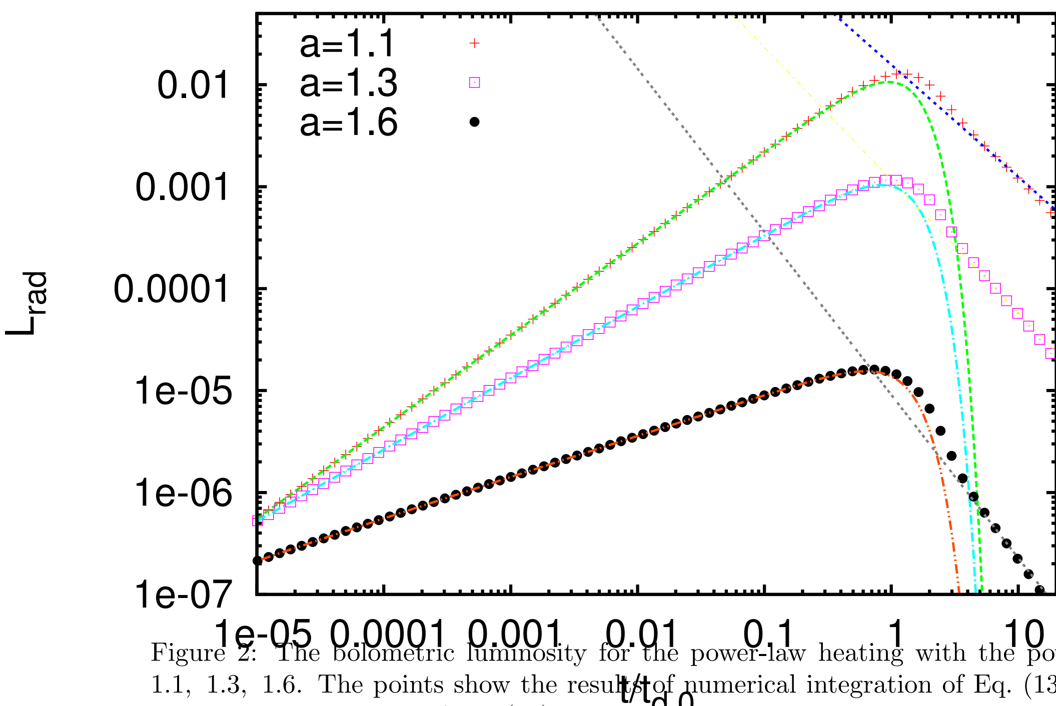

Figure 2: The bolometric luminosity for the power-law heating with the power-low indexes $a = 1.1,\ 1.3,\ 1.6$. The points show the results of numerical integration of Eq. (13) and the lines show the analytic expression of Eq. (29).