

Aspects of Lifshitz scaling in string theory

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- Lifshitz scaling, gravity duals, AdS null deformations
- Lifshitz scaling, hyperscaling violation
- Lifshitz singularities in string constructions

[arXiv:1005.3291, [Koushik Balasubramanian, KN](#), 1103.1279, [KN](#),
1202.5935, [KN](#), 1204.3506, [KN](#), work in progress, ...]

[see also AdS/CFT cosmo singularities: hep-th/0602107, 0610053, [Das, Michelson, KN, Trivedi](#);
arXiv:0711.2994, [Awad, Das, KN, Trivedi](#); arXiv:0807.1517, [Awad, Das, Nampuri, KN, Trivedi](#).]

Holography and Lifshitz scaling

Interesting to explore holography with reduced symmetries.

Generalizations of AdS/CFT to nonrelativistic systems

→ holographic condensed matter, ...

[Son; Balasubramanian,McGreevy; Adams et al; Herzog et al; Maldacena et al; ...]

→ Phases of string theory with non-relativistic symmetries.

Lifshitz symmetries: t, x_i -translations, x_i -rotations,

scaling $t \rightarrow \lambda^z t, x_i \rightarrow \lambda x_i$ [z : dynamical exponent].

[smaller than Galilean symmetries: *e.g.* Galilean boosts broken]

Landau-Ginzburg action (free $z = 2$ Lifshitz): $S = \int d^3x ((\partial_t \varphi)^2 - \kappa (\nabla^2 \varphi)^2)$.

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Landau-Ginzburg action (free $z = 2$ Lifshitz): $S = \int d^3x ((\partial_t \varphi)^2 - \kappa (\nabla^2 \varphi)^2)$.

Lifshitz spacetime: $ds_4^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2}$. Kachru,Liu,Mulligan

Scaling: $t \rightarrow \lambda^z t, x_i \rightarrow \lambda x_i, r \rightarrow \lambda r$ [$z = 1$: AdS].

Solution to 4-dim gravity with $\Lambda < 0$ and massive gauge field $A \sim \frac{dt}{r^z}$
(or alternatively gauge field + 2-form: dualize to get A -mass) Taylor

[Focus on zero temperature solutions]

Lifshitz scaling in string theory

Many natural guesses appear difficult to realize explicitly.

String construction for $z = 2$ Lifshitz [Balasubramanian,KN]:

x^+ -DLCQ of relativistic $\mathcal{N}=4$ SYM \longrightarrow

$z = 2$ nonrelativistic (Galilean) 2+1-dim system.

Gauge coupling $g_{YM}^2(x^+) = e^{\Phi(x^+)}$ varying in lightlike x^+ -direction
 \longrightarrow breaks x^+ -shift reducing to 2+1-dim Lifshitz symmetries.

Concrete bulk realization: null deformations of $AdS_5 \times S^5$ (more generally, $AdS \times X$) sourced by lightlike scalar (*e.g.* dilaton in IIB).

$$ds_{Einst}^2 = \frac{1}{r^2} [-2dx^+ dx^- + dx_i^2 + \frac{1}{4}r^2 (\partial_+ \Phi)^2 (dx^+)^2] + \frac{dr^2}{r^2} + d\Omega_5^2.$$

Lightlike dilaton: $\Phi = \Phi(x^+)$ (also 5-form). [Das,KN,Trivedi,et al]

[The scalar Φ could also arise from the compactification: other interpretations, *e.g.* fermion condensate etc? (recall Hartnoll,Polchinski,Silverstein,Tong, ...)]

AdS null deformations and Lifshitz

$$ds_{Einstein}^2 = \frac{1}{r^2} [-2dx^+ dx^- + dx_i^2 + \frac{1}{4} r^2 (\partial_+ \Phi)^2 (dx^+)^2] + \frac{dr^2}{r^2} + d\Omega_5^2, \quad \Phi = \Phi(x^+).$$

$\Phi = const$ i.e. $g_{++} = 0$: DLCQ x^+ of AdS in lightcone coordinates
— nonrelativistic, Schrodinger (Galilean) symmetries

[Goldberger, Barbon et al, Maldacena et al].

Regard $x^- \equiv t$ (time), $x^+ \equiv$ compact coordinate ($g_{++} \sim (\Phi')^2 > 0$).

Strictly: $x^+ = const \equiv$ null surfaces and $x^- = const$ surfaces spacelike ($g^{--} < 0$).

Symmetries, 2+1-dim Lifshitz: x^-, x_i -translations, x_i -rotations,
 $z = 2$ scaling $x^- \equiv t \rightarrow \lambda^2 t, x_i \rightarrow \lambda x_i, r \rightarrow \lambda r$ (x^+ , no scaling).

x^+ compact \Rightarrow lightlike boosts broken.

Galilean boosts $x_i \rightarrow x_i - v_i x^-, x^+ \rightarrow x^+ - \frac{1}{2}(2v_i x_i - v_i^2 x^-)$:

broken by g_{++} . Also broken $z = 2$ special conformal symmetry.

Nontrivial x^+ -dependence $\Rightarrow z = 2$ Galilean broken to Lifshitz.

AdS null deformations, Lifshitz

$$ds_{Einstein}^2 = \frac{1}{r^2} [-2dx^+ dx^- + dx_i^2 + \frac{1}{4} r^2 (\partial_+ \Phi)^2 (dx^+)^2] + \frac{dr^2}{r^2} + d\Omega_5^2, \quad \Phi = \Phi(x^+).$$

- (Das,KN,Trivedi,et al) Coord. transfmn. $r = w e^{-f/2}, \quad x^- = y^- - \frac{r^2 f'}{4} \rightarrow$

$$ds^2 = \frac{1}{w^2} [e^{f(x^+)} (-2dx^+ dy^- + dx_i^2) + dw^2] + d\Omega_5^2, \quad \Phi(x^+).$$

EOM: $R_{++} = \frac{1}{2} (f')^2 - f'' = \frac{1}{2} (\Phi')^2$. Function-worth of solutions (any $\Phi(x^+)$).

- Dual: 4-d $\mathcal{N}=4$ super Yang-Mills theory with gauge coupling

lightlike-deformed $g_{YM}^2(x^+) = e^{\Phi(x^+)} \rightarrow \text{DLCQ}_{x^+}$.

Lightlike (chiral) deformation \Rightarrow various physical observables (*e.g.* trace anomaly, anomalous dims) unaffected. 2-point correlator (conf. coords): operators \mathcal{O} dual to massive scalars $\varphi \rightarrow$

$$\langle \mathcal{O}(x_i) \mathcal{O}(x'_i) \rangle \sim \frac{1}{[\sum_i (\Delta x_i)^2]^\Delta}, \quad \text{and} \quad \langle \mathcal{O}(t) \mathcal{O}(t') \rangle \sim \frac{1}{(\Delta x^-)^\Delta}.$$

Agrees with equal-time 2-pt fn of 2 + 1-dim Lifshitz theory (Kachru,Liu,Mulligan).

Calculation difficult in form with $g_{++} \neq 0$: scalar wave eqn not straightforward to solve (later).

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- Can replace $S^5 \rightarrow X^5$ (Sasaki-Einstein). Preserve half susy.

These solutions can be generalized to a large family of solutions with axion-dilaton, 3-form field strength turned on (Donos,Gauntlett, various groups, ...).

- There also exist $AdS_4 \times X^7$ null deformations in M-theory, with scalar arising from G-flux on X^7 : presumably dual to lightlike deformations of Chern-Simons (ABJM-like) theories arising on M2-brane stacks. Also AdS_7 null deformations from M5-branes.

More on AdS null defmns, Lifshitz

$$ds^2 = \frac{1}{r^2} [-2dx^+ dx^- + dx_i^2 + \frac{1}{4} r^2 (\Phi')^2 (dx^+)^2] + \frac{dr^2}{r^2} + d\Omega_5^2, \quad \Phi = \Phi(x^+).$$

- Long wavelength geometry seen by massless bulk scalar: Lifshitz.

$$\left[\text{action } S = \frac{1}{G_5} \int d^5x \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \text{ for modes with no } x^+ \text{-dependence } (\partial_+ \varphi = 0), \right.$$

$$S \rightarrow \frac{1}{G_5} \int \frac{d^4x}{r^5} \left[-r^4 \left(\frac{\int dx^+ (\Phi')^2}{4} \right) (\partial_- \varphi)^2 + r^2 L (\partial_i \varphi)^2 + r^2 L (\partial_w \varphi)^2 \right]$$

$$L: \text{compactification size, rescale } x^- \rightarrow \sqrt{\frac{L}{\int dx^+ (\Phi')^2}} x^- .]$$

- Standard Kaluza-Klein x^+ -reduction does not work here in general: nontrivial x^+ -dependence thro' $(\Phi')^2$ means no clear separation of scales. However an argument for dimensional reduction of “off-shell” metric family containing these solutions suggests long wavelength geometry is $z = 2$ Lifshitz.

- $\Phi' = \text{const}$: **linear dilaton**. Not x^+ -periodic. Compactify in a “non-geometric” way, upto S-duality of IIB theory.

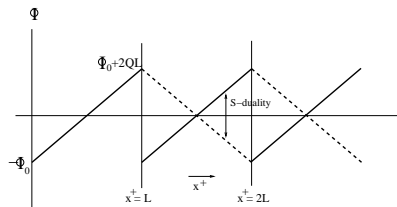
This can also be done in a linear axion solution.

(Interesting to study the noncompact solution independently: later.)

Non-geometric DLCQ

Linear dilaton not x^+ -periodic. Compactify upto S-duality: **KN**

$\Phi(x^+)$ continuous, not periodic: note S-duality symmetry, $\tau \rightarrow -\frac{1}{\tau}$, *i.e.* $\Phi \rightarrow -\Phi$.



Piecewise linear dilaton: $\Phi = \Phi_0 + 2Qx^+$, $x^+ \in [0, L]$,

$\Phi = \Phi(L) - 2Q(x^+ - L)$, $x^+ \in [L, 2L]$, ...

Φ periodic upto S-duality if $\Phi(x^+ + L) = -\Phi(x^+)$, *i.e.*

$$\Phi(L) = -\Phi(0) \Rightarrow \Phi_0 + 2QL = -\Phi_0, \quad \text{i.e.} \quad g_s = e^{\Phi_0} = e^{-QL}.$$

[Einstein metric $ds^2 = \frac{1}{r^2}(-2dx^+dx^- + dx_i^2 + r^2Q^2(dx^+)^2 + dr^2)$ smooth.]

Asymptotic string coupling fixed: large string corrections?

Lightlike deformations of $AdS_5 \times S^5$: no nonzero contractions (only $\partial_+ \Phi$ nonzero, $g^{++} = 0$), likely no higher derivative corrections. Preserves half susy. Dilaton bounded, no singularities.

Lower dim Lifshitz symmetries \rightarrow lightlike structure in 5-dim, possibly controlled corrections.

\Rightarrow non-geometric construction for dual $\mathcal{N}=4$ SYM too.

Linear axion: $c_0 = c_0^0 + 2Qx^+$, $ds^2 = ds_{AdS \times S}^2 + (\partial_+ c_0)^2(dx^+)^2$.

$(x^+ \rightarrow x^+ + L)$: $c_0 \rightarrow c_0 + 2QL$ maps to $\tau \rightarrow \tau + 1$ shift if $QL = \frac{1}{2}$.

Axion \rightarrow θ -angle in dual gauge theory.

Balasubramanian, McGreevy: Lifshitz-Chern-Simons gauge theory duals for some of these.

x^+ -noncompact, anisotropic Lifshitz

$$ds^2 = \frac{1}{r^2} [-2dx^+ dx^- + dx_i^2 + \frac{1}{4}r^2 (\Phi')^2 (dx^+)^2] + \frac{dr^2}{r^2} + d\Omega_5^2,$$

$$\Phi = \Phi(Qx^+), \quad Q \text{ parameter of mass dim one.}$$

Symmetries: time x^- -translations, spatial x_i -translations/rotations (x^+ -translations broken by nontrivial x^+ -dependence), and anisotropic

Lifshitz scaling $r \rightarrow \lambda r$, $x_i \rightarrow \lambda x_i$, $x^- \rightarrow \lambda^2 x^-$, $x^+ \rightarrow \lambda^0 x^+$,

$$i.e. \quad z = 2 \quad \{x_i, x^-\}: \quad x^- \rightarrow \lambda^2 x^-, \quad x_i \rightarrow \lambda x_i,$$

$$z = \infty \quad \{x^+, x^-\}: \quad x^- \rightarrow \lambda^2 x^-, \quad x^+ \rightarrow \lambda^0 x^+.$$

In addition, dilaton $\Phi(Qx^+)$ acts as spatial x^+ -potential.

$$[\text{Recall } z = 0 \text{ Schrodinger systems: } ds^2 = -dt^2 + \frac{dx_i^2 + dt d\xi + dr^2}{r^2}.$$

Here $x^- \equiv$ time (const- x^- surface is spacelike).]

Also recall D3-D7 anisotropic Lifshitz systems + radial scalars ([Azeyanagi, Li, Takayanagi](#)).

Linear dilaton, anisotropic Lifshitz

$$ds^2 = \frac{1}{r^2} [-2dx^+ dx^- + dx_i^2 + r^2 Q^2 (dx^+)^2] + \frac{dr^2}{r^2}, \quad \Phi = \Phi_0 + 2Qx^+.$$

Now metric also has x^+ -translations: linear dilaton acts as spatial x^+ -potential.

Dual gauge theory: d=4 $\mathcal{N}=4$ SYM theory living on flat spacetime (flat boundary), gauge coupling $g_{YM}^2(x^+) = g_s e^{2Qx^+}$ ($g_s = e^{\Phi_0}$).
 $x^+ \rightarrow -\infty$: weakly coupled gauge theory.

Consistent: string frame metric $ds_{str}^2 = e^{\Phi/2} ds^2$ highly curved.

Momentum space 2-pt correlator (operators dual to bulk scalars):

[Lorentzian calculation, closed form bulk-bndry propagator involving conf.hypergeom.fns.]

$$\begin{aligned} \langle \mathcal{O}(k) \mathcal{O}(-k) \rangle &\sim -\nu 2^\nu \alpha^\nu \frac{\Gamma(-\nu)}{\Gamma(\nu)} \frac{\Gamma(\frac{1+\nu}{2} + \frac{k^2}{4iQk_-})}{\Gamma(\frac{1-\nu}{2} + \frac{k^2}{4iQk_-})} \quad (\nu \notin \mathbb{Z}), \\ &\sim ((k_i^2 - 2k_+ k_-)^2 + 4Q^2 k_-^2) [\log(iQk_-) + \psi(\frac{3}{2} + \frac{k_i^2 - 2k_+ k_-}{4iQk_-})] \\ &(\nu = 2 \ (\Delta = 4)). \quad [k^2 = -2k_+ k_- + k_i^2, \ \Delta = 2 + \sqrt{4 + m^2} = 2 + \nu]. \end{aligned}$$

Structurally similar to **Kachru,Liu,Mulligan**: for $k_+ \sim 0$, (after Euclidean continuation) this agrees with **KLM**, as expected from dim.redn. of 5-d background to Lifshitz.

Linear dilaton, Lifshitz, correlators

$$\langle \mathcal{O}(k)\mathcal{O}(-k) \rangle \sim -\nu 2^\nu \alpha^\nu \frac{\Gamma(-\nu)}{\Gamma(\nu)} \frac{\Gamma(\frac{1+\nu}{2} + \frac{k^2}{4iQk_-})}{\Gamma(\frac{1-\nu}{2} + \frac{k^2}{4iQk_-})} \quad (\nu \notin \mathbb{Z}),$$

$$\sim ((k_i^2 - 2k_+k_-)^2 + 4Q^2k_-^2) (\log(iQk_-) + \psi(\frac{3}{2} + \frac{k_i^2 - 2k_+k_-}{4iQk_-})). \quad [\Delta = 4]$$

[Consistent with Lifshitz scaling: $(x^-, x^+, x_i) \rightarrow (\lambda^2 x^-, \lambda^0 x^+, \lambda x_i)$.]

[For $Q \rightarrow 0$, $\psi() \rightarrow \log()$: recover AdS 2-point correlation fn $k^4 \log k^2$.]

Note: $(k_i^2 - 2k_+k_-)^2 + 4Q^2k_-^2 = (k_i^2 - 2(k_+ - iQ)k_-)(k_i^2 - 2(k_+ + iQ)k_-)$

Effective x^+ -momentum shift: $k_+ \rightarrow k_+ \pm iQ$ ($e^{ik_+x^+} \rightarrow e^{ik_+x^+} e^{\pm Qx^+}$)

— reminiscent of Liouville-like wall in $c = 1$ string theory.

Also, for $k_i = 0$: $(k_i^2 - 2k_+k_-)^2 + 4Q^2k_-^2 \rightarrow (k_+^2 + Q^2)k_-^2$,

i.e. effective mass-gap in x^+ -direction.

Free SYM: $S = \int \frac{d^4x}{g_{YM}^2(x^+)} Tr F^2 \rightarrow \int d^4x e^{-\Phi(x^+)} [(\partial_j A_i)^2 - 2(\partial_+ A_i)(\partial_- A_i)]$.

Wave modes $e^{ik_+x^+ + ik_-x^- + ik_i x^i} \rightarrow k_i^2 + 2(k_+ + iQ)k_- = 0$, *i.e.*

$k_+ = -\frac{k_i^2}{2k_-} - iQ$. For generic k_i, k_- , x^+ -momentum k_+ nonzero, *i.e.*

generic waves move along x^+ -direction due to dilaton x^+ -potential.

Recap: $z = 2$ Lifshitz, null defmns

$d = 4$ $\mathcal{N}=4$ super Yang-Mills theory with gauge coupling
lightlike-deformed as $g_{YM}^2(x^+) = e^{\Phi(x^+)} \longrightarrow \text{DLCQ}_{x^+}$.

Lightlike deformation \rightarrow break Galilean symmetries \rightarrow Lifshitz.

$AdS_5 \times S^5 \rightarrow$ null deformations with lightlike dilaton $\rightarrow \text{DLCQ}_{x^+}$:

$$ds^2 = \frac{1}{r^2}[-2dx^+dx^- + dx_i^2] + \frac{1}{4}(\Phi')^2(dx^+)^2 + \frac{dr^2}{r^2} + d\Omega_5^2, \quad \Phi = \Phi(x^+).$$

$z = 2$ scaling $x^- \equiv t \rightarrow \lambda^2 t$, $x_i \rightarrow \lambda x_i$, $r \rightarrow \lambda r$ (x^+ compact, no scaling).

Note: so far $g_{++} \sim \frac{1}{r^2}(r^2)$ — non-normalizable null deformation.

Normalizable null deformations of AdS ? e.g. $g_{++} \sim \frac{1}{r^2}(r^4)$ [AdS_5].

Dim'nal reduction \longrightarrow Hyperscaling violation effects. **KN**

Hyperscaling violation

$$ds^2 = \frac{R^2}{r^2} [-2dx^+ dx^- + dx_i^2 + dr^2] + R^2 Q r^2 (dx^+)^2 + R^2 d\Omega_5^2$$

AdS shock wave: $R^4 \sim g_{YM}^2 N \alpha'^2$, $Q \sim$ energy-momentum density.

Has appeared in literature in many places.

[recently: **Singh**, “zero temperature” double-scaling limit of boosted black D3-branes]

(5-d) Scaling: $x_i \rightarrow \lambda x_i$, $r \rightarrow \lambda r$, $x^- \rightarrow \lambda^3 x^-$, $x^+ \rightarrow \lambda^{-1} x^+$

$$ds^2 = R^2 \left(-\frac{dt^2}{Qr^6} + \frac{dx_i^2 + dr^2}{r^2} + Qr^2 \left(dx^+ - \frac{dt}{Qr^4} \right)^2 \right) \rightarrow \text{dim.redn.}_{x^+} \rightarrow$$

4-dim Einstein metric ($x^- \equiv t$): $ds_E^2 = \frac{R^3 \sqrt{Q}}{r} \left(-\frac{dt^2}{Qr^4} + dx_i^2 + dr^2 \right)$,

Electric gauge field $A = -\frac{dt}{Qr^4}$, scalar $e^\phi \sim r$. Nontrivial IR scales R, Q .

$$[ds^2 = g_{\mu\nu}^D dx^\mu dx^\nu + h(x^\mu) d\sigma_{D_I}^2 \rightarrow ds_E^2 = h^{D_I/(D-2)} g_{\mu\nu}^D dx^\mu dx^\nu].$$

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$$[ds^2 = g_{\mu\nu}^D dx^\mu dx^\nu + h(x^\mu) d\sigma_{D_I}^2 \rightarrow ds_E^2 = h^{D_I/(D-2)} g_{\mu\nu}^D dx^\mu dx^\nu].$$

$$ds^2 = r^{2\theta/d} \left(-\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2} \right).$$

$d =$ “boundary” spatial dim (x_i).

$\theta =$ hyperscaling violation exponent.

Conformal to Lifshitz. Arise in effective gravity+vector+scalar theories.

Thermodynamics reflects effective space dim ($d - \theta$) [*e.g.* $S \sim T^{(d-\theta)/z}$].

Above: $d = 2$, $z = 3$, $\theta = 1$ ($d_{eff} = d - \theta = 1$).

(4-d) scaling: $t \rightarrow \lambda^z t$, $x_i \rightarrow \lambda x_i$, $r \rightarrow \lambda r$, $ds \rightarrow \lambda^{\theta/d} ds$ (Lifshitz symmetry broken)

Kachru et al: dim’nal reduction of black Dp -branes $\rightarrow z = 1, \theta \neq 0$ solutions.

Hyperscaling violation

$$ds^2 = \frac{R^2}{r^2} [-2dx^+ dx^- + dx_i^2 + dr^2] + R^2 Q r^2 (dx^+)^2 + R^2 d\Omega_5^2$$

AdS shock wave: $R^4 \sim g_{YM}^2 N \alpha'^2$, $Q \sim$ energy-momentum density. \rightarrow dim.redn. _{x^+}

- Using Ryu-Takayanagi expression, $\theta = d - 1$ gives logarithmic violation of entanglement entropy.

Gravitational dual of hidden Fermi surfaces? Takayanagi et al, Sachdev et al

- Holographic stress tensor calculated upstairs using standard

AdS/CFT prescription: $T_{++} = \frac{2Q}{8\pi G_5} \longrightarrow$ wave on boundary.

Lower dimensional point of view? Definition for holographic renormalization downstairs from upstairs definition?

- Often in condensed matter literature on critical phenomena, violation of hyperscaling relations is due to dangerously irrelevant operator that ruins simple scaling relations. Connections to present context?

Gauge theory interpretation: nontrivial shock wave state in CFT \longrightarrow dim'nal reduction \longrightarrow residual nontrivial IR scales. Explore ...

Hyperscaling violation

$$ds^2 = \frac{R^2}{r^2} [-2dx^+ dx^- + dx_i^2 + dr^2] + R^2 Q r^{D-3} (dx^+)^2$$

AdS_D shock wave, $D = d + 3$, Q energy-momentum density, $(D - 1)$ -dim.

Scaling: $x_i \rightarrow \lambda x_i$, $r \rightarrow \lambda r$, $x^- \rightarrow \lambda^{2+d/2} x^-$, $x^+ \rightarrow \lambda^{-d/2} x^+$

Dim'nal redn:
$$ds_E^2 = \frac{R^2 (R^2 Q)^{1/(D-3)}}{r} \left(-\frac{dt^2}{Q r^{D-1}} + dx_i^2 + dr^2 \right).$$

“boundary” spatial dimension $d = D - 3$, $z = \frac{d}{2} + 2$, $\theta = \frac{d}{2}$.

[General dim'nal reduction: $\int d^D x \sqrt{-g^{(D)}} (R^{(D)} - 2\Lambda) = \int dx^+ d^{D-1} x \sqrt{-g^{(D-1)}} (R^{(D-1)} - \# \Lambda e^{-2\phi/(D-3)} - \# (\partial\phi)^2 - \# e^{2(D-2)\phi/(D-3)} F_{\mu\nu}^2)$
Lower dim electric gauge field $A = -\frac{dt}{r^{D-1}}$, scalar $g_{DD} = e^{2\phi} = r^{D-3}$.]

(recall effective gravity+vector+scalar theories, *e.g.* [Trivedi et al](#), [Kiritsis et al](#), [Takayanagi et al](#))

In particular, for the conformal branes of M-theory:

$M2$ -brane stacks $\rightarrow AdS_4$ deformations, $d = 1$, $z = \frac{5}{2}$, $\theta = \frac{1}{2}$.

$M5$ -brane stacks $\rightarrow AdS_7$ deformations, $d = 4$, $z = 4$, $\theta = 2$.

Phase structure of these M-brane solutions includes IIA string/sugra phases of D2, D4-brane solutions with null deformations (new z, θ).

(recall D_p -brane phases, [Itzhaki, Maldacena, Sonnenschein, Yankielowicz](#))

Hyperscaling violation

More generally (along **IMSY** lines): Conformal M-theory brane stack

$ds^2 = ds^2_{AdS \times S} + g_{++}(dx^+)^2$, for g_{++} (non-)normalizable deformations, gives rise in appropriate IIA phase to hyperscaling violation and new z, θ with $ds^2_E = r^{2\theta/d} \left(-\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2} \right)$.

$ds^2_{M-th} \rightarrow$ 10-d ds^2_{st} + dilaton $\Phi \rightarrow ds^2_E \rightarrow$ dim.redn.^{sphere S} \rightarrow null-deformed solution, non-compact $x^+ \rightarrow$ dim.redn. ^{x^+} $\rightarrow z, \theta$.

[Note: 10-d solutions can be checked independently in IIA supergravity.]

D p -branes $\rightarrow d = p, z = 1, \theta \neq 0$. (**Dong,Harrison,Kachru,Torroba,Wang**)

M2 \rightarrow D2 null non-normalizable: $d = 1, z = \frac{7}{3}, \theta = 0$.

M2 \rightarrow D2 null normalizable: $d = 1, z = 3, \theta = \frac{2}{3}$.

M5 \rightarrow D4 null non-normalizable: $d = 3, z = 2, \theta = -1$.

M5 \rightarrow D4 null normalizable: $d = 3, z = 4, \theta = \frac{1}{3}$.

Note: some of these (with $d = 1$) have $d - 1 \leq \theta \leq d \rightarrow$

Violation of area law of entanglement entropy. Field theory duals nontrivial: explore?

Hyperscaling violation

(Following Dp -brane phase structure of [Itzhaki, Maldacena, Sonnenschein, Yankielowicz](#)):

These solutions are of the form of null-deformed Dp -brane systems, with rich phase structure. z, θ -values flow.

Null-deformed D2-brane phases: flow from $DLCQ_{x^+}$ of 2+1-dim perturbative SYM (UV) regime \rightarrow IIA supergravity region with null-deformed D2-solution (valid in some intermediate regime where x^+ -circle large relative to 11-th circle) \rightarrow 11-dim AdS_4 null-deformed M2-brane IR phase (dual to DLCQ of null-defmn of Chern-Simons ABJM-like theory).

Null-deformed D4-brane phases: flow from AdS_7 null-deformed M5-brane UV phase (dual to null-deformation of $(2, 0)$ M5-theory) \rightarrow intermediate IIA supergravity region with null-deformed D4-solution \rightarrow $DLCQ_{x^+}$ of 4+1-dim perturbative SYM IR phase.

Lifshitz singularities, string theory

Mild singularities present in Lifshitz spacetimes $ds_4^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2}$:
curvature invariants finite, diverging tidal forces as $r \rightarrow \infty$ (interior).

Expect that these are zero temperature limits of Lifshitz black holes with regular horizons: however zero temperature limit singular.

[Kachru et al: stringy cloak, de-singularizing horizon by higher derivative corrections.]

Singularities also reflected in above string constructions exhibiting exact Lifshitz symmetries (Horowitz,Way): origins? resolution?

$$ds^2 = \frac{1}{r^2} [-2dx^+ dx^- + dx_i^2] + \frac{1}{4} (\partial_+ \Phi)^2 (dx^+)^2 + \frac{dr^2}{r^2} + d\Omega_5^2, \quad \Phi(x^+).$$

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Recall coord. transfmn. $r = we^{-f/2}$, $x^- = y^- - \frac{r^2 f'}{4} \rightarrow$

$$ds^2 = \frac{1}{w^2} [e^{f(x^+)} (-2dx^+ dy^- + dx_i^2) + dw^2] + d\Omega_5^2, \quad \Phi(x^+).$$

$$\text{EOM: } R_{++} = \frac{1}{2} (f')^2 - f'' = \frac{1}{2} (\Phi')^2. \quad (\text{Das,KN,Trivedi et al})$$

This lies in general family of AdS -deformations (also 5-form)

$$ds_{Einst}^2 = \frac{R^2}{w^2} (\tilde{g}_{\mu\nu}(x^\mu) dx^\mu dx^\nu + dw^2) + R^2 d\Omega_5^2, \quad \Phi = \Phi(x^\mu),$$

solutions if $\tilde{R}_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi$, $\tilde{\square} \Phi \equiv \partial_\mu (\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\nu \Phi) = 0$.

Above null solutions: $\tilde{g}_{\mu\nu} = e^{f(x^+)} \eta_{\mu\nu}$.

Lifshitz singularities, string theory

$$ds_{Einstein}^2 = \frac{R^2}{w^2} (\tilde{g}_{\mu\nu}(x^\mu) dx^\mu dx^\nu + dw^2) + R^2 d\Omega_5^2, \quad \Phi = \Phi(x^\mu)$$

Potential singularities on Poincare horizon $w \rightarrow \infty$:

$$R_{ABCD} R^{ABCD} = w^4 \tilde{R}_{\mu\nu\alpha\beta} \tilde{R}^{\mu\nu\alpha\beta} + \dots \quad \text{diverges.}$$

For null solutions, invariants vanish: no nonzero contraction.

Diverging tidal forces as $w \rightarrow \infty$. [Horowitz, Way](#)

(Possible the metric tidal force calculation in IR of string construction is incomplete: sources.)

Holographic stress tensor ([Awad, Das, KN, Trivedi](#))

$$T^{\mu\nu} \sim K^{\mu\nu} - K h^{\mu\nu} - 3h^{\mu\nu} + \frac{1}{2} G^{\mu\nu} - \frac{1}{4} \partial^\mu \Phi \partial^\nu \Phi + \frac{1}{8} h^{\mu\nu} (\partial\Phi)^2$$

Vanishes identically for null solutions (either coords).

Recall: source for bulk field turned on \Rightarrow generically expect response.

Here metric source $\tilde{g}_{\mu\nu}$, so expect $\langle T^{\mu\nu} \rangle \neq 0$. At variance with above.

Requiring vanishing $T^{\mu\nu}$ gives constraint $\tilde{R}_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi$ above.

Maybe the usual *AdS* prescription needs to be modified for Lifshitz

boundary conditions (*e.g.* uplift of downstairs prescriptions, [Ross et al?](#)).

Lifshitz singularities, string theory

$$ds_{Einstein}^2 = \frac{R^2}{w^2} (\tilde{g}_{\mu\nu}(x^\mu) dx^\mu dx^\nu + dw^2) + R^2 d\Omega_5^2, \quad \Phi = \Phi(x^\mu)$$

Consider Fefferman-Graham expansion of asymptotically locally AdS space (Poincare coords) about boundary $w = 0$:

$$ds^2 = \frac{dw^2}{w^2} + \frac{1}{w^2} [g_{\mu\nu}^0(x^\mu) + w^2 g_{\mu\nu}^2(x^\mu) + \dots] dx^\mu dx^\nu$$

In this perspective, above solutions seem constrained: $g_{\mu\nu}^n = 0, n > 0$.

In general, with leading source $g_{\mu\nu}^0 = \tilde{g}_{\mu\nu}$, holographic RG methods give relations between $g_{\mu\nu}^n, \dots$, stress tensor etc. [Skenderis et al](#)

Iteratively solve $R_{MN} = -4g_{MN} + \frac{1}{2}\partial_M\Phi\partial_N\Phi$, using Fefferman-Graham expn for 5-d metric and massless scalar $\Phi = \Phi^0 + r^2\Phi^2 + \dots$:

$$g_{\mu\nu}^2 \sim R_{\mu\nu}^0 - \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2(d-1)}\left(R - \frac{1}{2}(\partial\Phi)^2\right)g_{\mu\nu}^0, \quad \Phi^2 \sim \square^0\Phi^0.$$

$$g_{\mu\nu}^2 = 0 \Rightarrow R_{\mu\nu}^0 = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi, \quad \Phi^2 = 0 \Rightarrow \square^0\Phi^0 = 0.$$

Higher order coefficients also vanish.

[Analogous arguments for AdS -cosmologies ([Das,KN,Trivedi et al](#)) and also for similar deformations of de Sitter space (Poincare slices) with late-time singularity ([1204.3506](#); [Das,KN, in progress](#)): initial conditions appear fine-tuned.]

Lifshitz singularities, string theory

$$ds_{Einstein}^2 = \frac{R^2}{w^2} (\tilde{g}_{\mu\nu}(x^\mu) dx^\mu dx^\nu + dw^2), \quad \Phi = \Phi(x^\mu) : \quad g_{\mu\nu}^0, \Phi^0 \text{ nonzero.}$$
$$g_{\mu\nu}^2 = 0 \Rightarrow R_{\mu\nu}^0 = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi, \quad \Phi^2 = 0 \Rightarrow \square^0 \Phi^0 = 0. \quad \text{Higher coeffs vanish.}$$

The above conditions on the Fefferman-Graham coefficients

$g_{\mu\nu}^n, \phi^n, n > 0$ vanishing in these string configurations are non-generic.

Appear to be nontrivial constraints fine-tuning dual gauge theory state.

[Earlier coordinates with $g_{++} \neq 0$? Hol. RG relations?]

This might suggest that Lifshitz state in these string constructions is a “constrained” state, leading to bulk singularity in the interior.

[Unclear if higher derivative corrections become large: dilaton regular.]

[Alternatively, “unconstrained” state: Lifshitz-like UV boundary, interior must be specified.]

Speculation: IR singularities in field theory? Lifshitz state unstable?

(recall Gregory-Laflamme).

Technical resolution: turn on the subleading coefficients $g_{\mu\nu}^n$.

Physical interpretations? Lifshitz symmetries break.

New (zero temperature) phases, perhaps with hyperscaling violation, clumped phases?

work in progress ...

Conclusions, open questions

- AdS null deformations \rightarrow dim'nal redux
 \rightarrow Lifshitz scaling, hyperscaling violation.

Lifshitz state appears constrained in string constructions.

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- Finite temperature?
- Lifshitz singularities in string constructions? Resolutions?
- Explore hyperscaling violation from field theory point of view.

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