Active Brownian Motion in 2d

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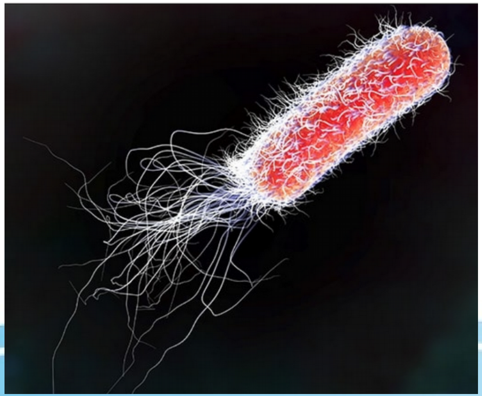
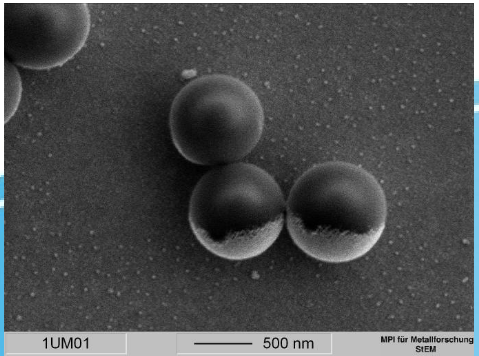
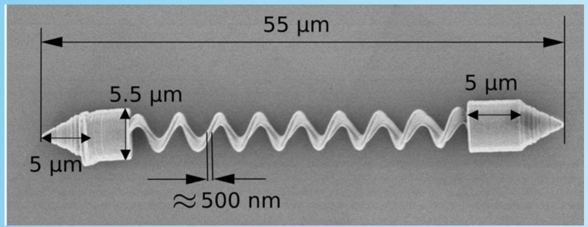
Active Matter

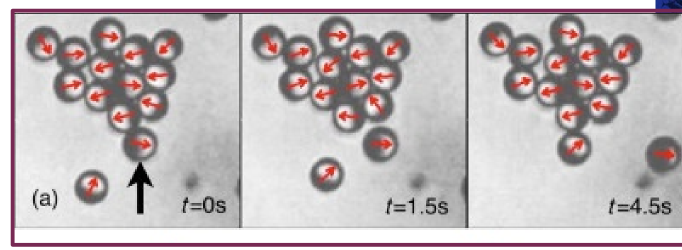
- Collection of self-propelled 'active' agents
- Generate directed motion consuming energy from environment
- Intrinsically driven *out of equilibrium*
- Relevant in a wide range of disciplines
- Many examples : nature and artificial







- Novel collective behaviour
  - Swarming/flocking
  - Motility induced phase separation
  - Absence of well defined pressure



- Even at the single particle level: Rich behaviour
  - Non-Boltzmann stationary state
  - Anomalous dynamics

Study a simple model ...

Active Brownian Particle (ABP)

- Overdamped particle moving with a constant speed v_0
- Two dimensions: position (x,y) and orientation φ
- Orientation undergoes rotational diffusion

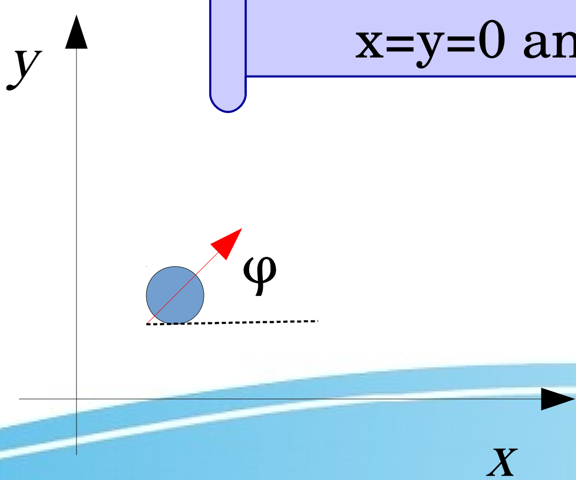
Langevin equations

$$
\begin{aligned}
\dot{x} &= v_0 \cos\phi(t) \\
\dot{y} &= v_0 \sin\phi(t) \\
\dot{\phi} &= \sqrt{2D_R}\,\eta_\phi(t)
\end{aligned}
$$

Rotational diffusion constant

White noise (delta-correlated)

Starting at origin, oriented along x: x=y=0 and φ=0



[Bechinger et. al. Rev Mod Phys 2016]

[Cates & Tailleur, EPL 2013]

and many more...

- Recast ABP dynamics

$$\dot{x} \;=\; \xi_x(t) = v_0 \cos\phi(t)$$

$$\dot{y} \;=\; \xi_y(t) = v_0 \sin\phi(t)$$

$$\boxed{\dot{\phi} \;=\; \sqrt{2D_R}\,\eta_\phi(t)}$$

- Effective noise is different than white-noise

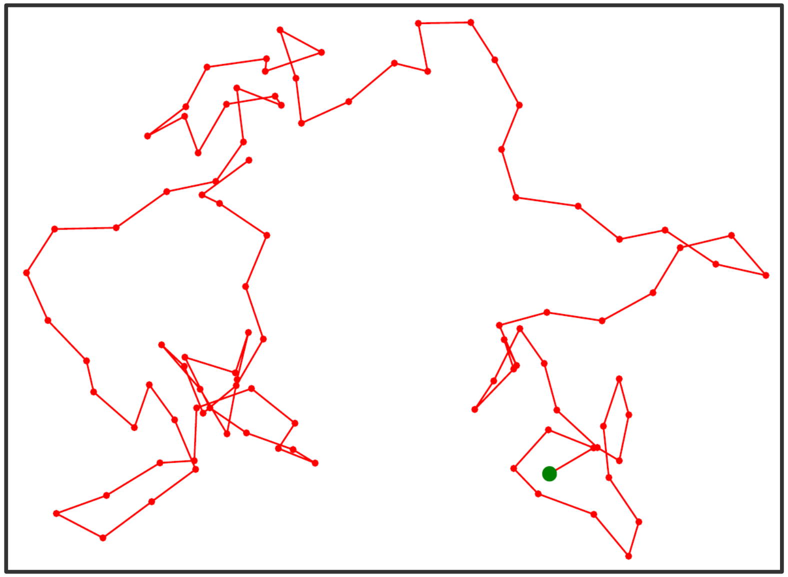
Bounded $\quad |\xi| \leq v_0$

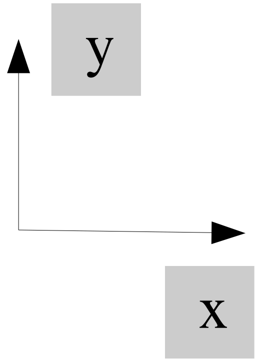
Correlated $\quad \langle \xi_x(t)\xi_x(t')\rangle \simeq v_0^2 e^{-D_R|t-t'|}$

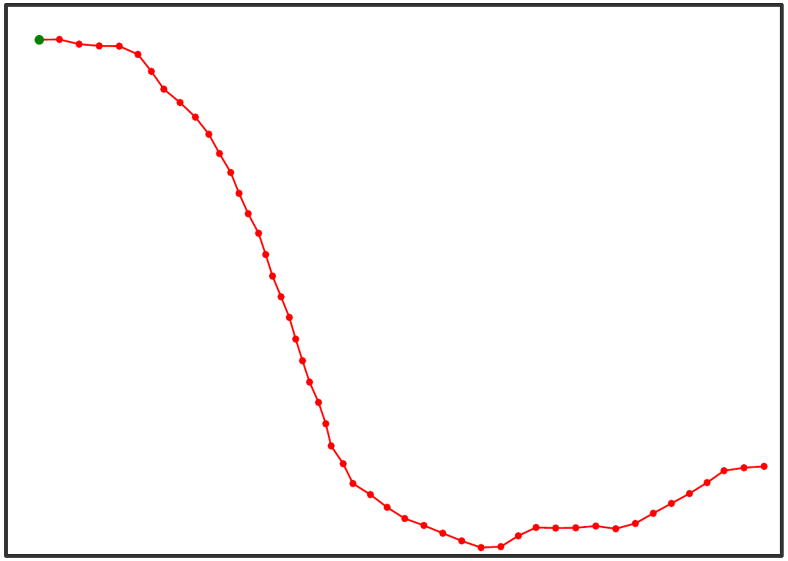
Similar form for the y-y and x-y noise correlations

Time-scale: $1/D_R$

Trajectories







Long time ($t \gg 1/D_R$)

Diffusive, similar to passive Brownian motion

Effective diffusion constant

$$\boxed{D_{\rm eff} = v_0^2/2D_R}$$

Short time ($t \ll 1/D_R$)

Very different:
signature of activity

- Explore and characterize ...

Short-time Regime ($t \ll 1/D_R$)

U.B., S. N. Majumdar, A. Rosso, G. Schehr, Phys. Rev. E 2018

Short-time: Anisotropy

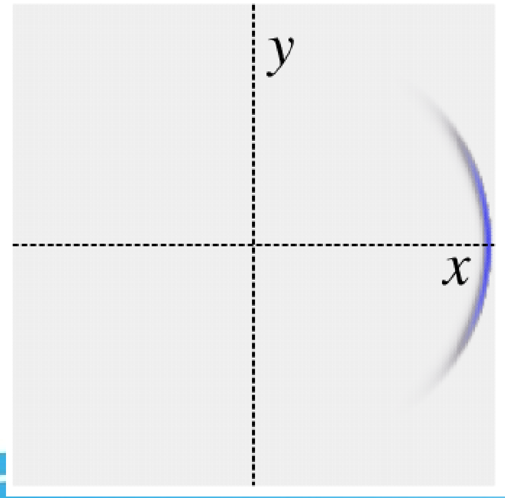
- Effective equation for short-time t << 1/D_R

$$\begin{aligned} \dot{x} &= v_0 \cos\phi(t) \\ \dot{y} &= v_0 \sin\phi(t) \\ \dot{\phi} &= \sqrt{2D_R}\,\eta_\phi(t) \end{aligned}$$

Small t,
small φ

$$\begin{aligned} \dot{x} &\approx v_0\left(1 - \frac{\phi^2}{2}\right) \\ \dot{y} &\approx v_0\phi \end{aligned}$$

Position
distribution
P(x,y;t)



D_R=0.1
t=1

Analyze marginal distributions
P(x,t) and P(y,t) separately

- Use Path Integral techniques ...

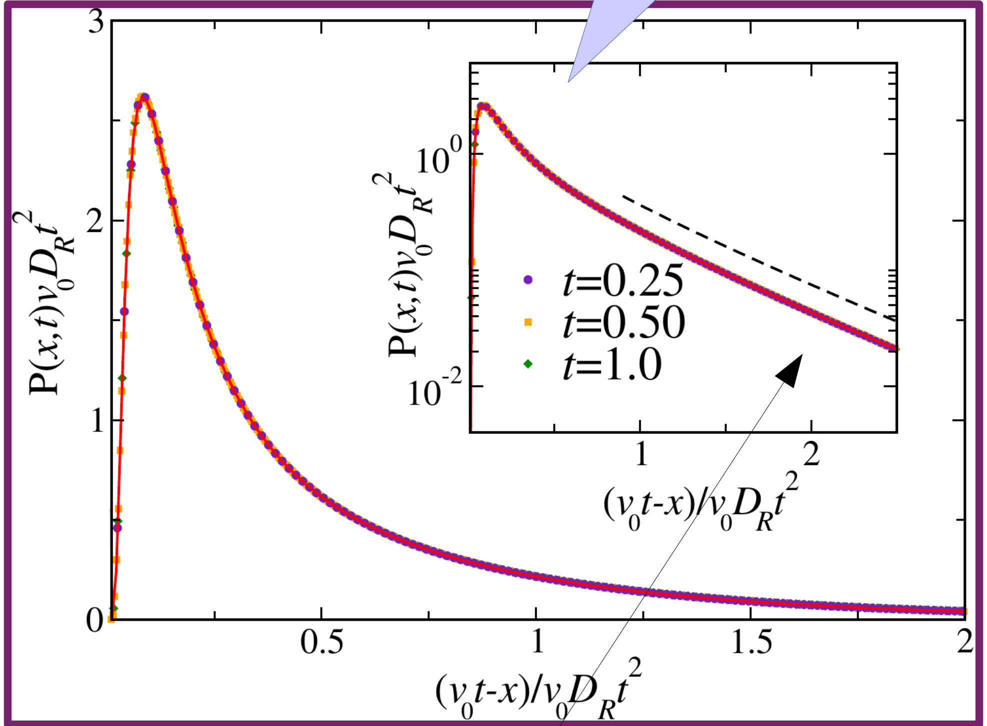
Short-time: P(x,t)

D_R=0.01
v_0=1

Exact probability distribution

$$P(x,t) = \frac{1}{v_0 D_R t^2} f\left(\frac{v_0 t - x}{v_0 D_R t^2}\right)$$

$$f(u) = \frac{1}{2\sqrt{\pi u^3}} \sum_{m=0}^{\infty} (-1)^m \frac{(4m+1)}{2^{2m}} e^{-\frac{(4m+1)^2}{8u}}$$



Asymptotes

$$\lim_{u\to 0} f(u) \simeq \frac{1}{2\sqrt{\pi u^3}} e^{-\frac{1}{8u}}$$

$$\lim_{u\to\infty} f(u) \simeq \frac{1}{\sqrt{2u}} e^{-\frac{\pi^2 u}{8}}$$

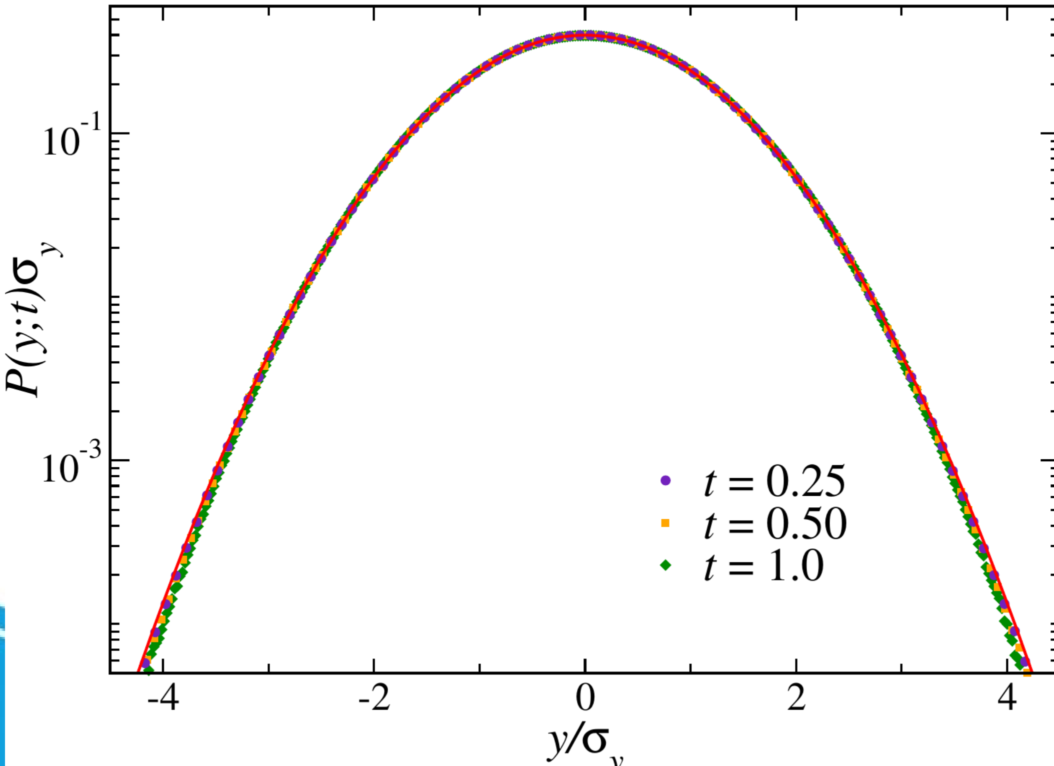
Variance: $\sigma_x^2 \approx \frac{1}{3} v_0^2 D_R^2 t^4$

Short-time: P(y,t)

$$\dot{y} \approx v_0 \phi$$

- Mapping to Random Acceleration Process
- Gaussian distribution

$$P(y,t) = \sqrt{\frac{3}{4\pi v_0^2 D_R t^3}} \exp\left[-\frac{3y^2}{4v_0^2 D_R t^3}\right]$$



Variance:

$$\sigma_y^2 \approx \frac{2}{3} v_0^2 D_R t^3$$

D_R=0.01

v_0=1

Short-time: Radial distribution

- Radial distance r:

$$\boxed{P(r^2,t) = \frac{1}{2D_R v_0^2 t^3} f_r\left(\frac{v_0^2t^2 - r^2}{2D_R v_0^2 t^3}\right)}$$

$$f_r(z) = \frac{1}{4\pi z^{3/2}} \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2\, 2^{2k}} \sqrt{4k+1}\, e^{-\frac{(4k+1)^2}{16z}} \left(\left(\frac{(4k+1)^2}{4z} - 1\right) K_{1/4}\left(\frac{(4k+1)^2}{16z}\right) + \frac{(4k+1)^2}{4z} K_{3/4}\left(\frac{(4k+1)^2}{16z}\right)\right)$$

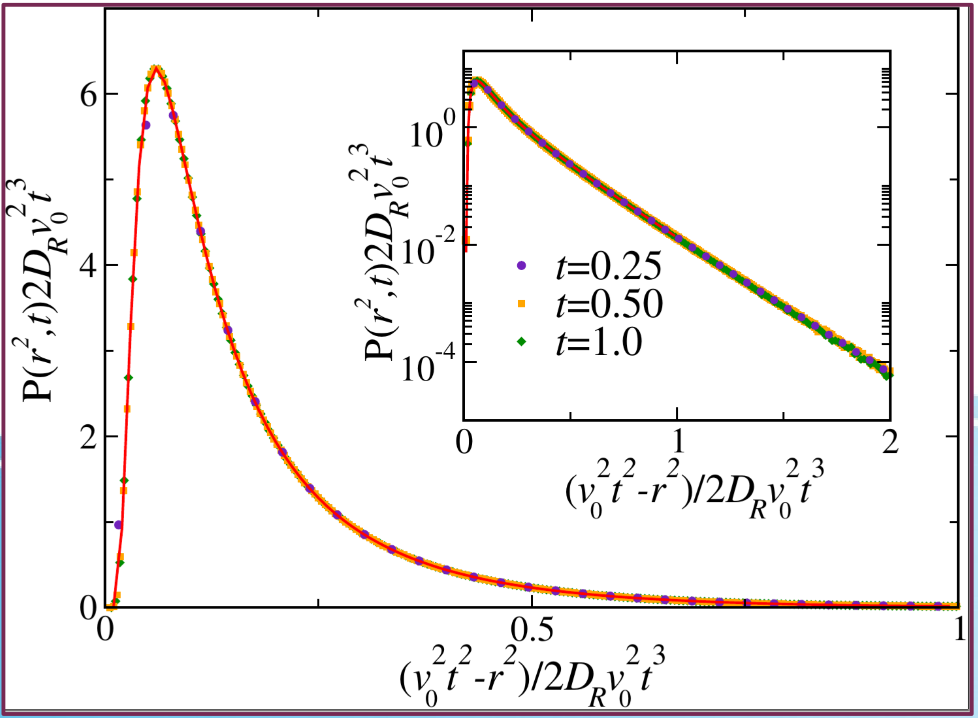
D_R=0.01

v_0=1

Asymptotes

$$f_r(z) \simeq \frac{1}{2\sqrt{2\pi}\, z^2}\, e^{-\frac{1}{8z}}, \quad \text{as} \quad z \to 0$$

$$f_r(z) \simeq \sqrt{\frac{\pi}{z}}\, e^{-\frac{\pi^2}{2} z} \quad \text{as} \quad z \to \infty$$



Summary so far...

- Strong signatures of activity at short-times

  Anisotropy

  Non-trivial position distribution

- Crossover to passive Brownian behaviour at times larger than $1/D_R$

How to see signatures of activity in the long-time regime?

Presence of external potential

- ABP diffusing in an external potential
  - Numerical studies
  - Experiments
- Strong signatures of activity at long-times: Non-Boltzmann stationary state
- Crossover to Boltzmann-like state

[Takatori et. al., Nat. Comm. 2015]

[Pototsky & Stark, EPL 2012] [Solon, Cates & Tailleur, EPJ Spl. Topics 2015]

Qualitative understanding, few exact results

[Malakar et. al., arXiv: 1902.04171]

ABP in a Harmonic trap

- 2d harmonic potential $\boxed{U(x,y) = \frac{\mu}{2}(x^2 + y^2)}$
- Langevin equation

$$\boxed{\begin{aligned} \dot{x} &= -\mu x + v_0 \cos\phi(t) \\ \dot{y} &= -\mu y + v_0 \sin\phi(t) \\ \dot{\phi} &= \sqrt{2D_R}\,\eta_\phi(t). \end{aligned}}$$

- Two competing time-scales: trap ($1/\mu$) and rotational diffusion ($1/D_R$)

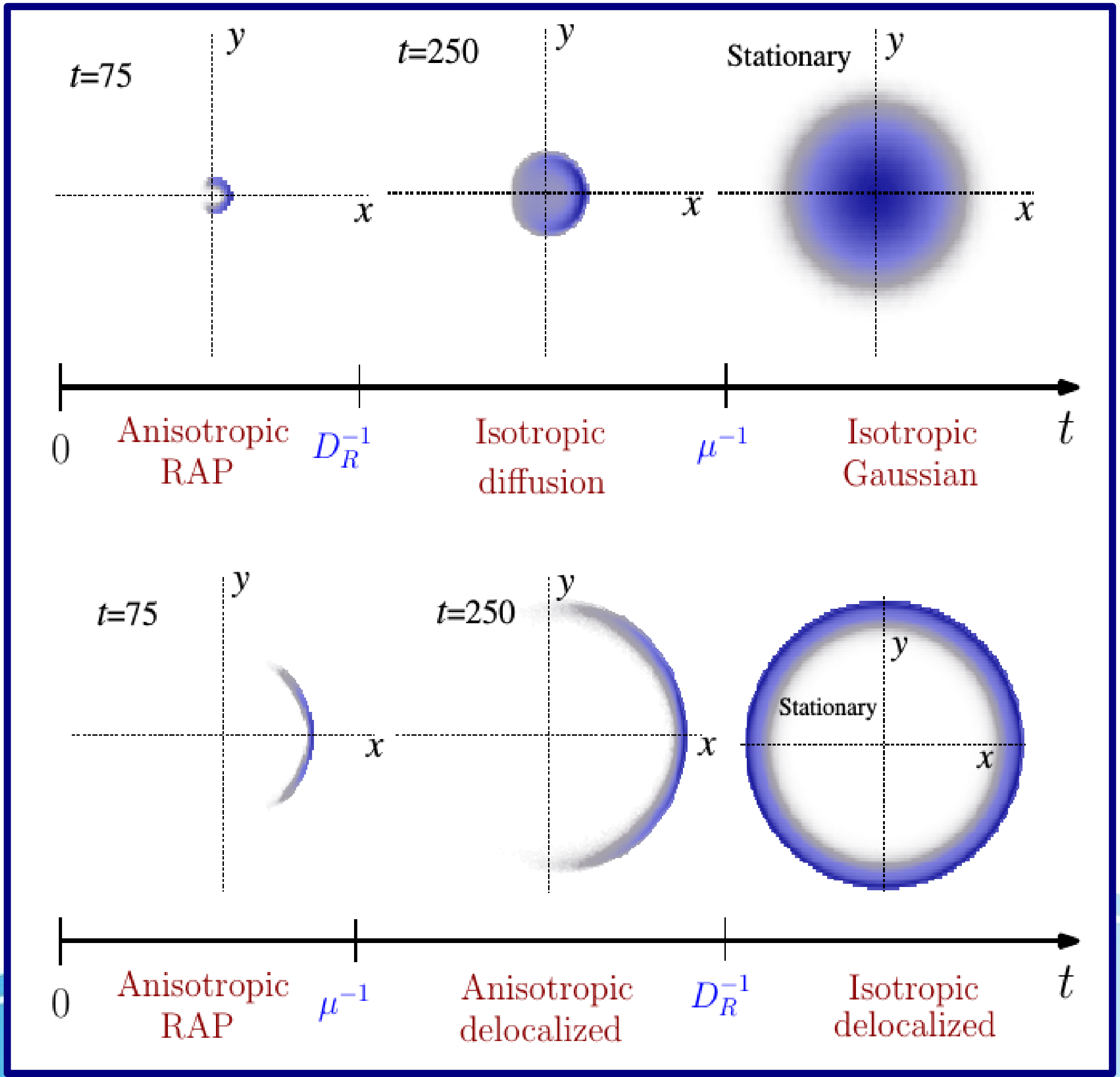
★ Passive Brownian particle → Ornstein-Uhlenbeck process

Dynamical Evolution

$$\mu^{-1} > D_R^{-1}$$

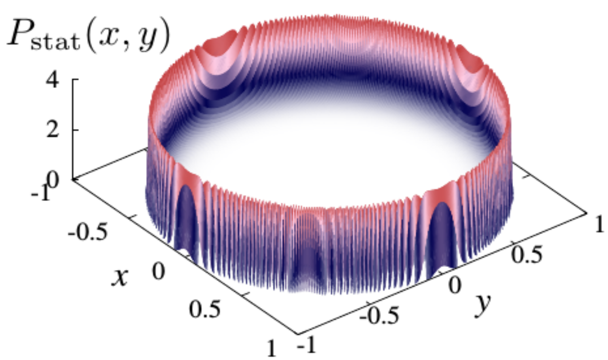
Two regimes

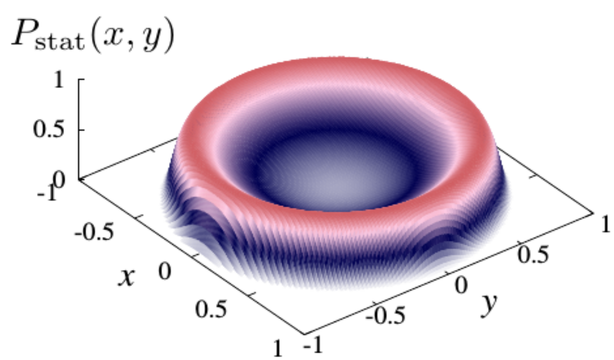
$$\mu^{-1} < D_R^{-1}$$

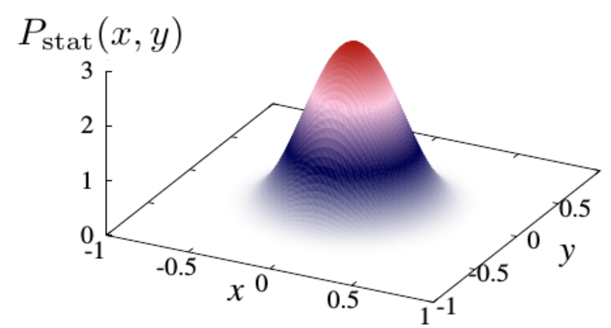


Stationary state

- Crossover from single peak to delocalized ring away from trap center







D_R=0.1

D_R=1

D_R=10

Analytical understanding ?

μ=1, v_0=1

Position distribution

- Moment evolution equation

$$\boxed{\dot{M}_{k,l} = -D_R(k-l)^2 M_{k,l} + v_0 e^{-\mu t}[k\,M_{k-1,l} + l\,M_{k,l-1}]}$$

- Difficult to solve in general, even in the stationary state
- Limiting cases:
  - Strongly passive : $D_R \to \infty$ Boltzmann-like
  - Strongly active : $D_R \to 0$ Non-Boltzmann

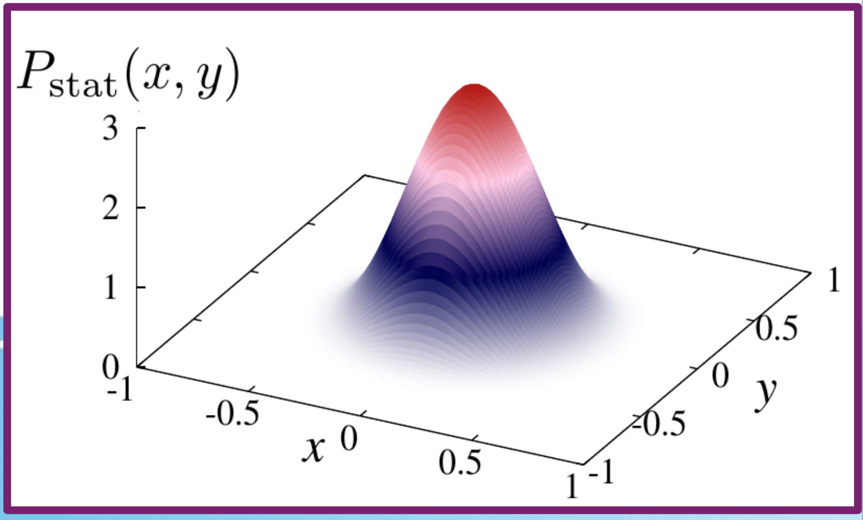
Passive limit ($D_R \to \infty$)

- Radial distribution: Gaussian *at all times*

$$\boxed{P_{\mathrm{rad}}(r,t) \simeq \frac{2\,\mu D_R}{v_0^2(1-e^{-2\mu t})}\exp\left[-\frac{\mu D_R r^2}{v_0^2(1-e^{-2\mu t})}\right]}$$

- Stationary state given by large t limit
- Rotationally symmetric

$$\boxed{P_{\mathrm{stat}}(x,y) = \frac{\mu D_R}{\pi v_0^2}\exp\left[-\frac{\mu D_R(x^2+y^2)}{v_0^2}\right]}$$



Strongly active limit ($D_R \to 0$)

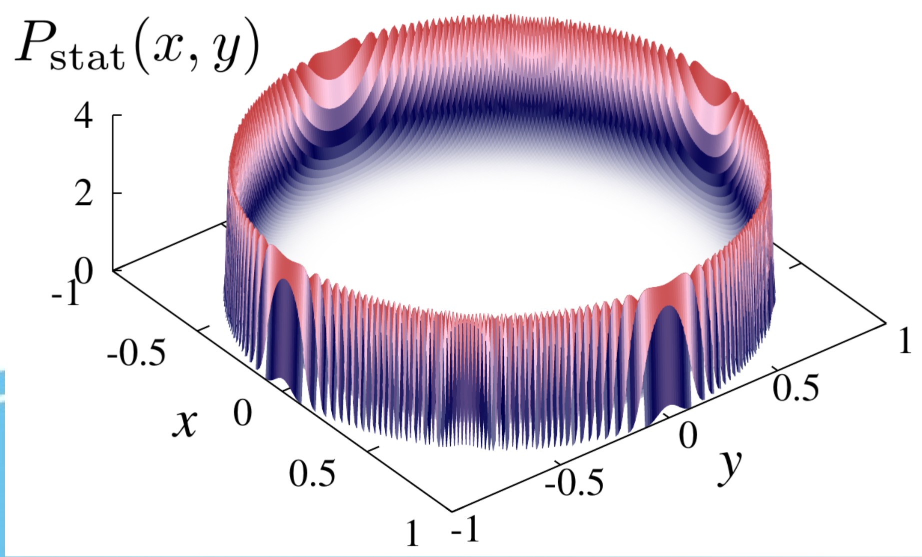
- Time-dependent radial distribution

$$\boxed{P_{\mathrm{rad}}(r,t) = \frac{\mu}{v_0(1-e^{-\mu t})}\delta\left[r - \frac{v_0(1-e^{-\mu t})}{\mu}\right]}$$

The particle is likely to be away from the center of the trap, the distance growing with time until 'boundary' is reached

- Stationary distribution: rotationally symmetric for $D_R \to 0+$

$$\boxed{P_{\mathrm{stat}}(x,y) = \frac{\mu}{2\pi v_0}\delta\left[\sqrt{x^2+y^2} - \frac{v_0}{\mu}\right]}$$



Conclusions

- Active Brownian motion in 2d
  - Diffusive behaviour at long-times
  - Non-trivial anisotropic short-time properties
- Exact distribution at short-times
- ABP in presence of a trap
  - Exact results in two limiting cases

Thank you!