Domain walls and layers in Ising spin glasses

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H. Khoshbakht and MW, Phys. Rev. B 97, 064410 (2018)
M.-S. Vaezi, G. Ortiz, MW, and Z. Nussinov, Phys. Rev. Lett. 121, 080601 (2018)

What is a spin glass?

Enlarge

LITTLE TIPSY SPIN GLASSES

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DESCRIPTION | GIFT INCLUDES

Serve and sip your favorite beverage with a unique stemless design that allows each glass to oxygenate while swirling and swiveling spill free. Hand blown from lead free Italian crystal, each elegant little glass holds 3 ounces and the set of four comes packaged in Olive & Cocoa® gift wrap with ribbon.

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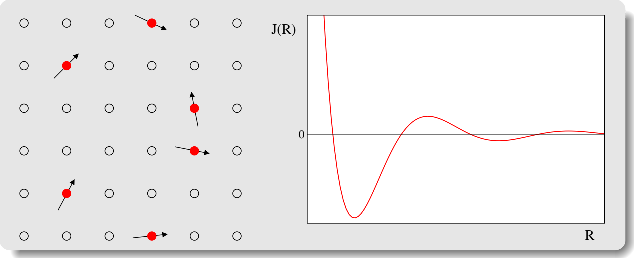
Spin glass history

Classical example of spin glass: noble metals weakly diluted with transition metal ions, coupled via the RKKY interaction,

$$J(\boldsymbol{R}) = J_0 \frac{\cos(2k_F R + \phi_0)}{(k_F R)^3}$$

Emergent properties:

- no long-range order down to $T = 0$
- phase transition to short-range ordered, "glassy" phase
- diverging relaxation times, memory, rejuvenation etc.

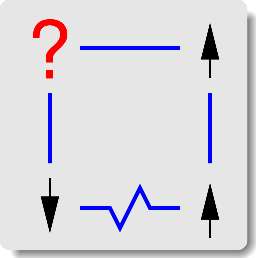


The EA model

Simplify to the essential properties, **disorder** and **frustration** to yield the Edwards-Anderson (EA) model,

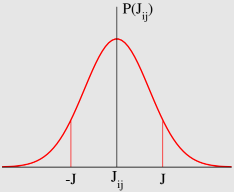
$$\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij}\, s_i s_j, \quad s_i = \pm 1$$

where J_{ij} are *quenched*, random variables.

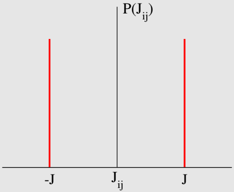


Coupling distributions

Gaussian



bimodal



Universality

A glass phase only exists at $T = 0$ for this model. Is the critical behavior the same for both coupling distributions?

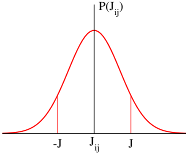
At finite temperatures:

Gaussian	**bimodal**
vanishing energy gap a	finite gap $4J$
continuous scaling	"freezing"
$\xi \sim T^{-\nu}$	$\xi \sim \exp(cT)$?
$\nu \approx 3.6$	$\nu = \infty$?, $\nu = 4.8$?
$\eta = 0$	$\eta > 0$?
entropy exponent	$\theta_S = 0.5$

Degeneracies

At $T = 0$ physics is described by the ground states.

Gaussian



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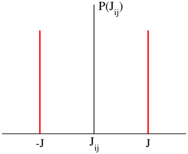
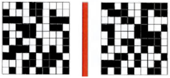
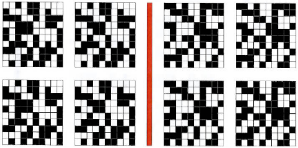






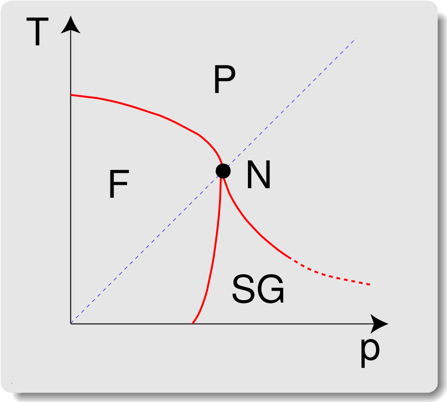
Unique ground state.

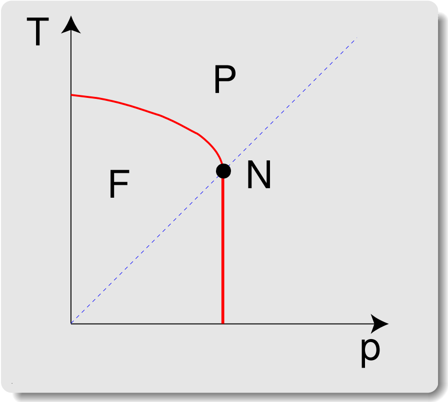
Exponentially many ground states,

$$N_{\mathrm{GS}} \sim \exp(L^2 s_0).$$

Phase diagrams

Finite-temperature transition in 3D, but spin-glass order only at $T = 0$ in 2D.





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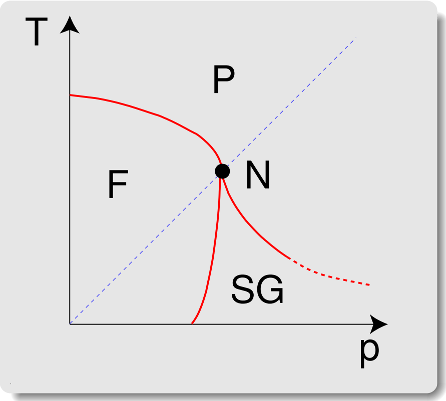
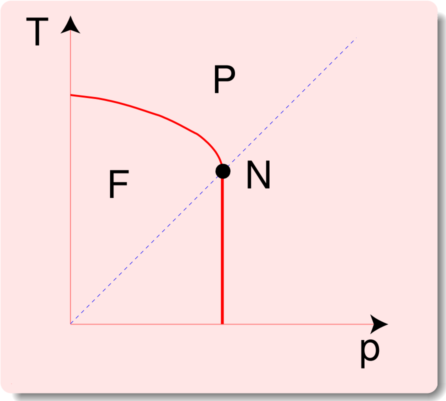




Zero temperature

Behavior at $T = 0$ is quite clearly not consistent.

Gaussian	bimodal
unique ground state	exponentially many ground states
stiffness exponent $\theta \approx -0.3$	$\theta = 0$
domain-wall fractal dimension $d_\mathrm{f} \approx 1.3$	?
entropy exponent	$\theta_S = 0.5$

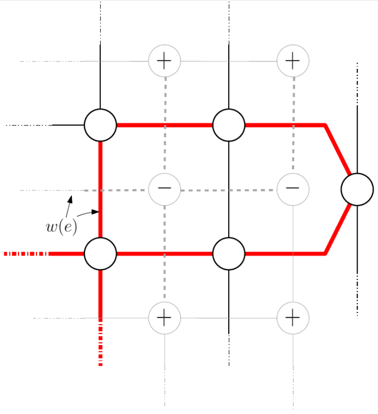
We should aim to:

- be able to **determine ground states** for large systems
- be able to **sample degenerate ground states** for the bimodal model

Matching on auxiliary graph

Use mapping of the Ising problem to minimum-cut:

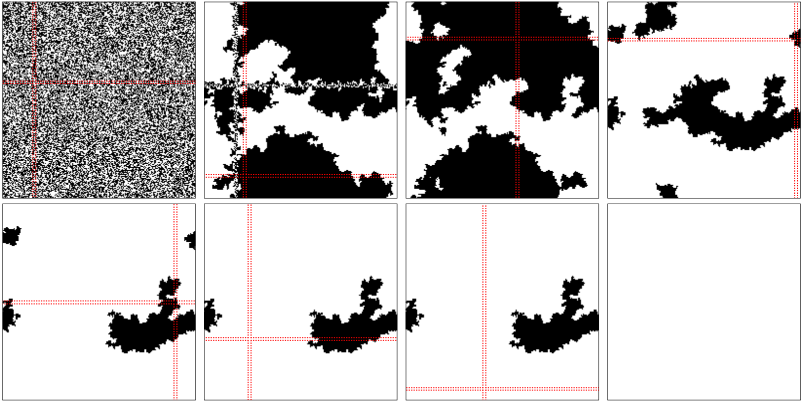
$$-\mathcal{H} = \sum_{\langle ij \rangle} J_{ij}\, s_i s_j = W^{+} + W^{-} - W^{\pm} = K - 2W^{\pm},$$



- GS search again corresponds to **minimum-weight perfect matching problem** (Thomas & Middleton, 2007; Pardella & Liers, 2008)
- matching solution always corresponds to spin configuration for **planar** graphs
- we use a windowing technique to also treat fully periodic boundaries
- **space complexity is O(V)**

Windowing technique

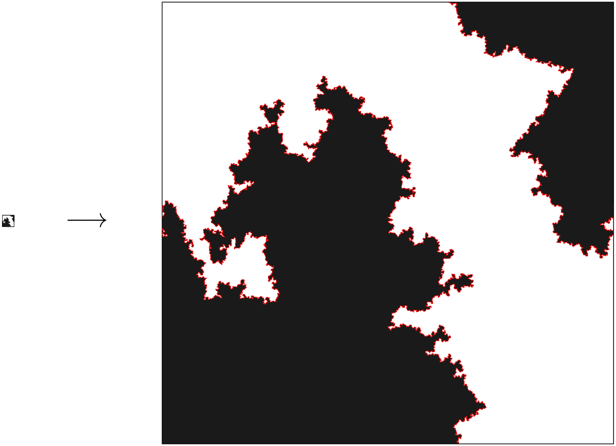
Determine exact ground-states for fully periodic systems in polynomial time.



Ising spin glass in 2D

Complex energy landscape leads to slow relaxation: sizes restricted to $L \approx 128$ (MC) or maybe $L = 256$ (GS techniques).

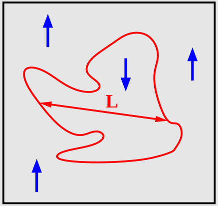
A newly developed **combinatorial optimization** method allows us to treat large system sizes up to $10\,000 \times 10\,000$ spins **exactly** (for $T = 0$).



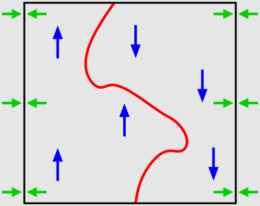
Spin stiffness and zero-temperature scaling

Edwards-Anderson model: $\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j, \quad s_i = \pm 1$

Ferromagnet (Peierls)



$$\Delta E \sim L^{d-1}$$



Spin glass (Bray/Moore, 1987)

Distribution of couplings evolving under RG transformations, asymptotic width scales as

$$J(L) \sim J L^{\theta(d)}.$$

Spin-stiffness exponent θ determines lower critical dimension. For $\theta < 0$,

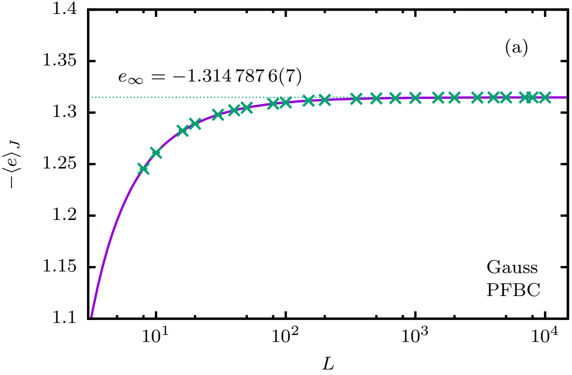
$$\xi \sim T^{-\nu}, \quad \nu = -1/\theta.$$

Numerically, θ can be determined from inducing droplets or domain walls with a change of *boundary conditions*,

$$\Delta E = |E_{\mathrm{AP}} - E_{\mathrm{P}}| \sim L^{\theta}.$$

Ground-state energy

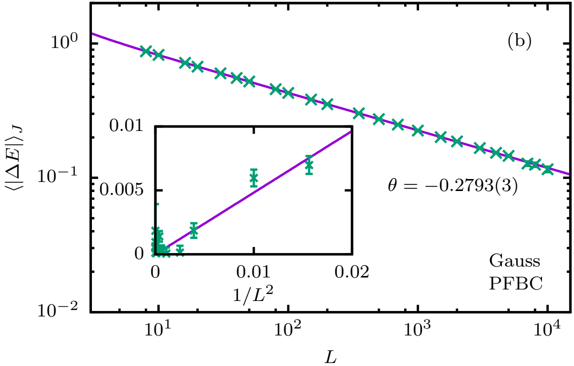
Average ground-state energy per spin.



$$\langle e(L) \rangle_J = e_\infty + \hat{A}_E L^{-(d-\theta)} + (\hat{C}_E - e_\infty/2) L^{-1} - (\hat{C}_E/2) L^{-2}$$

Defect energies

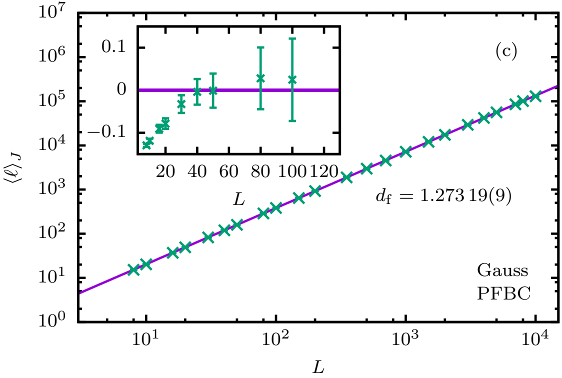
Defect energy.



$$\langle|\Delta E(L)|\rangle_J(L) = A_\theta L^\theta(1 + B_\theta L^{-\omega}) + \frac{C_\theta}{L} + \frac{D_\theta}{L^2} + \cdots ,$$

Fractal dimension

Fractal dimension of domain wall.



$$\langle \ell \rangle_J(L) = A_\ell L^{d_\mathrm{f}}(1 + B_\ell L^{-\omega}) + \frac{C_\ell}{L} + \frac{D_\ell}{L^2} + \cdots .$$

Results

Perform calculations for periodic-free and periodic-periodic boundary conditions.

	PFBC	PPBC
$-e_\infty$	1.3147876(7)	1.314788(3)
θ	-0.2793(3)	-0.2788(11)
d_f	1.27319(9)	1.2732(5)

Results are fully consistent with each other.

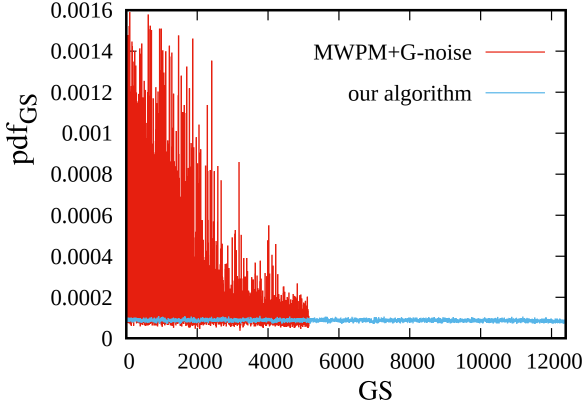
Based on SLE and further assumptions, Amoruso et al. (2006) proposed

$$d_\mathrm{f} = 1 + \frac{3}{4(3+\theta)}.$$

$d_\mathrm{f} = 1.27319(9)$ would imply $\theta = -0.2546(9)$ which is **not compatible** with the direct estimate.

Sampling degenerate states

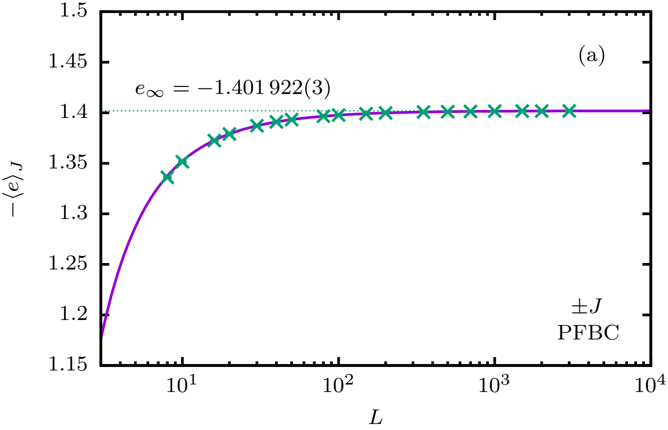
How well does it work?



Problems:

- breaking of degeneracies depends on cluster size
- clusters cannot be flipped independently

Bimodal results



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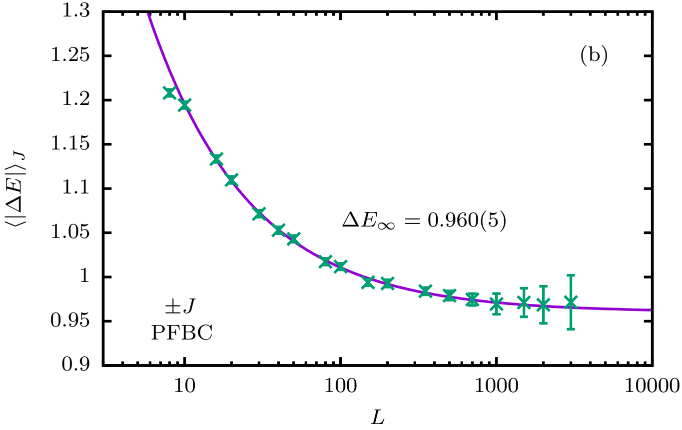
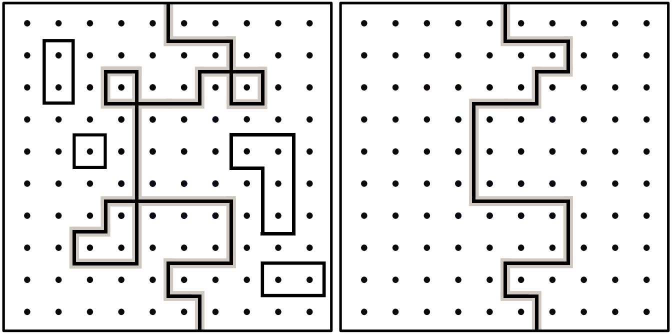


$\theta = 0$

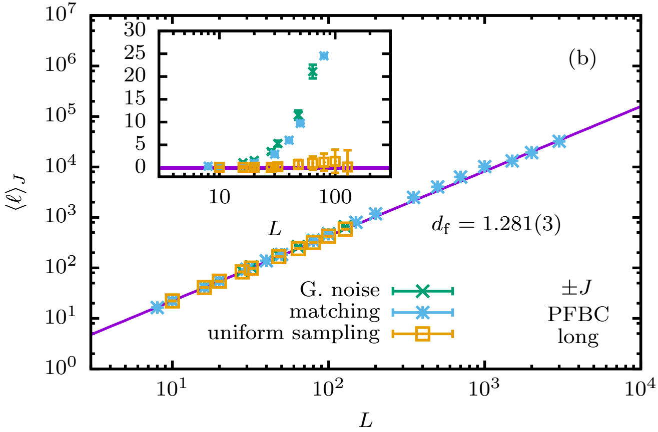
Domain walls again

For the bimodal model, definition of domain wall is not unique.



Fractal dimension

Uniform sampling makes a difference.



short walls: $d_f = 1.279(2)$, long walls: $d_f = 1.281(3)$

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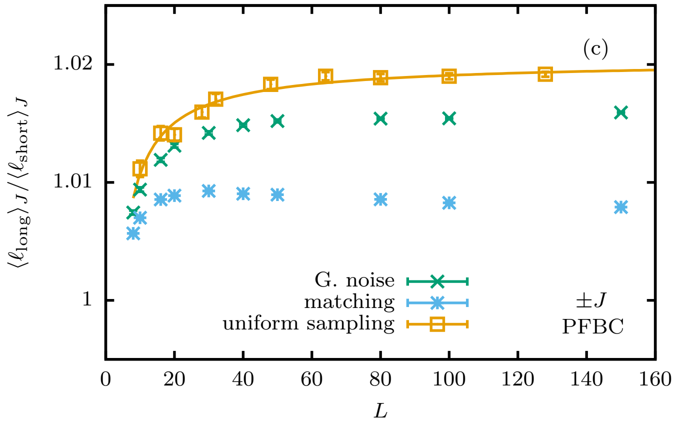

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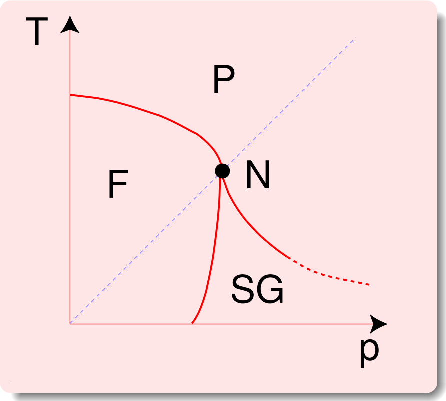

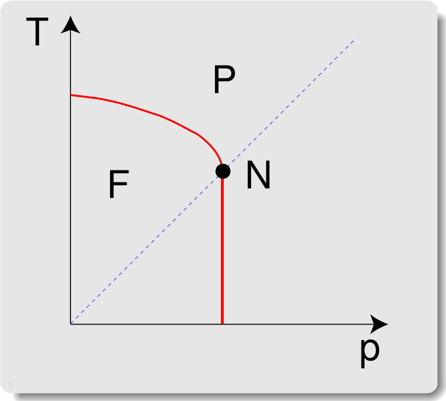

The layered model

- Gaussian and bimodal model are very different at $T = 0$
- how could we connect (and hence understand) these cases?
- construct models with **tunable degeneracy**

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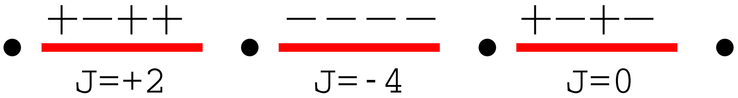
Put m layers of bimodal couplings on each bond:



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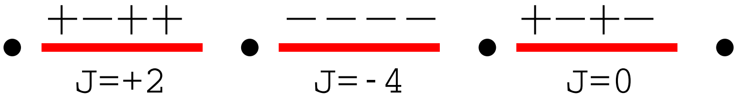
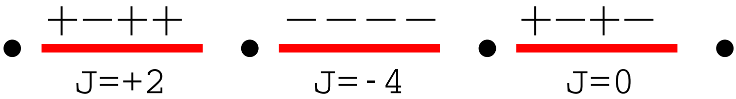


$$H_m = -\sum_{ij} \mathcal{J}_{ij}^m s_i s_j, \quad \mathcal{J}_{ij}^m \equiv \frac{1}{\sqrt{m}} \sum_{k=1}^{m} J_{ij}^{(k)}, \quad J_{ij}^{(k)} = \pm 1.$$

The layered model (cont'd)



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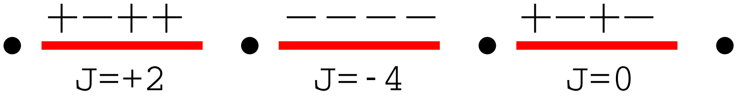

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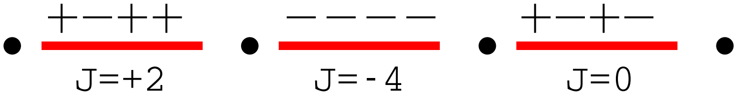

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The probability distribution of $\mathcal{J}_{ij}^m$ is then a *binomial*,

$$\tilde{P}(\mathcal{J}_\alpha^m) = \sum_{j=0}^{m} \binom{m}{j} p^{m-j} (1-p)^j \delta\left(\mathcal{J}_\alpha^m - \frac{m-2j}{\sqrt{m}}\right)$$

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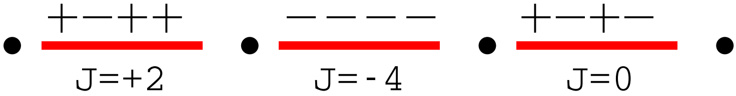

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The case $m \to \infty$ corresponds to the Gaussian model, and $m = 1$ to the bimodal case. The binomial model is hence an **interpolation** between these extremes.

The layered model: degeneracies

How does the binomial model behave in terms of degeneracies?

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We can show rigorously that on d-dimensional hypercubic lattices the entropy per spin of *any* energy level is bounded by

$$\mathcal{S}_0 \leq (\sqrt{d/2m} + 1/N) \ln 2,$$

where $N = L^d$ is the number of spins.

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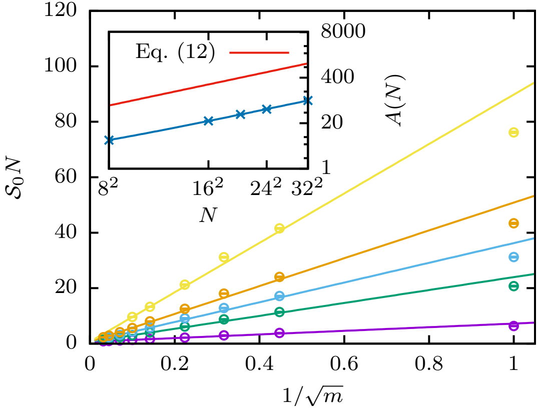
Hence

- there is a **unique ground-state pair** for $m \to \infty$, N finite
- **degenerate ground-state pairs** are expected if $N \to \infty$, $m \to \infty$ with $N/\sqrt{m}$ fixed.

The result depends on the order of taking limits.

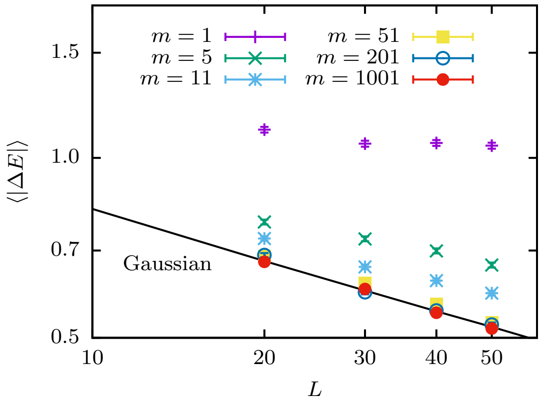
The layered model: entropy



$$\mathcal{S}_0 N = \left(\frac{A(N)}{\sqrt{m}} + 1 \right) \ln 2, \quad A(N) = aN + b.$$

The layered model: defect energies

Model interpolates the different behaviors of $m = 1$ and $m \to \infty$ in defect-energy scaling.



There is an m dependent crossover length $L^*(m)$.

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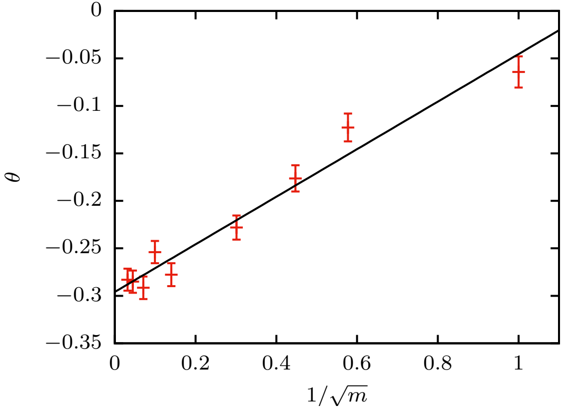

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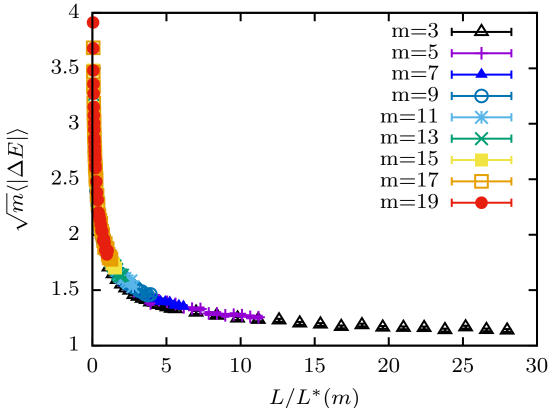

Crossover length

$$L^*(m) \sim m^{-1/2\theta} \approx m^{1.8}$$

Model appears continuous for $L < L^*(m)$ and discrete for $L > L^*(m)$.

Conclusions

Domain walls:

- new techniques allow to study systems up to $10\,000 \times 10\,000$ spins
- windowing method enables ground-state calculations for toroidal graphs
- careful FSS analysis yields $e_\infty = -1.3147876(7)$, $\theta = -0.2793(3)$ and $d_\mathrm{f} = 1.27319(9)$ for the Gaussian model
- not consistent within error bars with $d_\mathrm{f} = 1 + 3/[4(3+\theta)]$
- cluster updating technique allows uniform sampling of degenerate ground states
- $\theta = 0$ and $d_\mathrm{f} = 1.279(2)$ for bimodal model
- additional results (not shown) for distributions

Layered model:

- m layers of bimodal couplings
- continuous interpolation between bimodal and Gaussian model
- can prove uniqueness of ground states in continuous limit,

$$\mathcal{S}_0 \leq (\sqrt{d/2m} + 1/N) \ln 2$$

- interpolating behavior of θ with crossover length $L^*(m)$

H. Khoshbakht and MW, Phys. Rev. B 97, 064410 (2018)
M. Vaezi, G. Ortiz, MW, and Z. Nussinov, Phys. Rev. Lett. 121, 080601 (2018)