Quantization of Geometric Phase with Integer and Fractional Topological Characterization in a Quantum Ising Chain

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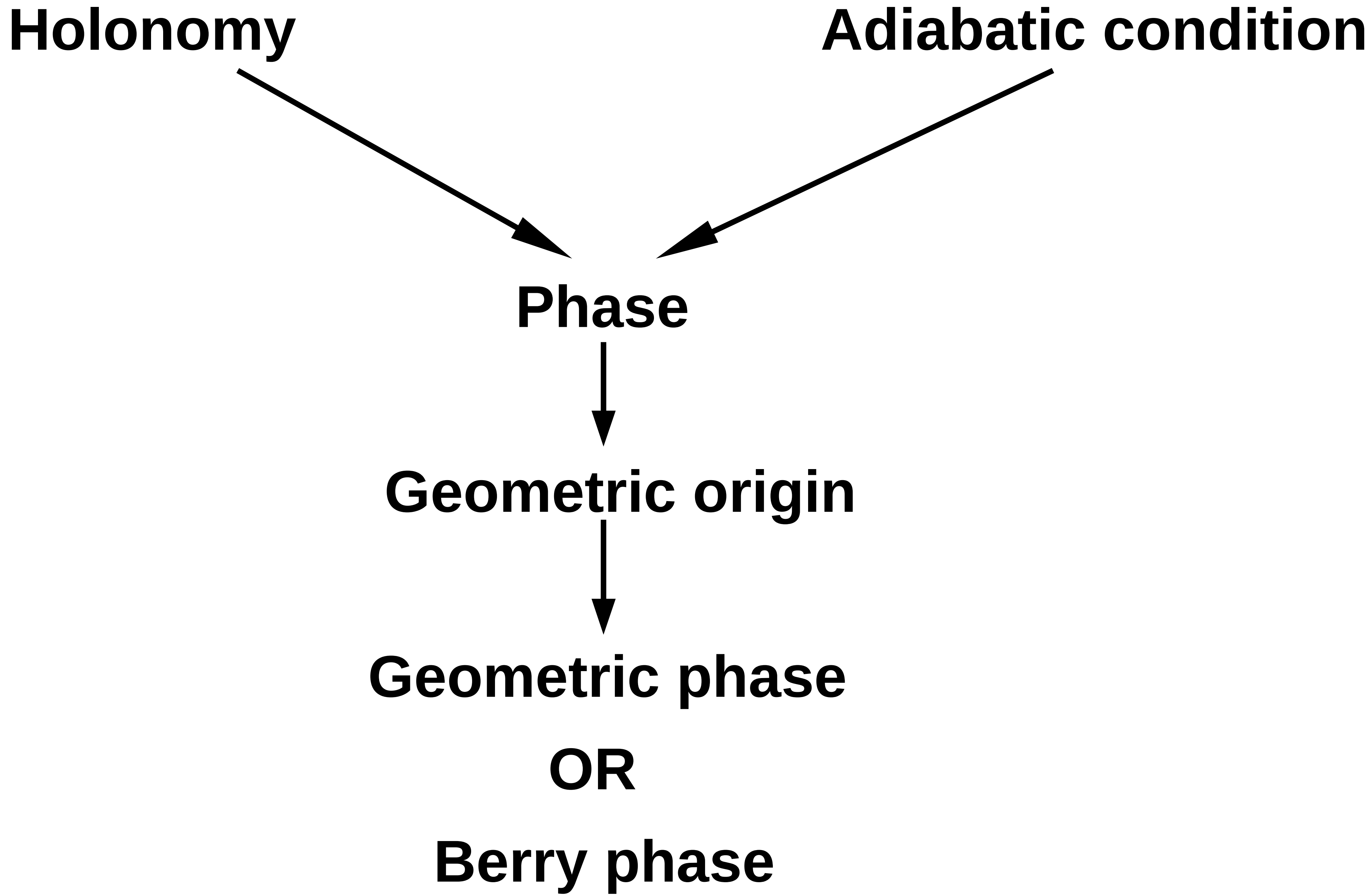
Outline

1. Introduction

2. Basic Aspects of Model Hamiltonian

3. Results

4. Conclusions


Holonomy
Adiabatic condition
Phase
Geometric origin
Geometric phase
OR
Berry phase

Berry phase

We consider the adiabatic holonomic process,

How does the final state differ from the initial state, if the parameters in the Hamiltonian are carried adiabatically around some closed cycle.?

Berry phase

$$\Psi_n(t) = e^{i[\theta_n(t) + \gamma_n(t)]} \psi_n(t)$$

- Dynamical phase

$$\theta_n(t) = \frac{-1}{\hbar} \int_0^t E_n(t') dt'$$

- Geometric phase

$$\gamma_n(T) = i \oint \langle \psi_n | \nabla_R \psi_n \rangle dR$$

Berry phase

- Berry phase depends only on the path taken, not on the rate at which the path is traversed.

- Berry phase is invariant with respect to the guage transformation. So it cannot be ignored.

Berry phase

Berry connection **A(R)**	**Vector potential** **A(r)**
Berry curvature **F(R)**	**Magnetic field** **B(r)**
Berry phase **γ(C)**	**Magnetic flux** **Φ(C)**

Difference between topological quantum phase transition and quantum phase transition

Topological quantum phase transition describes with the topologically invariant number while the quantum phase transition describes by order parameter. Topological number changes by an integer number during the topological quantum phase transition from topological state to the non-topological state, which is related to the appearance of Majorana zero modes localized at the edge of the system. This detailed study based on exact calcula-

Landau's theory of Phase Transition provide the phenomenological footing on which symmetry broken states can be explained.

Properties of Topological Phases

- The bulk of the system is gapped, namely, there is a finite energy gap between the ground state and the excited states. Hence the bulk is an insulator at low temperatures

- The band structure of the bulk of the system is characterized by a topological invariant which is a non-zero integer

- There are gapless states at the boundaries of the system; these contribute to electronic transport

- Bulk-boundary correspondence: The number of boundary states is equal to the topological invariant; it does not change if the parameters in the Hamiltonian are changed a bit or if a small amount of disorder is present

Model Hamiltonian and geometric phase calculations

$$H = -\sum_i (\mu\sigma_i{}^x + \lambda_2\sigma_i{}^x\sigma_{i-1}{}^z\sigma_{i+1}{}^z + \lambda_1\sigma_i{}^z\sigma_{i+1}{}^z).$$

This Hamiltonian transfer to the spinless fermion Hamiltonian through the Jordan-Wigner transformation, $\sigma_i{}^x = (1-2c_i{}^\dagger c_i)$, $\sigma_i{}^z = \Pi_{j<i}(1-2c_j{}^\dagger c_j)(c_j + c_j{}^\dagger)$.

After the Jordan-Wigner transformation, the Hamiltonian is reduced to,

$$H = -\mu\sum_{i=1}^{N}(1-2c_i{}^\dagger c_i) - \lambda_1\sum_{i=1}^{N-1}(c_i{}^\dagger c_{i+1} + c_i{}^\dagger c_{i+1}{}^\dagger + h.c) - \lambda_2\sum_{i=1}^{N-1}(c_{i-1}{}^\dagger c_{i+1} + c_{i+1}c_{i-1} + h.c). \quad (4)$$

Finally, after the Fourier transform, the Hamiltonian become,

$$H = \sum_k (2\mu - 2\lambda_1\cos k - 2\lambda_2\cos 2k)c_k{}^\dagger c_k$$
$$+i\sum_k (2\lambda_1 \sin k c_k{}^\dagger c_{-k}{}^\dagger + 2\lambda_2 \sin 2k c_k{}^\dagger c_{-k}{}^\dagger). \quad (5)$$

Defination and Analytical Expression for Zak Phase

definition of Zak phase is the following. The Berry's phase picked up by a particle moving across the Brillouin zone. Here, the Brillouin zone is in the one dimension as treated by the Zak , and, therefore, the natural choice for the cyclic parameter is the crystal momentum (k). The geometric phase in the momentum space is defined as

$$\gamma_n = \int_{-\pi}^{\pi} dk < u_{n,k} | i\partial_k | u_{n,k} >, \tag{6}$$

where $|u_{n,k}>$ is the Bloch states which are the eigenstates of the n^{th} band of the Hamiltonian.

The model Hamiltonian of the present problem is Z type topological invariant and the system has an anti-unitary particle hole symmetry

$$\gamma = W\pi \qquad \mod (2\pi).$$

Anderson Pseudo Spin Approach for the Determination of Winding Number

The BdG equation for this Hamiltonian is

$$H_{BdG} = \begin{pmatrix} \chi_z(k) & i\chi_y(k) \\ -i\chi_y(k) & -\chi_z(k) \end{pmatrix}$$

where,

$$\chi_y(k) = 2\lambda_1 \sin k + 2\lambda_2 \sin 2k,$$

$$\chi_z(k) = -2\lambda_1 \cos k - 2\lambda_2 \cos 2k + 2\mu,$$

$$E_{(k)} = \sqrt{\chi_z(k)^2 + \chi_y(k)^2}.$$

Topological phase transition can be ascribed by the topological invariant quantity.

$$H_{BdG}(k) = \sum_i \vec{\chi}(k)^i . \vec{\tau}^i,$$

Continuation

One can write the vector $\hat{\chi}(k)$ as,

$$\hat{\chi}(k) = \cos\theta_k \hat{y} + \sin\theta_k \hat{z}.$$

and $\theta_k = tan^{-1}(\frac{\chi_y(k)}{\chi_z(k)})$.

$$W = \frac{1}{2\pi}\int \frac{d\theta(k)}{dk}dk.$$

$$\gamma = \frac{1}{2}\int \frac{d\theta(k)}{dk}dk \mod (2\pi).$$

The winding number (W) presents the number of times $\hat{\chi}(k)$ rotates in the YZ plane

Analytical Expressions for Geometric (Zak) Phase

The general expression for the geometric phase (Zak phase) for this model Hamiltonian is

$$\gamma = \int_{-\pi}^{\pi} \frac{\lambda_1(6\lambda_2 - 2\mu)\cos k + 2({\lambda_1}^2 + 2{\lambda_2}^2 - 2\mu\lambda_2 \cos 2k)}{(4{\lambda_1}^2 + 4{\lambda_2}^2 + 4\mu^2 + 4\lambda_1(2\lambda_2 - 2\mu)\cos k - 8\mu\lambda_2 \cos 2k)}\, dk \qquad \mathrm{mod}\ (2\pi).$$

Results:

Here, we study the quantization of geometric phase in the different regime of the parameter space of the system , which are, (1) $\lambda_2 = 0$ and $\lambda_1 \neq 0$, (2) $\lambda_1 = 0$ and $\lambda_2 \neq 0$, (3) $\lambda_2 = -\mu$. (4) $\lambda_2 = \mu + \lambda_1$, (5) $\lambda_2 = \mu - \lambda_1$, The last three lines are the quantum critical lines.

Integer Topological Characterization

$$\gamma^{(1)} = \int_{-\pi}^{\pi} \frac{-2\lambda_1 \mu \cos k + 2{\lambda_1}^2}{(4{\lambda_1}^2 + 4\mu^2 - 8\lambda_1 \mu \cos k)} \, dk \qquad \mod (2\pi),$$

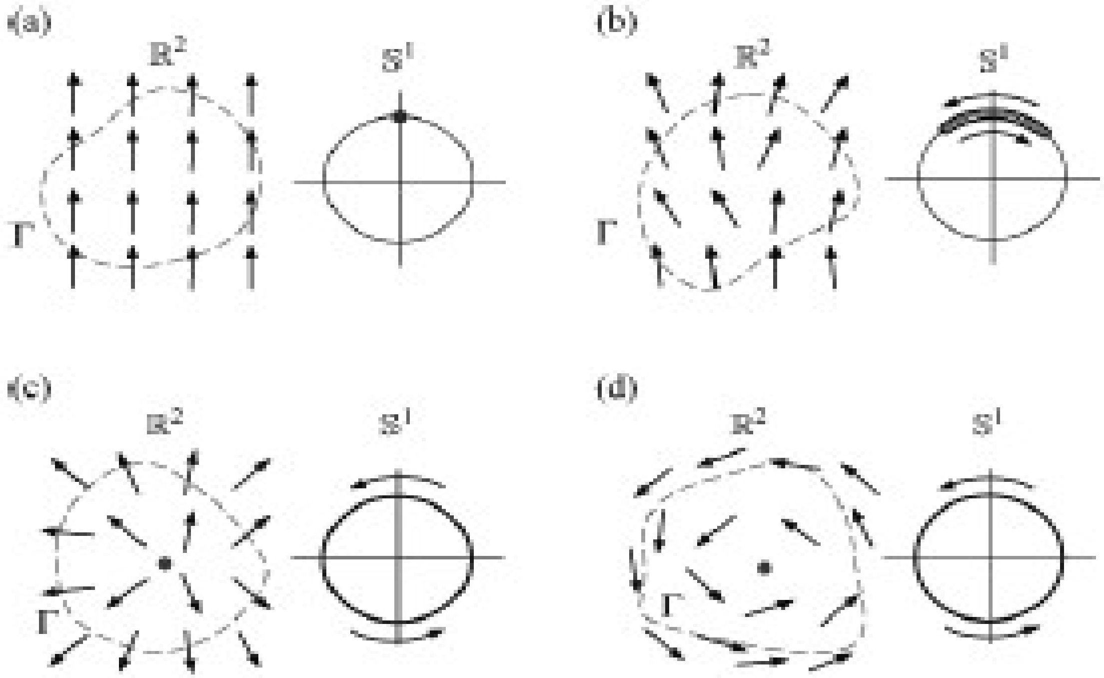
where $\lambda_1 \neq 0$ but $\lambda_2 = 0$.

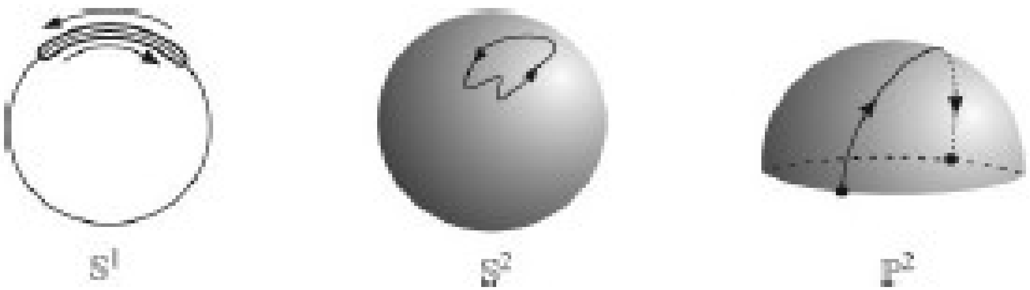
$$\gamma^{(2)} = 2\int_{-\pi}^{\pi} \frac{-2\lambda_2 \, \mu \, \cos 2k + 2{\lambda_2}^2}{(4{\lambda_2}^2 + 4\mu^2 - 8 \, \lambda_2 \, \mu \cos 2k)} \, dk \qquad \mod (2\pi),$$

where $\lambda_2 \neq 0$ but $\lambda_1 = 0$.

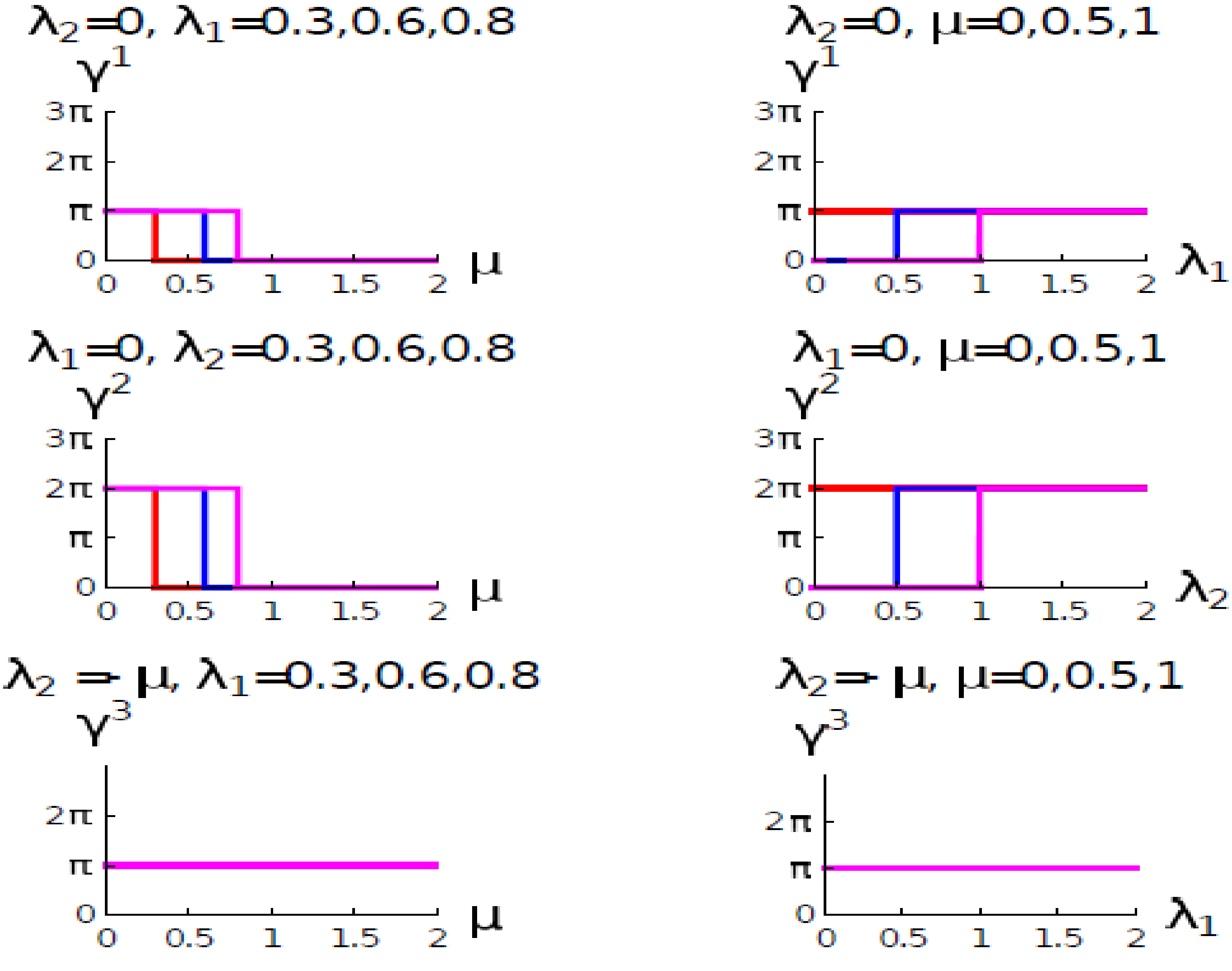
$$\gamma^{(3)} = \int_{-\pi}^{\pi} \frac{-8 \, \lambda_1 \, \mu \cos k + 2({\lambda_1}^2 + 2\mu^2 + 2\mu^2 \cos 2k)}{(4{\lambda_1}^2 + 8\mu^2 - 16 \, \lambda_1 \, \mu \cos k + 8\mu^2 \cos 2k)} \, dk \qquad \mod (2\pi),$$

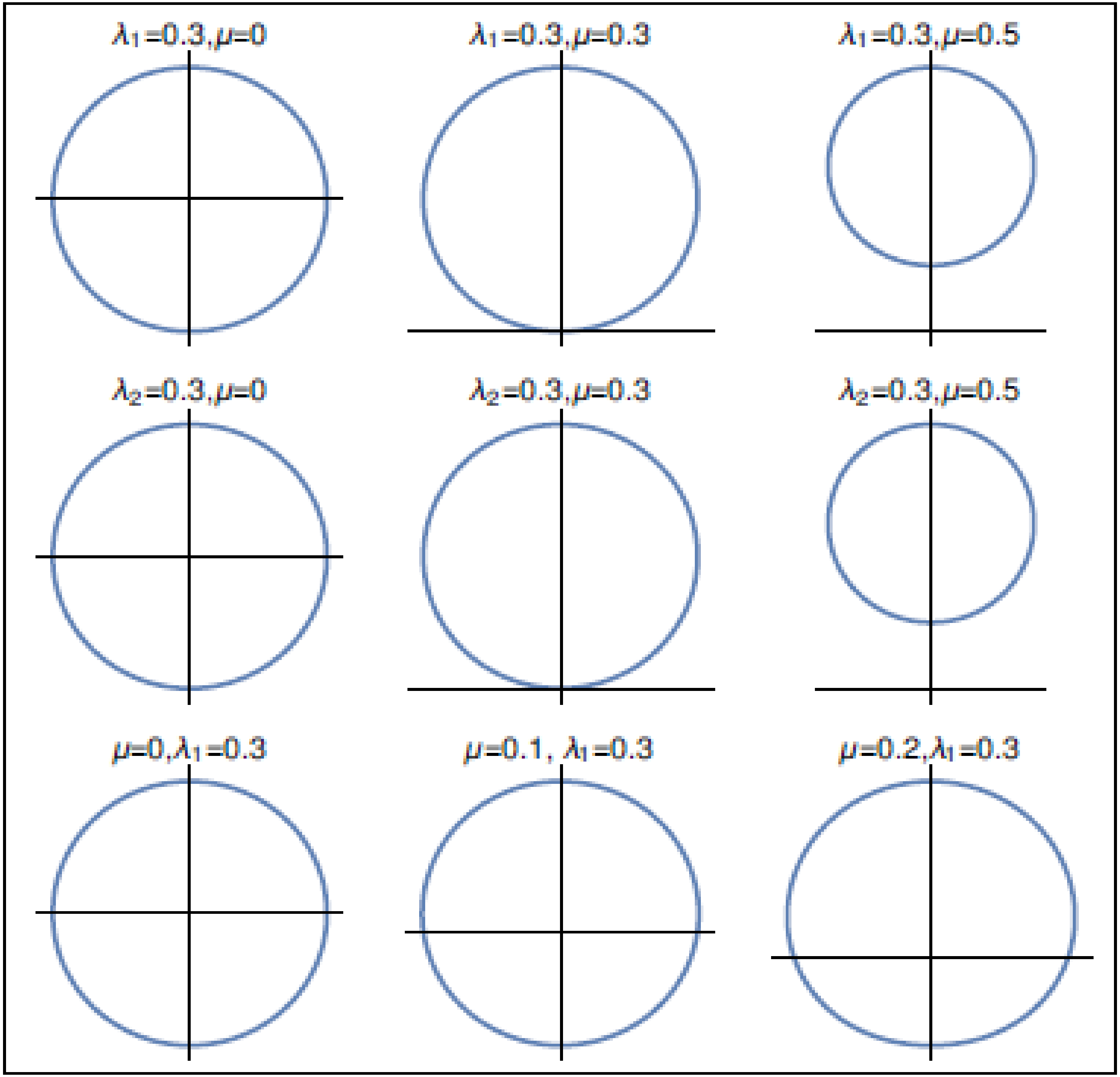
Pictorial Representation of Non-Topological and Topological State



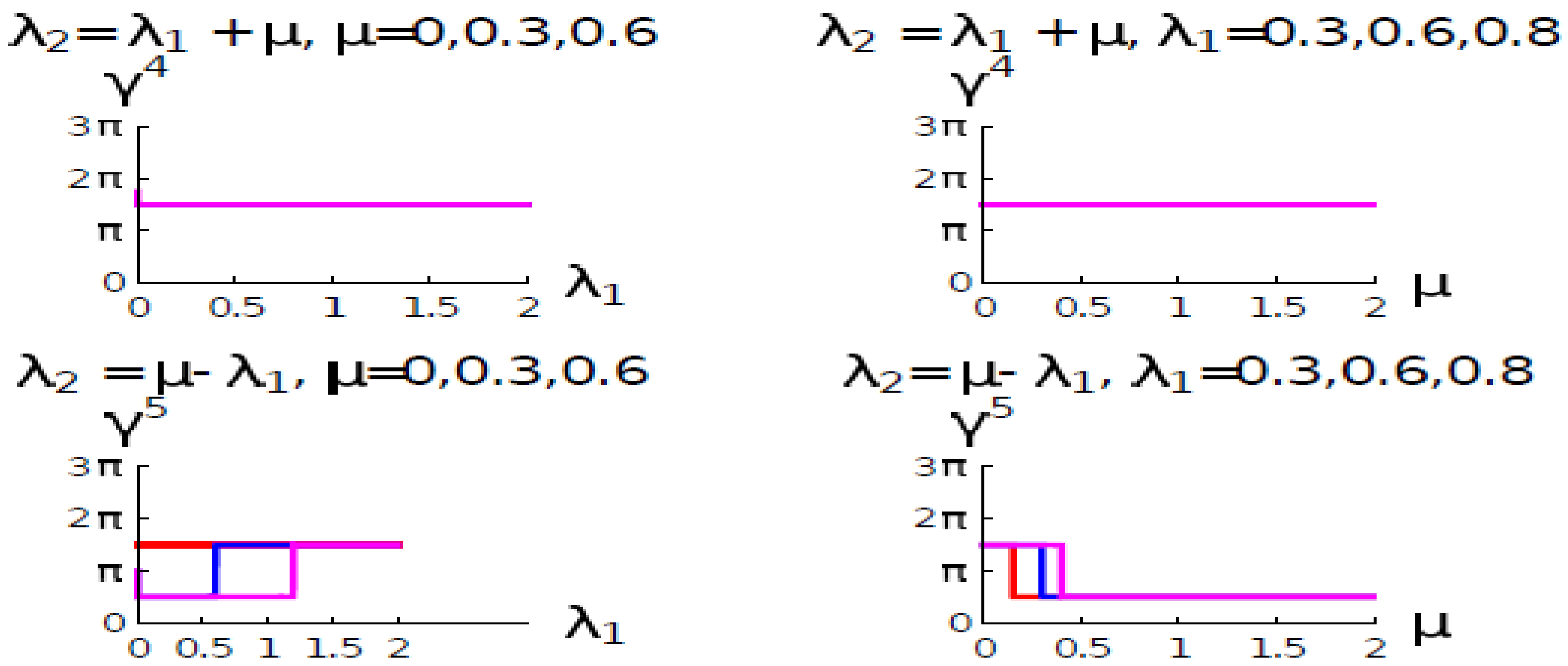


Integer topological characterization



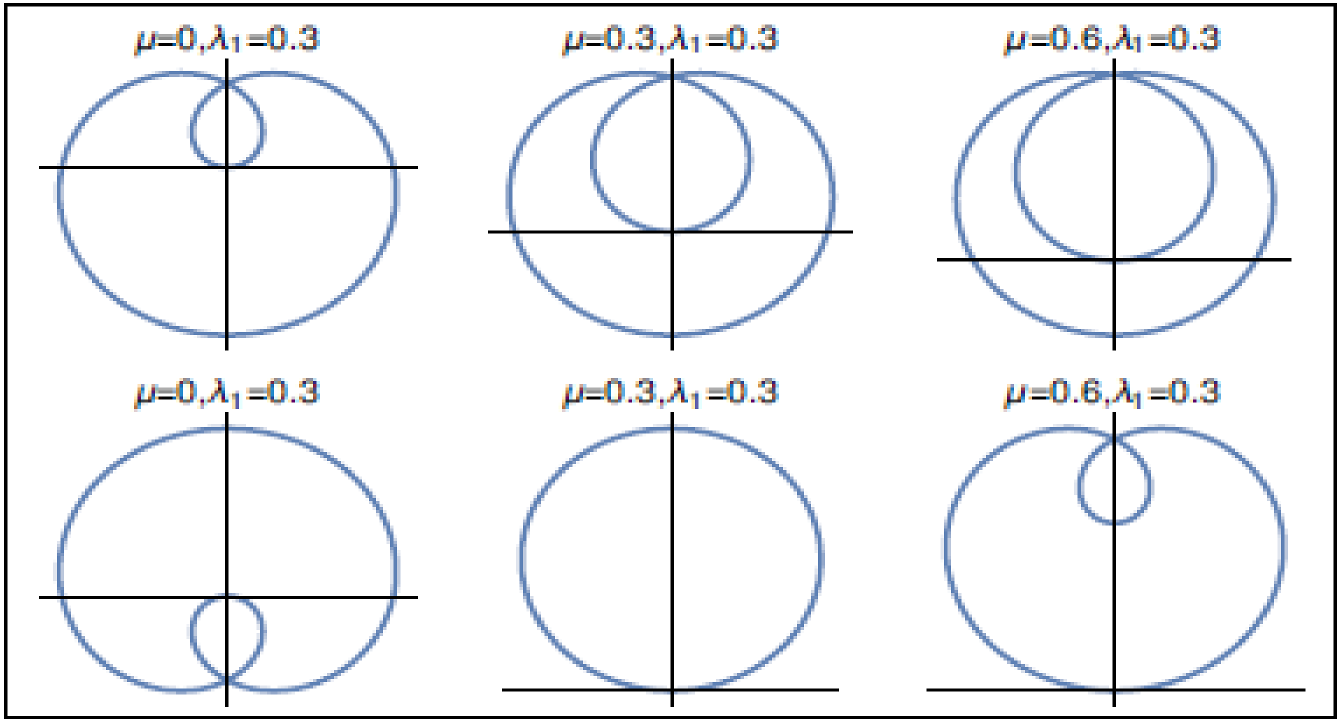


λ1=0.3,μ=0
λ1=0.3,μ=0.3
λ1=0.3,μ=0.5
λ2=0.3,μ=0
λ2=0.3,μ=0.3
λ2=0.3,μ=0.5
μ=0,λ1=0.3
μ=0.1, λ1=0.3
μ=0.2,λ1=0.3



Fractional
Topological
Characterization

Non-Hermitian
Quantum
System?



Conclusions

(1). We have presented the behaviour of geometrical phase for integer and fractional topological characterization.

(2). We have also explained the behaviour of geometric phase in the auxiliary space.

(3). We have also present the behaviour of quantum critical lines are different from the perspective of topological characterization.

Thank You