Resonance Oscillation of a Damped Driven Simple Pendulum

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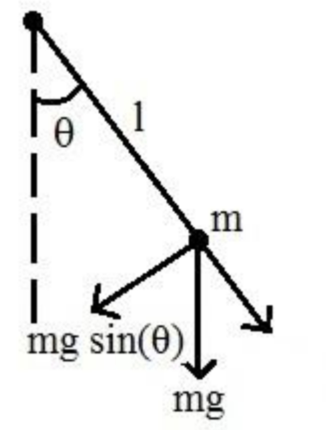
Undamped simple pendulum

Eqn. of motion: $I\frac{d^2\theta}{dt^2} = -mg.l.\sin(\theta)$



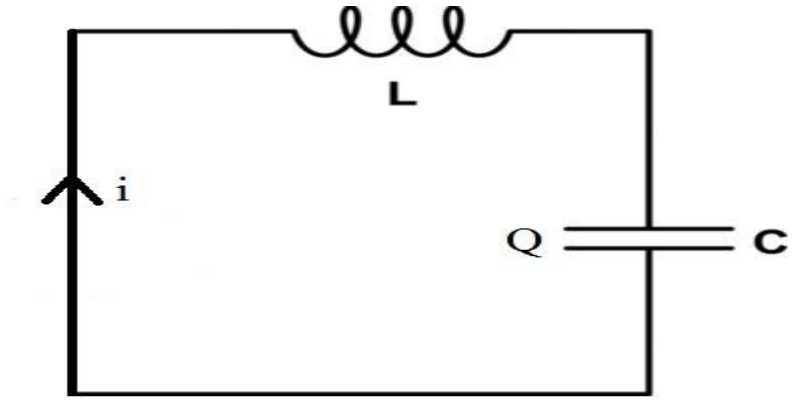
For small θ → $I\frac{d^2\theta}{dt^2} = -mgl\theta$

$\Rightarrow$ $\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta = -\frac{g}{l}\theta$ where $I = ml^2$

$\Rightarrow$ $\frac{d^2x}{dt^2} = -\omega^2 x$ in familiar notation.

ω is the natural frequency of oscillation.

$$L\frac{di}{dt}+\frac{Q}{C}=0$$



⇒ $$L\frac{d^2Q}{dt^2}=-\frac{1}{LC}Q$$

Again natural frequency of oscillation

$$\omega=\frac{1}{\sqrt{LC}}$$; Also the resonance frequency.

⇒ Same as a simple pendulum with very small amplitude of oscillation.

LC circuit is no longer like simple pendulum

if x not small, $\sin(x)\neq x$

If Force= - sin(x)
Then,

U(x)=-cos(x) → potential is sinusoidal, not quadratic.

Sinusoidal potential is quadratic (harmonic) only at positions close to the minimum of potential.

When the potential is sinusoidal (simple pendulum), what is the solution?

*A. Sommerfeld, Lectures on Theoretical Physics: MECHANICS, Levant Books (Indian Reprint), Kolkata (2003).
* Kittel, Knight, and Ruderman, MECHANICS, Berkley Physics Course-Volume 1, 1962, Ch. 7
(Advanced problem 1: Exact solution, problem of the simple pendulum)
* A. Belendez, et. al, Eur. J. Phys. 32, 1303 (2011).

Kittel's Approx.

$$\frac{\omega_1}{\omega_0} \approx 1 - \frac{x_0^2}{16}$$

Belendez's Approx.

(A. Belendez, et. al., Eur. J. Phys. 32, 1303 (2011))

$$\frac{\omega_1}{\omega_0} \approx \frac{1}{4}\left(1 + \sqrt{\cos\frac{x_0}{2}}\right)^2$$

Exact Result

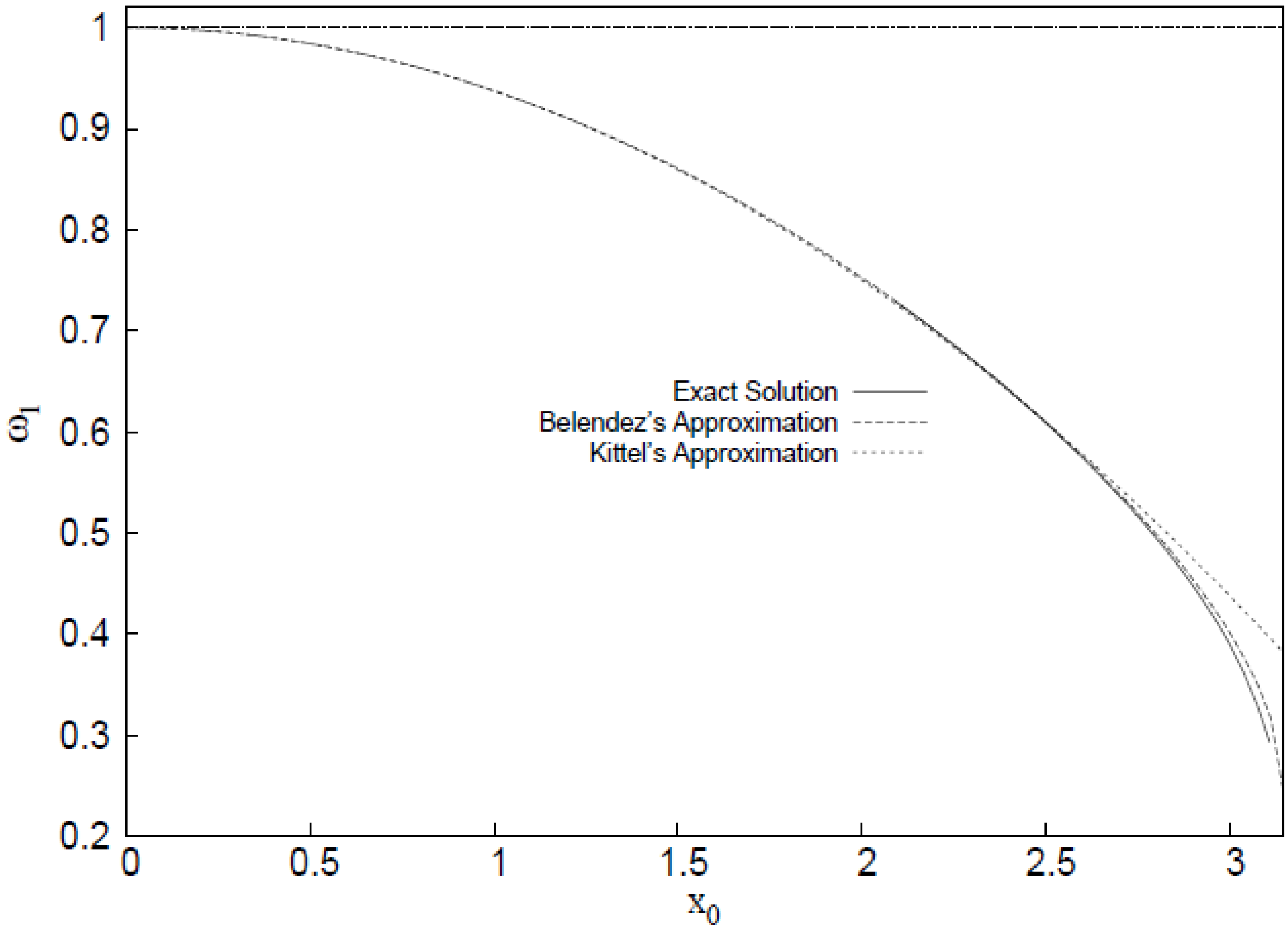
(Euler, 1736,
A. Sommerfeld, MECHANICS)

$$\frac{\omega_1}{\omega_0} = \frac{\pi}{2K(k)}$$

with

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k\sin^2\phi}}$$

$$k = \sin^2\frac{x_0}{2}$$



Dash-dotted horizontal line is the corresponding frequency of oscillation of a driven-damped harmonic oscillator as a function of initial position.

We transform this integral into a standard form by introducing a new variable ψ (Greek psi), defined by:

$$\sin\psi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)}. \tag{140}$$

(*a*) Show that

$$T = \frac{4}{\omega_0}\int_0^{\pi/2} \frac{d\psi}{[1 - \sin^2(\theta_0/2)\sin^2\psi]^{1/2}} = \frac{2\pi}{\omega}. \tag{141}$$

The integral

$$\int_0^{\pi/2} \frac{d\psi}{(1 - K^2\sin^2\psi)^{1/2}}$$

is called a *complete elliptic integral of the first kind;* extensive tables exist for its values (corresponding to different values of K^2); see, for example, Dwight 773.1 and 1040. Here K is just a constant.

The solution of the elliptic integral in series form is:

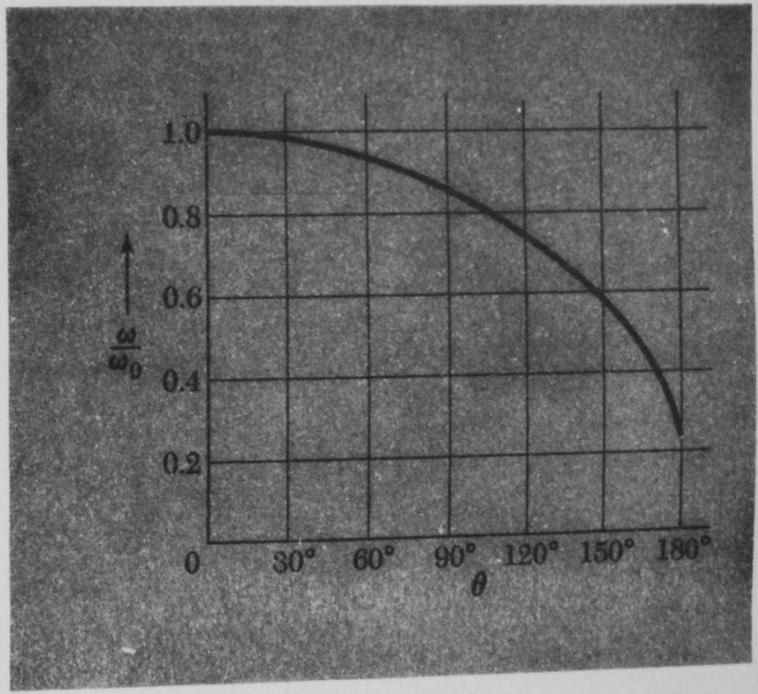
$$\int_0^{\pi/2} \frac{d\psi}{(1 - K^2\sin^2\psi)^{1/2}} = \frac{\pi}{2}\left\{1 + \sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot(2n)}\right]^2 K^{2n}\right\}. \tag{142}$$

If we write out the first few terms of (142),

$$\begin{aligned}\omega &= \omega_0\left[1 + \frac{1}{4}\sin^2\left(\frac{\theta_0}{2}\right) + \frac{9}{64}\sin^4\left(\frac{\theta_0}{2}\right) + \cdots\right]^{-1}\\ &\cong \omega_0\left(1 - \frac{1}{16}\theta_0^2 + \cdots\right),\end{aligned} \tag{143}$$

in agreement with (38).

(*b*) Show that $\omega = 0.847\,\omega_0$ at $\theta_0 = \pi/2$.



Sinusoidal potential and Presence of damping ?

Conclusion:

1. Presence of friction makes analytical solution difficult → Approx. solution given by Jahannessen, Eur. J. Phys. 35, 035014 (2014).
2. Oscillation die out in time.
3. If periodically driven, maximum power absorption close to resonance frequency.

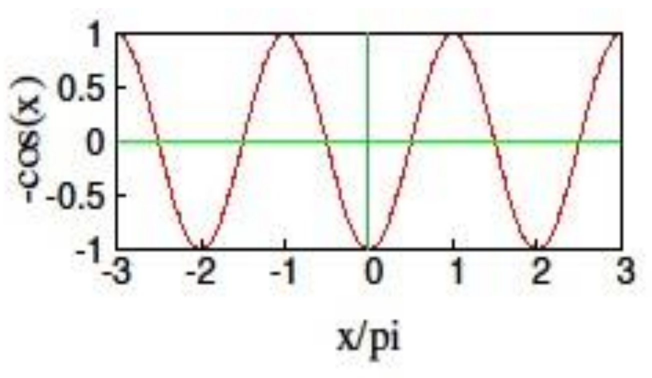
⇨ maximum power absorption → indicator of resonance oscillation and resonance frequency.

Equation of motion with potential $U = -U_0 \cos(kx)$

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - U_0 k \sin(kx) + F_0 \cos(\omega t)$$

In dimensionless form,

$$\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - \sin(x) + F_0 \cos(\omega t)$$



Given F_0 (small) we have only two parameters γ and ω(or τ).

Analytical solution – NOT POSSIBLE.

Find Numerical SOLUTION.

Fix γ, calculate power absorption for different ω or τ.

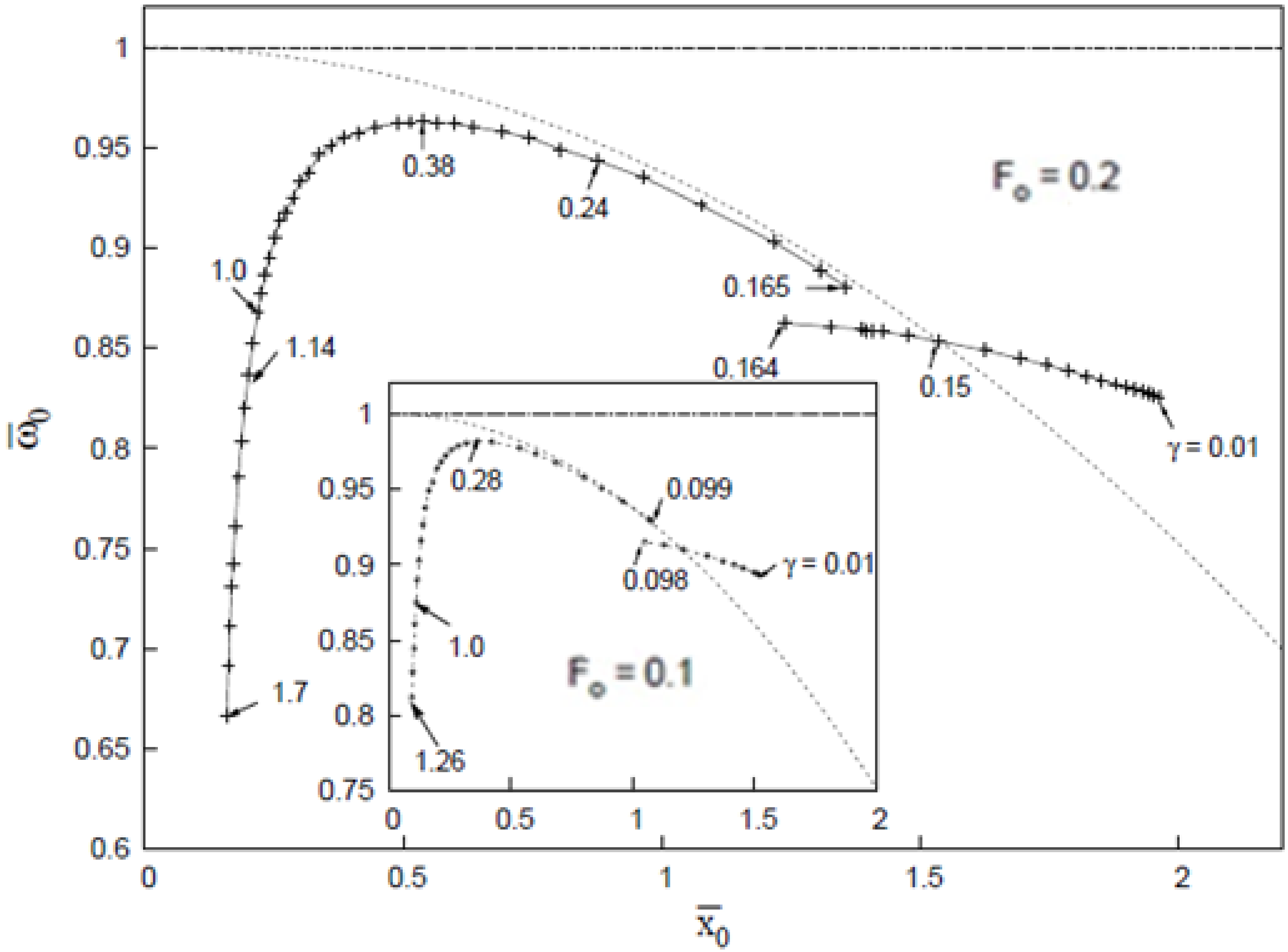
Find τ for which power absorption is maximum.

That ω, we take as resonance frequency $\overline{\omega_0}$

→Calculate the amplitude of oscillation $\overline{x_0}$

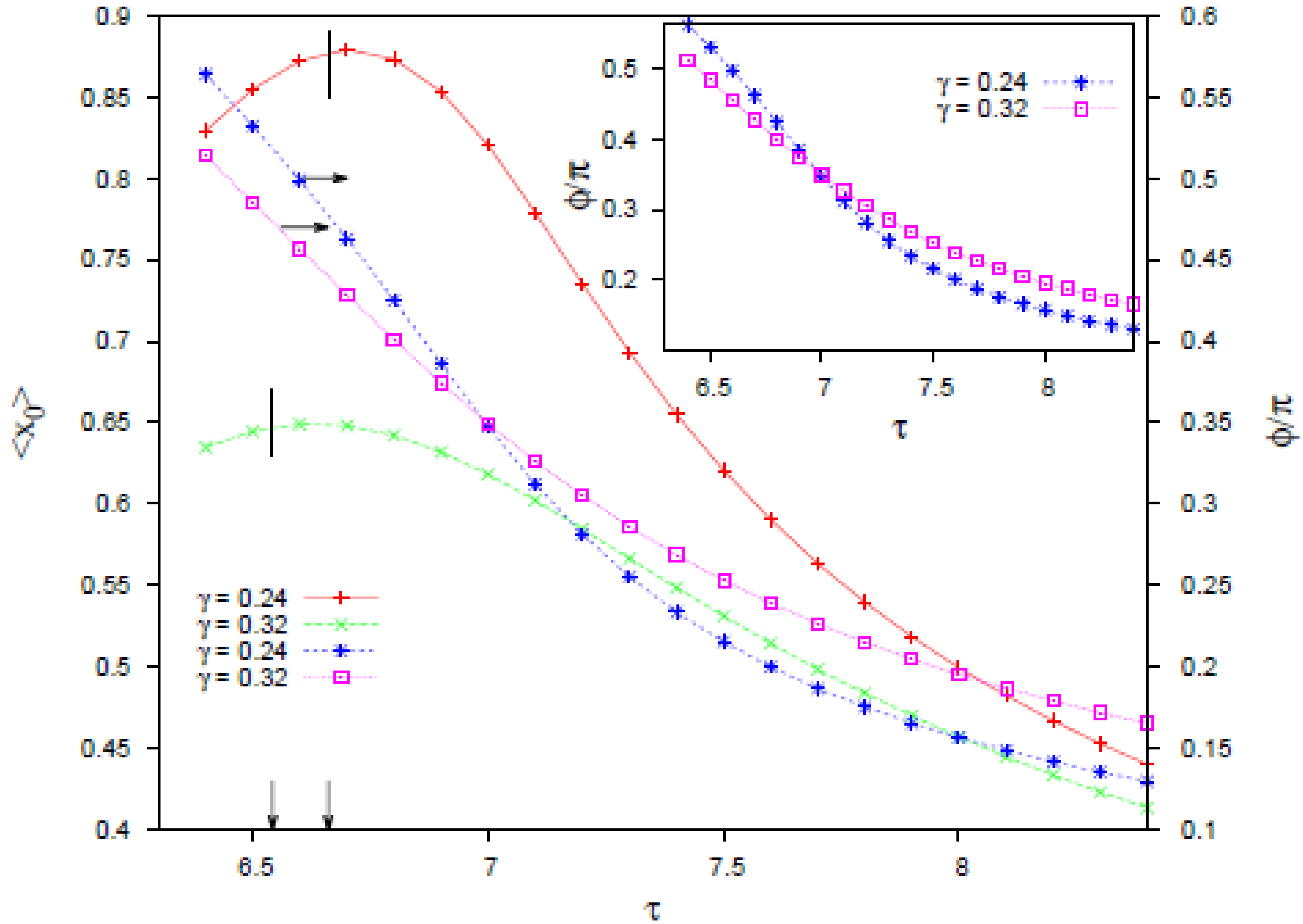
Find $\omega_0(\overline{x_0})$ for different γ.

Plot ω_0 as a function of $\overline{x_0}$

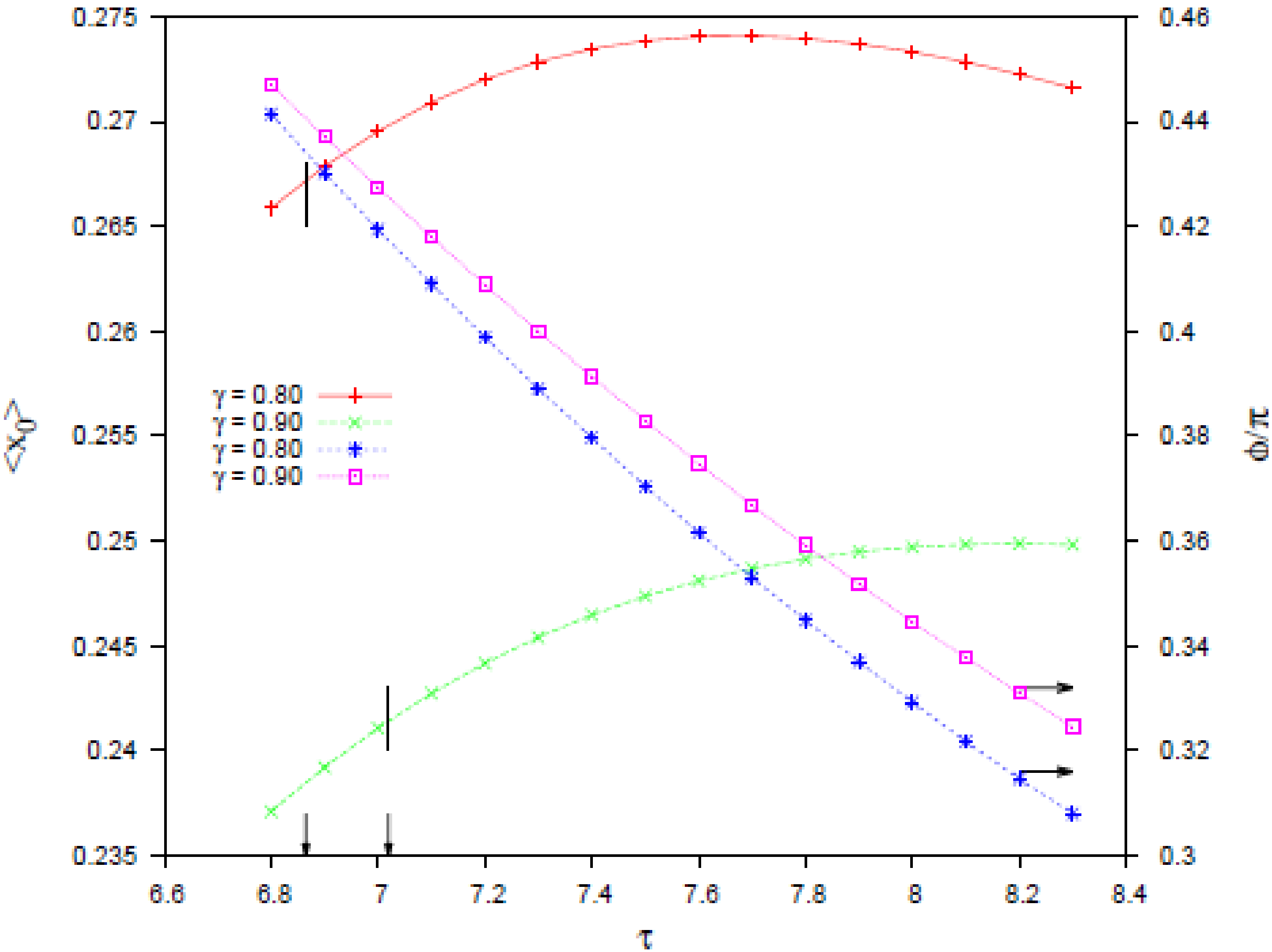


Two very distinct regions!

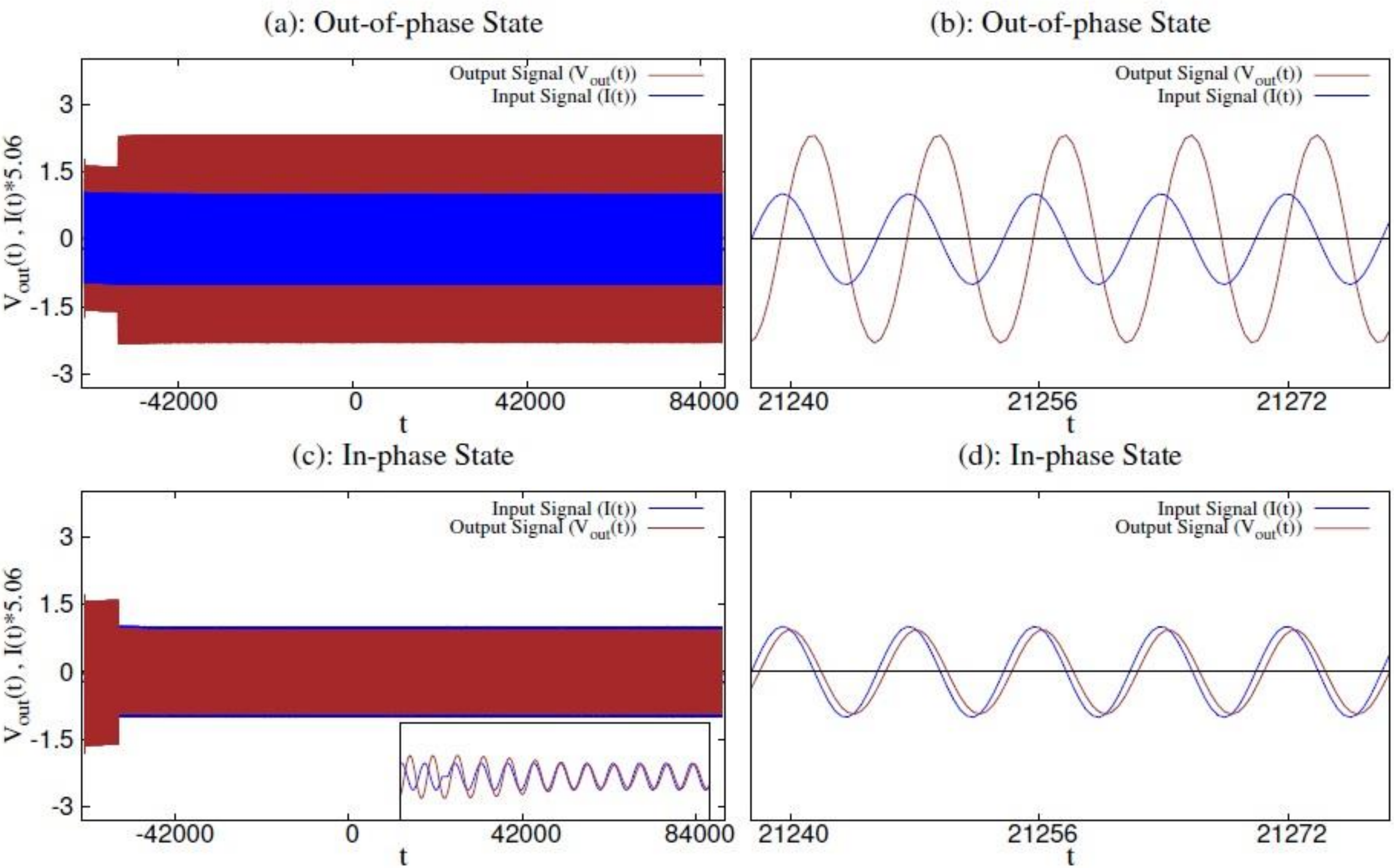
A. Intermediate range of γ values

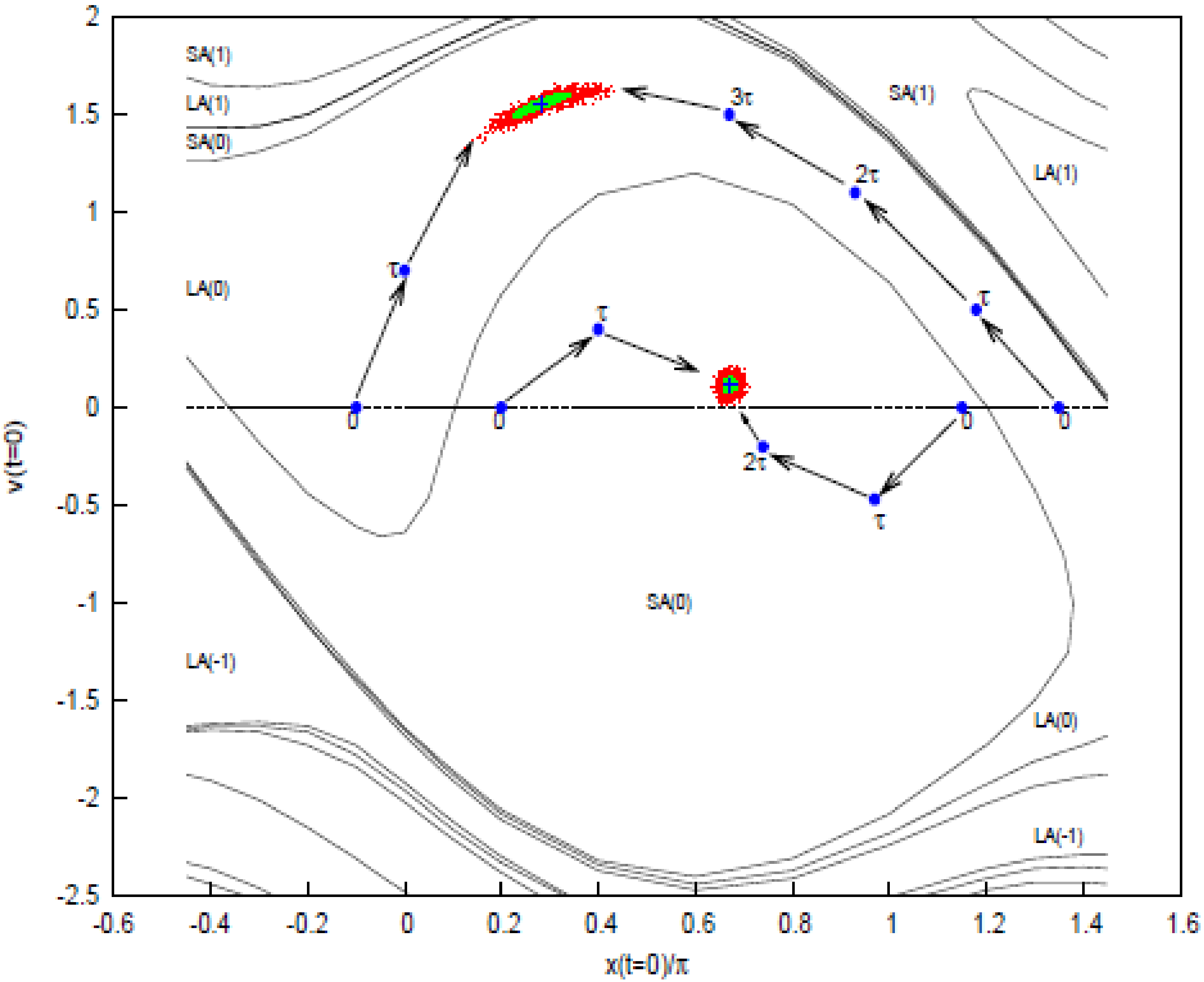


B. Large γ regime

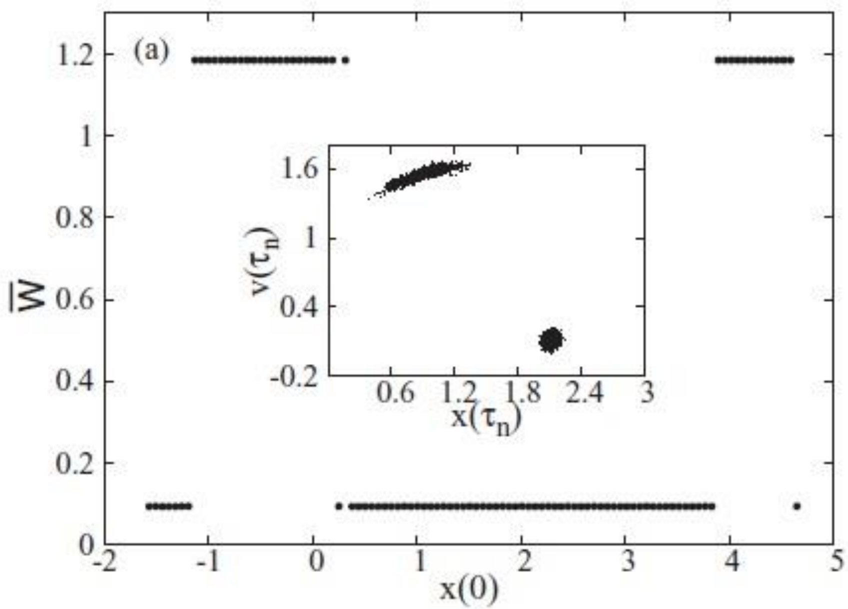


C. Small γ regime

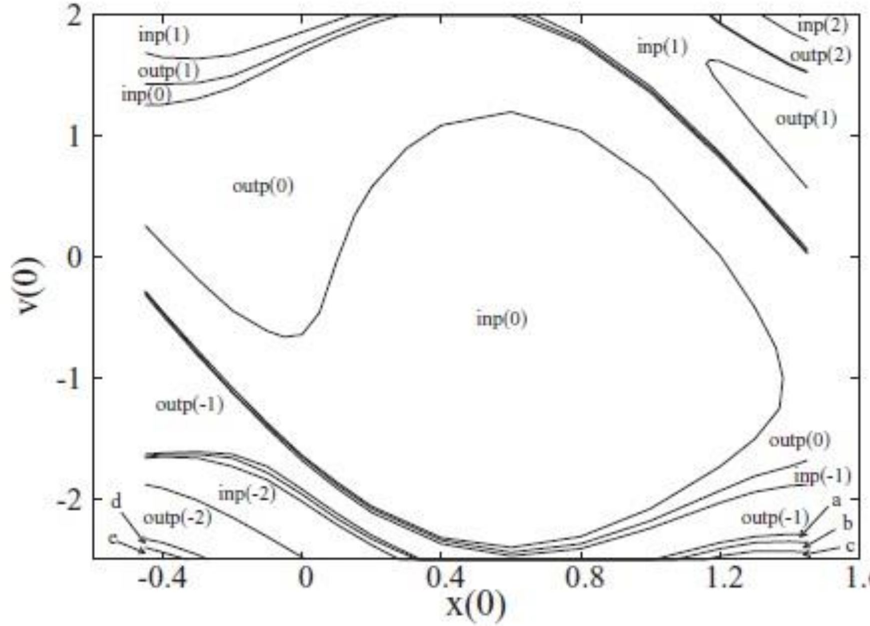




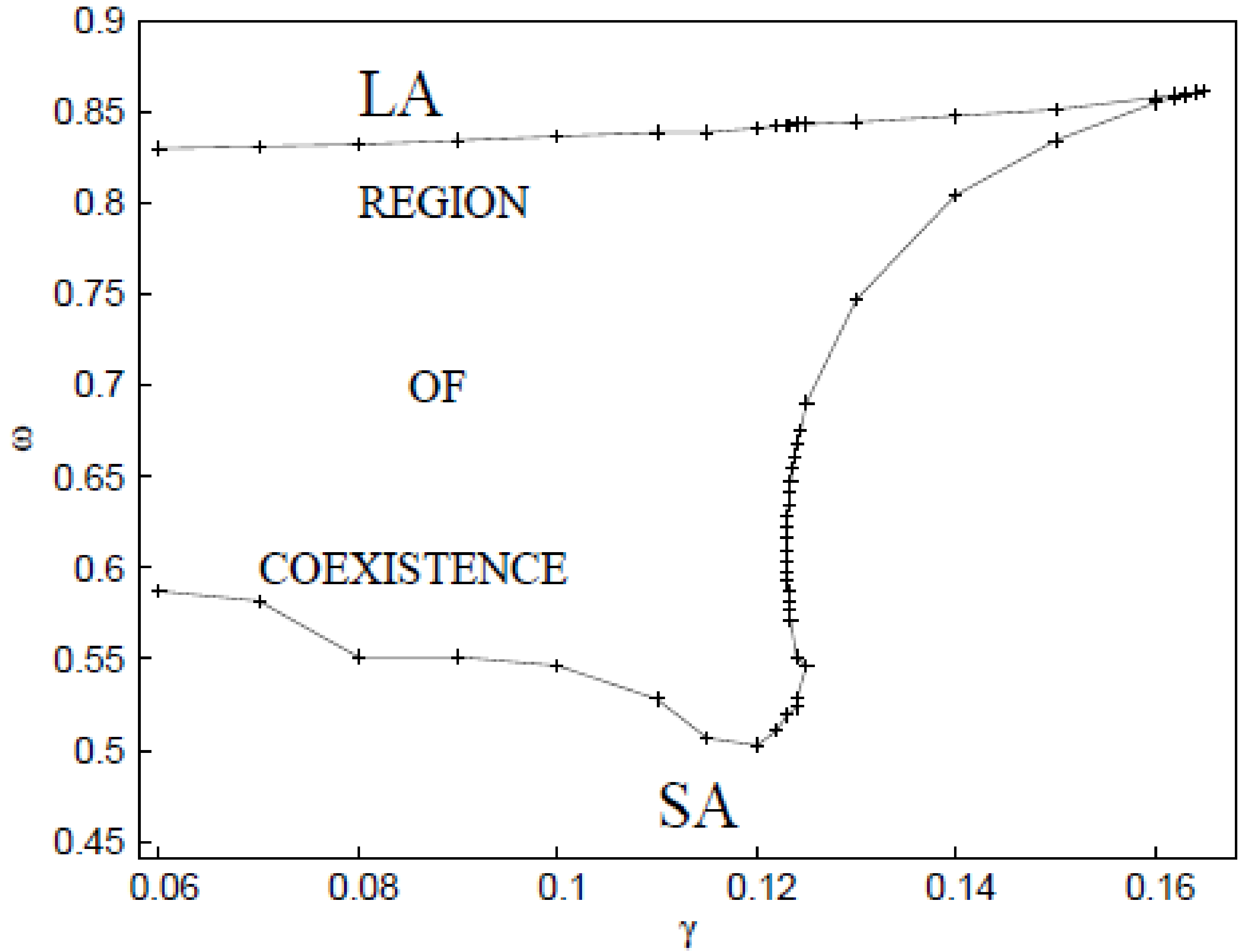
SA(1)
LA(1)
SA(0)
LA(0)
SA(1)
LA(1)
SA(0)
LA(-1)
LA(0)
LA(-1)
τ
3τ
2τ
0
x(t=0)/π
v(t=0)



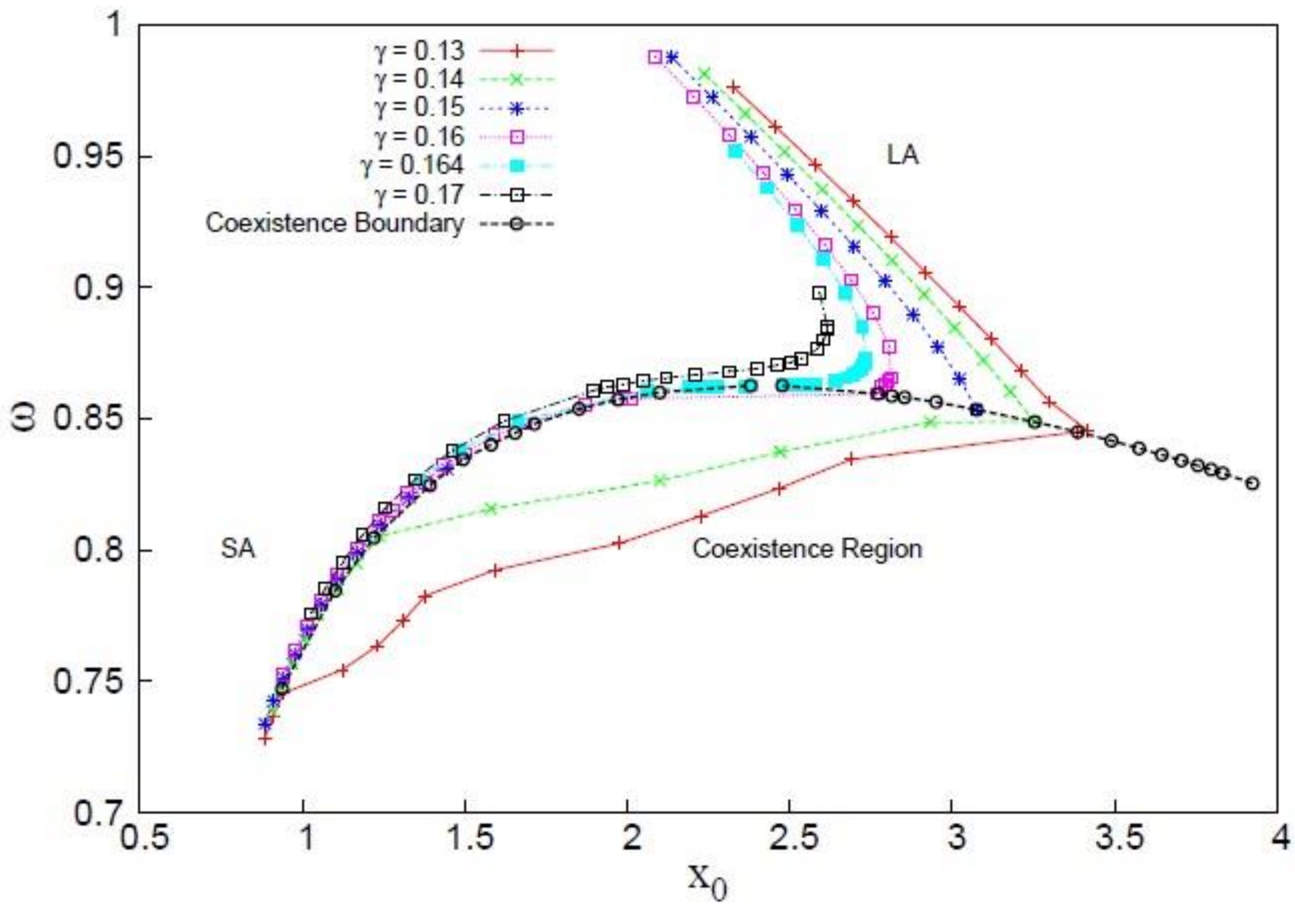
(a)
$\overline{W}$
x(0)
$v(\tau_n)$
$x(\tau_n)$



inp(1)
outp(1)
inp(0)
inp(1)
inp(2)
outp(2)
outp(1)
outp(0)
inp(0)
outp(-1)
outp(0)
inp(-1)
inp(-2)
outp(-2)
outp(-1)
a
b
c
d
e
v(0)
x(0)



LA
REGION
OF
COEXISTENCE
SA
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0.45
ω
0.06
0.08
0.1
0.12
0.14
0.16
γ



The figure is seemingly analogous to the liquid gas pressure-density diagram.

Conclusion:

1. The problem of simple pendulum is not at all simple.
2. Its analytical solution is still elusive.
3. The numerical solution indicates the richness of the problem.
4. The applicability of the problem as a prototype is quite vast.

Thank you for your attention!