Exact solution of a left-permeable open ASEP

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The Model

- Two-species semipermeable ASEP with L sites.
- Particles of type $1, 2$ as well as vacancies.
- Number n_1 of 1's conserved.
- Bulk dynamics

$$2\,0 \underset{q}{\overset{1}{\rightleftharpoons}} 0\,2$$

$$1\,0 \underset{q}{\overset{1}{\rightleftharpoons}} 0\,1$$

$$2\,1 \underset{q}{\overset{1}{\rightleftharpoons}} 1\,2$$

- Left boundary

$$0 \underset{\gamma}{\overset{\alpha}{\rightleftharpoons}} 2$$

- Right boundary

$$2 \underset{\delta}{\overset{\beta}{\rightleftharpoons}} 0$$

Matrix Ansatz

- Let $\tau = (\tau_1, \dots, \tau_L)$ be a configuration.
- The steady state can be computed using

$$P(\tau) = \frac{1}{Z(L,n)} \langle W| \prod_{i=1}^{L} (\delta_{\tau_i,2} D + \delta_{\tau_i,1} A + \delta_{\tau_i,0} A)\, |V\rangle.$$

- The algebra of the matrices and boundary vectors is given by

$$\begin{aligned} DE - qED &= D + E, \\ DA - qAD &= A, \\ AE - qEA &= A, \\ (\beta D - \delta E)|V\rangle &= |V\rangle, \\ \langle W|(\alpha E - \gamma D) &= \langle W|. \end{aligned}$$

- Solved by M. Uchiyama (*Chaos, Solitons and Fractals*, 2007)

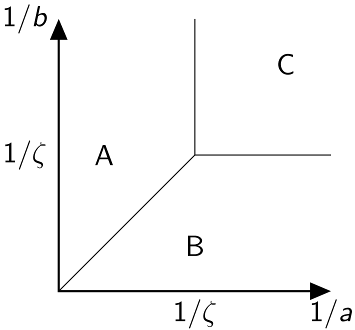
The Phase Diagram

Take $L \to \infty$ such that $n_1/L \to \rho$.

$$a = \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha},$$

$$b = \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta},$$

$$\zeta = \frac{1+\rho}{1-\rho}.$$



Densities and Currents

Region	Density of 2's	Current of 2's
A	$\dfrac{1}{1+a}$	$(1-q)\dfrac{a}{(1+a)^2}$
B	Piecewise constant	$(1-q)\dfrac{b}{(1+b)^2}$
C	$\dfrac{1-\rho}{2}$	$(1-q)\dfrac{1-\rho^2}{4}$

The two-species left-permeable ASEP

- One-dimensional lattice of size L
- No conservation
- Bulk rules same as before

$$j\,i \quad \underset{q}{\overset{1}{\rightleftharpoons}} i\,j \quad \text{if } j > i$$

- Left boundary

$$0 \to 1 \text{ with rate } \gamma,$$
$$0, 1 \to 2 \text{ with rate } \alpha,$$
$$2 \to 1 \text{ with rate } \widetilde{\gamma}.$$

where $\widetilde{\gamma} = \dfrac{\alpha + \gamma + q - 1}{\alpha + \gamma}\gamma$. Note that $\alpha + \gamma + q > 1$.

- Right boundary

$$2 \underset{\delta}{\overset{\beta}{\rightleftharpoons}} 0$$

Matrix Ansatz

- Denote the matrices for $2, 1, 0$ as D, A, E.
- Representation for this model is given by bulk relations

$$\begin{aligned} DE - qED &= D + E, \\ AE - qEA &= A, \\ DA - qAD &= A, \end{aligned}$$

and boundary relations

$$\begin{aligned} (\alpha+\gamma)\langle W|E &= \langle W|, \\ \langle W|\left(\gamma E - \alpha A + \widetilde{\gamma} D\right) &= 0, \\ \left(-\delta E + \beta D\right)|V\rangle &= |V\rangle. \end{aligned}$$

- We define matrices $\mathbf{e}$, $\mathbf{d}$, satisfying the q-deformed harmonic oscillator algebra,

$$\begin{aligned} \mathbf{de} - q\mathbf{ed} &= 1 - q, \\ \langle W|\mathbf{e} + ac\langle W|\mathbf{d} &= (a+c)\langle W|, \\ \mathbf{d}|V\rangle + bd\mathbf{e}|V\rangle &= (b+d)|V\rangle, \end{aligned}$$

  where a, b, c, d are expressed in terms of the boundary rates.

- Then the operators

$$D = \frac{1}{1-q}(1+\mathbf{d}), \qquad E = \frac{1}{1-q}(1+\mathbf{e}),$$
$$A = \lambda(DE - ED) = \frac{\lambda}{1-q}(1-\mathbf{ed}),$$

  satisfy the desired algebra, where $\lambda = \gamma/\alpha$.

Continuous big q-Hermite polynomials

- The q-shifted factorial is given by

$$(a_1, \ldots, a_s; q)_n = \prod_{r=1}^{s} (a_r; q)_n,$$

where

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k) = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}),$$

valid also for $n \to \infty$ when $q < 1$.

- The basic hypergeometric series is given by

$${}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(q, b_1, \ldots, b_s; q)_k} \times \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} z^k.$$

Continuous big q-Hermite polynomials

- Define

$$F_n(u,v;\lambda)=\sum_{k=0}^{n}\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}(\lambda u;q)_k v^k u^{n-k},$$

  which satisfies a three-term recurrence relation.
- Specialisation of the parameters u, v gives the continuous big q-Hermite polynomial,

$$H_n(\cos\theta;\lambda|q)=F_n(e^{\mathrm{i}\theta},e^{-\mathrm{i}\theta};\lambda).$$

Representation of the algebra

- The operators can be represented as

$$\mathbf{d} = \sum_{n=1}^{\infty} \sqrt{1-q^n}|n-1\rangle\langle n|,$$

$$\mathbf{e} = \sum_{n=0}^{\infty} \sqrt{1-q^{n+1}}|n+1\rangle\langle n|,$$

- Writing the boundary vectors as

$$\langle W| = \sum_{n=0}^{\infty} w_n \langle n|, \qquad |V\rangle = \sum_{n=0}^{\infty} v_n |n\rangle,$$

we find that

$$w_n = \frac{F_n(a,c;0)}{\sqrt{(q;q)_n}}, \qquad v_n = \frac{F_n(b,d;0)}{\sqrt{(q;q)_n}}.$$

Partition function

- Define the nonequilibrium partition function

$$Z_L(\xi^2,\zeta) = \langle W| \left(E + \xi^2 D + \zeta A\right)^L |V\rangle,$$

so that ξ^2, ζ are fugacities for type 1 and 2 particles.

- We then obtain

$$Z_L(\xi^2,\zeta) = \int_0^\pi \frac{\mathrm{d}\theta}{2\pi} w(\cos\theta; \lambda\zeta\xi^{-1})\, \Theta(\cos\theta; 0, \xi^{-1}c|\lambda\zeta\xi^{-1}) \\ \times \Theta(\cos\theta; \xi b, \xi d|\lambda\zeta\xi^{-1}) \left(\frac{1+\xi^2+2\xi\cos\theta}{1-q}\right)^L,$$

where Θ is an explicit ${}_2\phi_2$ and w is the weight for the orthogonal polynomials.

Observables

- The bulk densities are given by

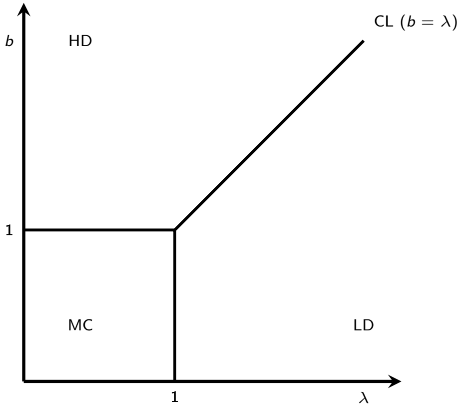
$$\rho^{(1)} = \frac{1}{L}\frac{\partial}{\partial \zeta} \log Z_L(\xi^2,\zeta)\big|_{\xi^2=\zeta=1},$$
$$\rho^{(2)} = \frac{1}{L}\frac{\partial}{\partial \xi^2} \log Z_L(\xi^2,\zeta)\big|_{\xi^2=\zeta=1}.$$

- Clearly, the current of 1's is 0. The current of 2's can be shown to be

$$J^{(2)} = \frac{Z_{L-1}}{Z_L}.$$

Phase diagram in the thermodynamic limit

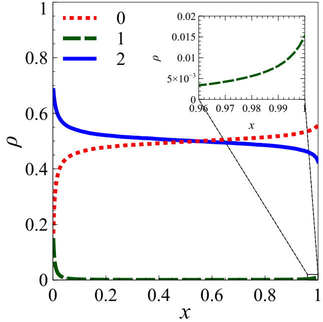
$$b = \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}, \lambda = \gamma/\alpha,$$



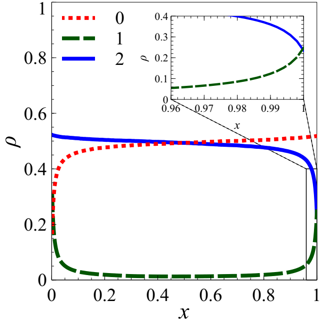
Densities and Currents

Phase	$\rho^{(1)}$	$\rho^{(2)}$	Current $J^{(2)}$
Maximal current (MC)	$\mathcal{O}\left(\frac{1}{L}\right)$	$\frac{1}{2}$	$\frac{1-q}{4}$
Low density (LD)	$\frac{\lambda-1}{\lambda+1}$	$\frac{1}{\lambda+1}$	$\frac{(1-q)\lambda}{(1+\lambda)^2}$
High density (HD)	$\mathcal{O}\left(\frac{1}{L}\right)$	$\frac{b}{1+b}$	$\frac{(1-q)b}{(1+b)^2}$
Coexistence line (CL)	linear	linear	$\frac{(1-q)b}{(1+b)^2}$

Simulations: MC phase

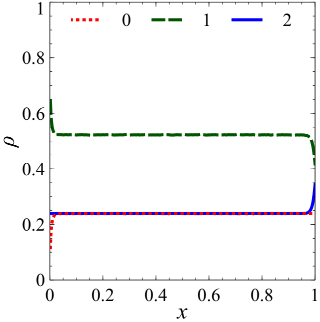


(a) $L = 500, \alpha = 0.62, \gamma = 0.23, \beta = 0.83, \delta = 0.37, q = 0.41$

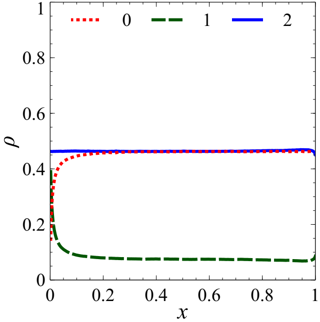


(b) $L = 500, \alpha = 0.45, \gamma = 0.41, \beta = 0.85, \delta = 0.1, q = 0.41$

Simulations: LD phase

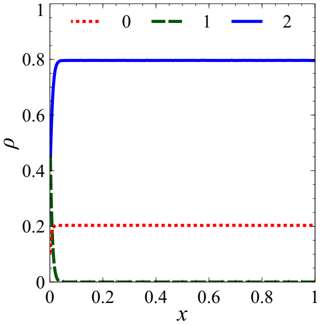


(c) $L = 500, \alpha = 0.22, \gamma = 0.7, \beta = 0.64, \delta = 0.47, q = 0.41$

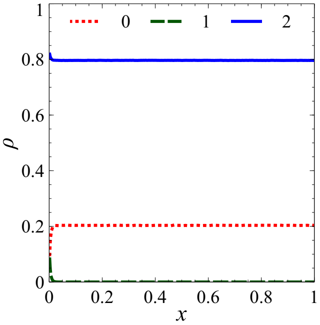


(d) $L = 500, \alpha = 0.5, \gamma = 0.58, \beta = 0.95, \delta = 0.61, q = 0.41$

Simulations: HD phase

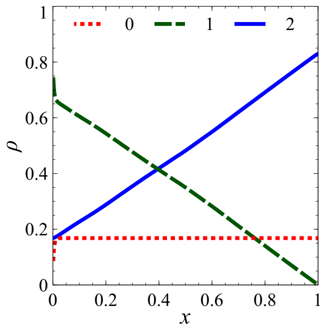


(e) $L = 500, \alpha = 0.32, \gamma = 0.56, \beta = 0.35, \delta = 0.9, q = 0.41$

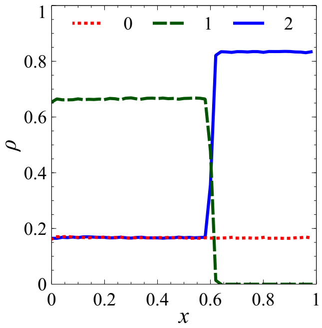


(f) $L = 500, \alpha = 0.81, \gamma = 0.16, \beta = .35, \delta = 0.9, q = 0.41$

Simulations: CL



(g) Time-averaged densities for $L = 500$ with $\alpha = 0.15, \gamma = 0.74, \beta = 0.28, \delta = 0.89, q = 0.41$.



(h) Instantaneous density profiles for $L = 2500$, coarse-grained over 50 sites with $\alpha = 0.15, \gamma = 0.74, \beta = 0.28, \delta = 0.89, q = 0.41$.

Thank you for your attention!

For more details, please see Dipankar's poster!