Extremal statistics in 1d Coulomb gas

Anupam Kundu

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ICTS | International Centre for Theoretical Sciences
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

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Joint work with :

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- Satya N. Majumdar (LPTMS, Orsay)
- Sanjib Sabhapandit (RRI, Bangalore)
- Gregory Scher (LPTMS, Orsay)

AD, AK, SNM, SS, GS, PRL 119, 060601, (2017)

Universality

Universality

It is a common concept in statistical physics.

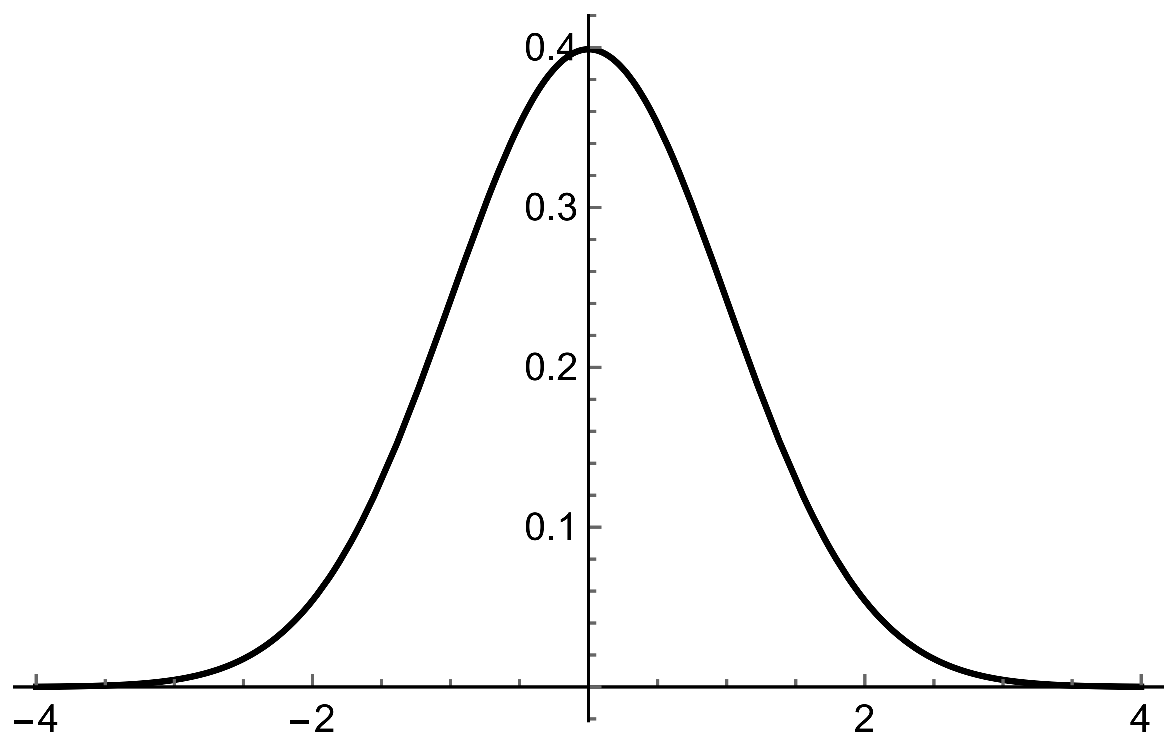
Central Limit Theorem:

Let $(x_1, x_2, ..., x_N)$ are N independent and identically distributed random variables chosen from $p(x)$ with finite moments

Sum: $X_N = \sum_i x_i$ $\quad P(X_N = X) \xrightarrow[N\to\infty]{} G\left(\frac{X-\mu N}{\sqrt{N}}\right)$

$$G(z) = \frac{\exp(-z^2/2)}{\sqrt{2\pi}}$$

Irrespective of $p(x)$!!!



Extreme value statistics:

Let $(x_1, x_2, ..., x_N)$ are N independent and identically distributed random variables chosen from $p(x)$

$$X_{max} = \max_{1 \leq i \leq N} \{x_i\} \qquad Q_N(X) = Prob.[X_{max} \leq X]$$

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$$Q_N(X) \xrightarrow[N\to\infty]{} F\left(\frac{X-a_N}{b_N}\right) \quad \text{or} \quad Q_N(a_N + b_N z) \xrightarrow[N\to\infty]{} F(z)$$

$F'(z)$ Universal scaling function: Only of 3 possible types depending on the tails of $p(x)$

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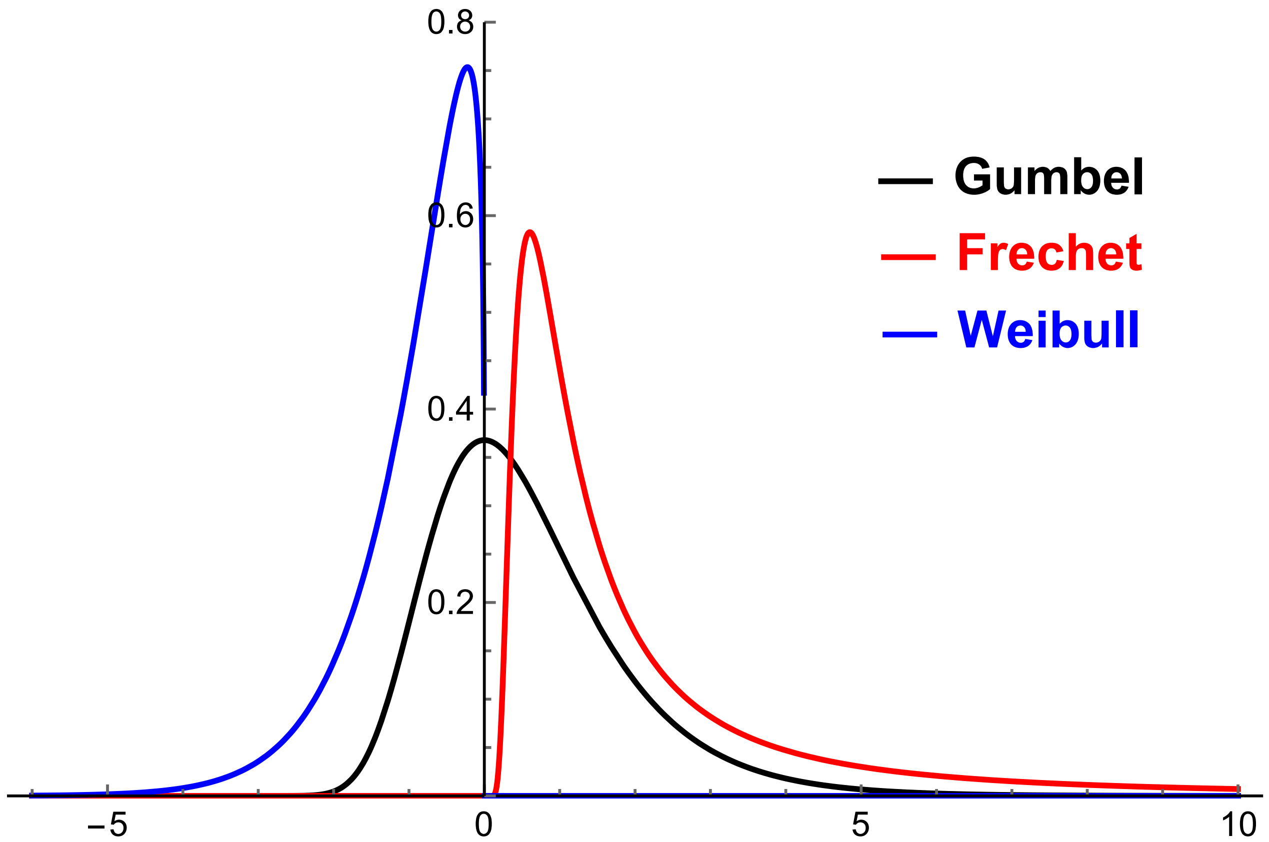

Extreme value statistics:

If $(x_1, x_2, ..., x_N)$ are N strongly correlated random variables?

Given $P(x_1, x_2, ..., x_N)$

what is the distribution of $X_{max} = \max_{1 \leq i \leq N} \{x_i\}$?

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Random Matrix theory:

NxN Gaussian random Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}$$

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$$\begin{aligned} Prob.[A] & \propto \exp\left[-\beta \frac{N}{2} \sum_{ij} a_{ij}^2\right] \\ & = \exp\left[-\beta \frac{N}{2} Tr(A^\dagger A)\right] \end{aligned}$$

Invariant under rotations

Spectral statistics?

Spectral statistics in RMT:

N real eigenvalues (scaled) $(\lambda_1, \lambda_2, ..., \lambda_N)$ strongly correlated random variables

$$P(\lambda_1, \lambda_2, ..., \lambda_N) = \frac{1}{Z_N} \exp\left[-\beta \frac{N}{2} \sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

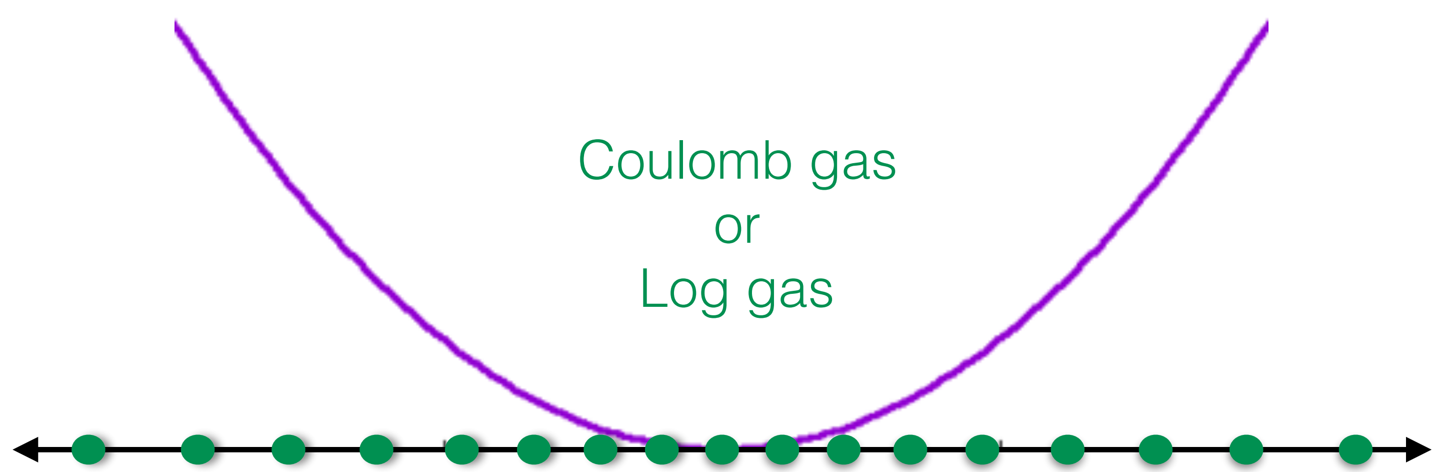
(Wigner, 1951)

Dyson index $\beta = 1$ (GOE), $\beta = 2$ (GUE), $\beta = 4$ (GSE)

Coulomb gas interpretation: (Dyson, 1962)

$$P(\lambda_1, \lambda_2, ..., \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2}\left(N \sum_{i=1}^{N} \lambda_i^2 - \sum_{j \neq k} \log|\lambda_j - \lambda_k|\right)\right] = \frac{1}{Z_N} \exp[-\beta E(\{x_i\})]$$

Boltzmann weight of a gas of N pairwise repelling charges confined in an external harmonic potential $V(\lambda) \sim \lambda^2$



Spectral Density:

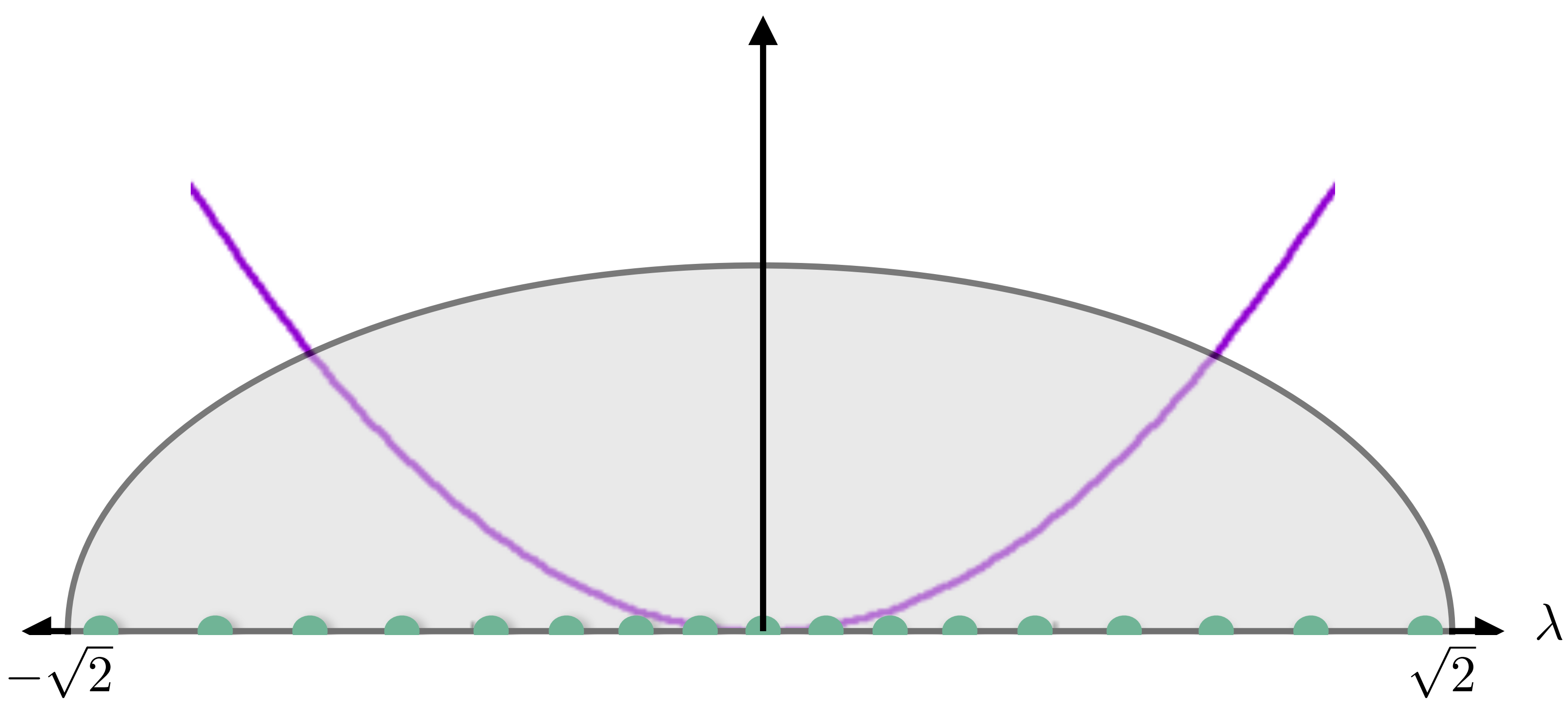
$$P(\lambda_1, \lambda_2, ..., \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2}\left(N\sum_{i=1}^{N}\lambda_i^2 - \sum_{j\neq k}\log|\lambda_j - \lambda_k|\right)\right]$$

Average density:

$$\rho_N(\lambda) = \tfrac{1}{N}\textstyle\sum_{i=1}^{N}\langle\delta(\lambda - \lambda_i)\rangle$$

Wigner semi circle:

$$\rho_N(\lambda) \xrightarrow[N\to\infty]{} \rho(\lambda) = \tfrac{1}{\pi}\sqrt{2-\lambda^2}$$



Top eigenvalue:

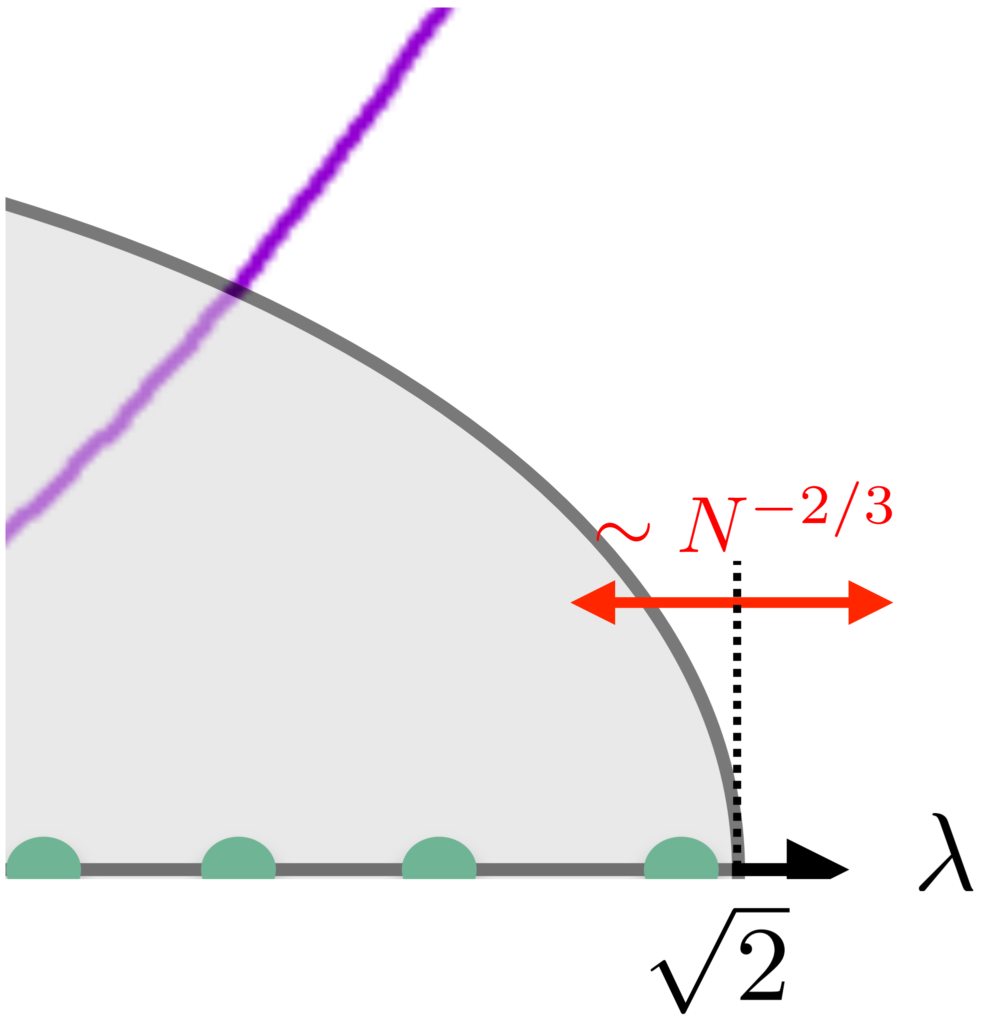
$$\lambda_{max} = \max_{1\leq i\leq N}\{\lambda_i\}$$

Average: $\langle\lambda_{max}\rangle = \sqrt{2}$

Typical fluctuation: $\sim N^{-2/3}$

Distribution of λ_{max}

$$P_N(\lambda_{max} = w) = ?$$



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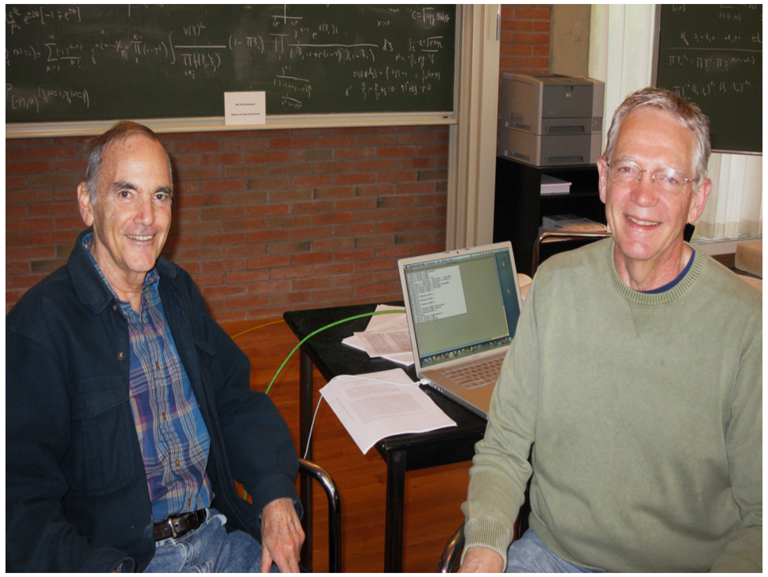


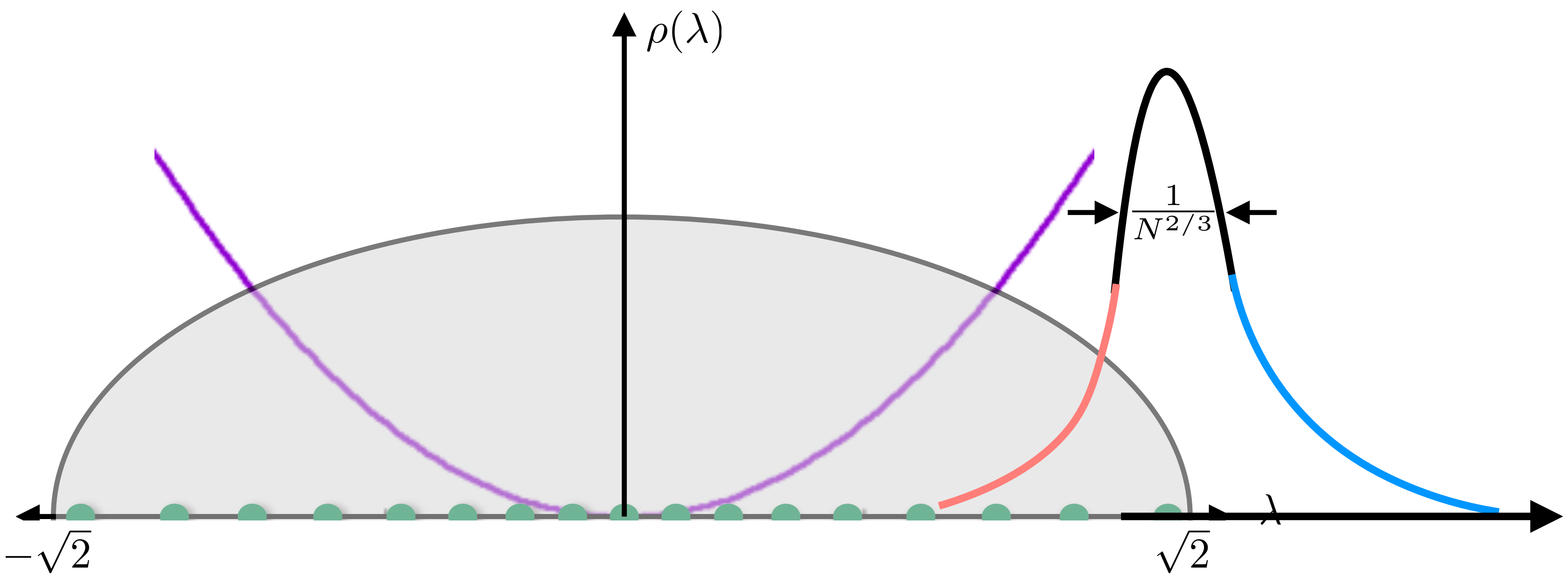
H. Widom and C. Tracy (1994)

Distribution of λ_{max} : Typical fluctuation

$$P_N(\lambda_{max} = w) \quad \xrightarrow[N\to\infty]{} \quad \sqrt{2}N^{2/3}\ f_\beta\left(\sqrt{2}N^{2/3}(w-\sqrt{2})\right) \quad for\ \ |w-\sqrt{2}| \sim \mathcal{O}(N^{-2/3})$$

$$f_\beta(z) \quad \leftarrow \quad Tracy - Widom\ distribution$$

C. Tracy and H. Widom (1994)



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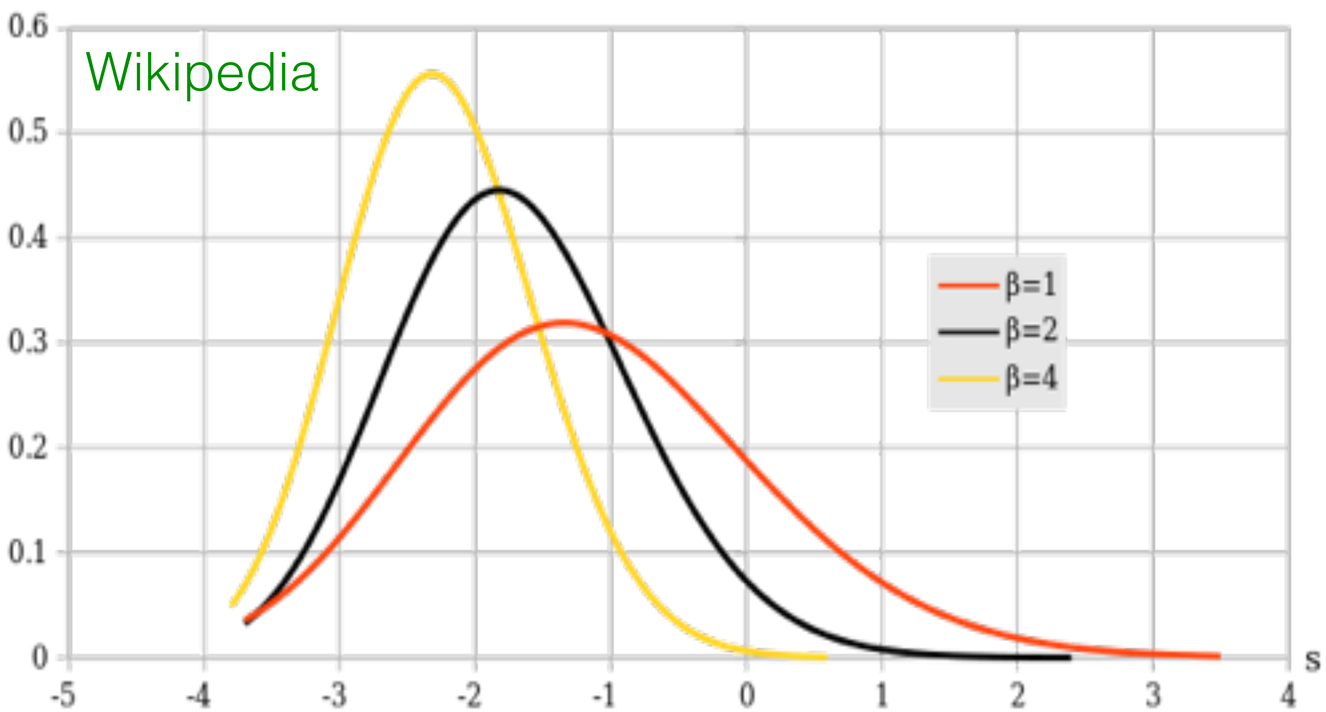

$$f_\beta(z)=\frac{dF_\beta(z)}{dz}$$

Expressed in terms of the H-M solutions of Painlevé II equation

$$f_2(z) = \frac{dF_2(z)}{dz}$$

$$F_2(z) = \exp\left(-\int_x^\infty (y-z)q^2(y)dy\right)$$

$$\frac{d^2q(y)}{dy^2} = 2q^3(y)+y\ q(y)$$

$$q(y\to\infty)\to Ai(y)$$

Distribution of λ_{max} : Typical fluctuation

$$P_N(\lambda_{max} = w) \quad \xrightarrow[N\to\infty]{} \quad \sqrt{2}N^{2/3}\ f_\beta\left(\sqrt{2}N^{2/3}(w-\sqrt{2})\right) \quad for \ \ |w-\sqrt{2}| \sim \mathcal{O}(N^{-2/3})$$

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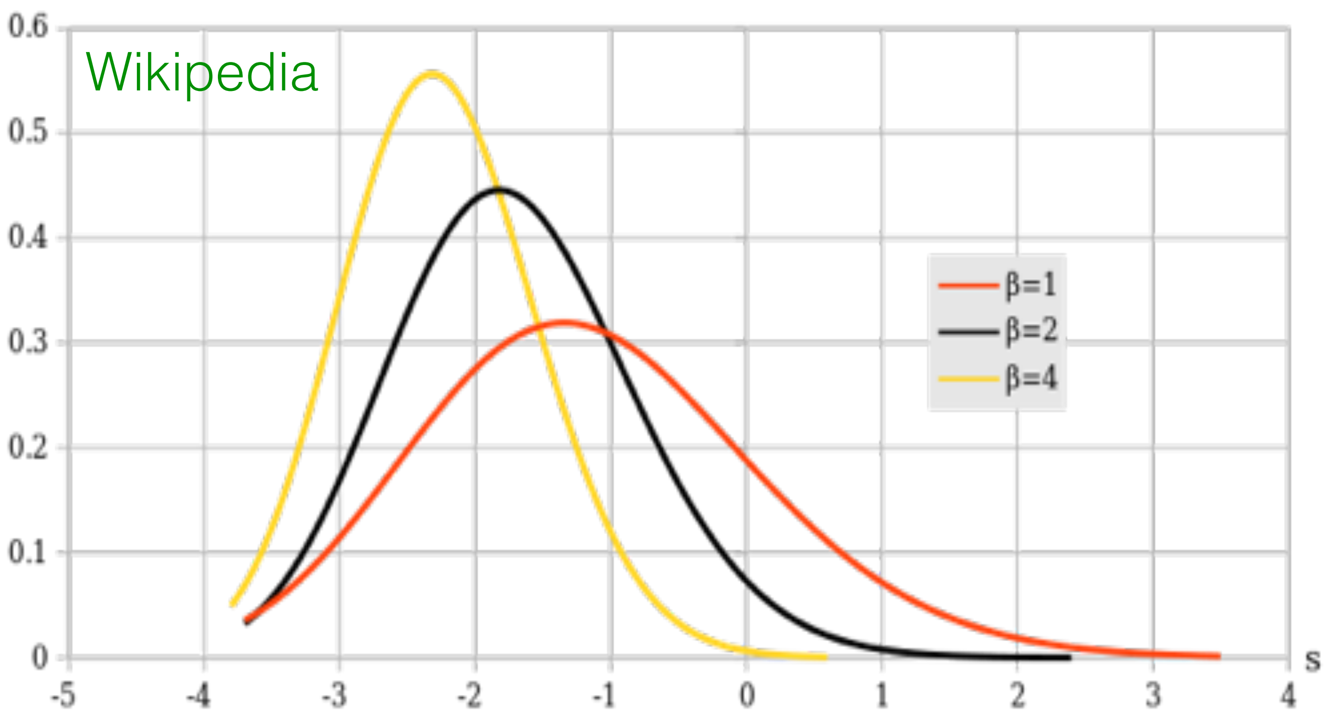


$$f_\beta(z) = \frac{dF_\beta(z)}{dz}$$

Expressed in terms of the H-M solutions of Painlevé II equation

Asymptotic behaviour:

$$f_\beta(z) \quad \sim \quad \exp\left(-\frac{\beta}{24}|z|^3\right) \ \ as\ z\to-\infty$$

$$\sim \quad \exp\left(-\frac{2\beta}{3}z^{3/2}\right) \ \ as\ z\to\infty$$

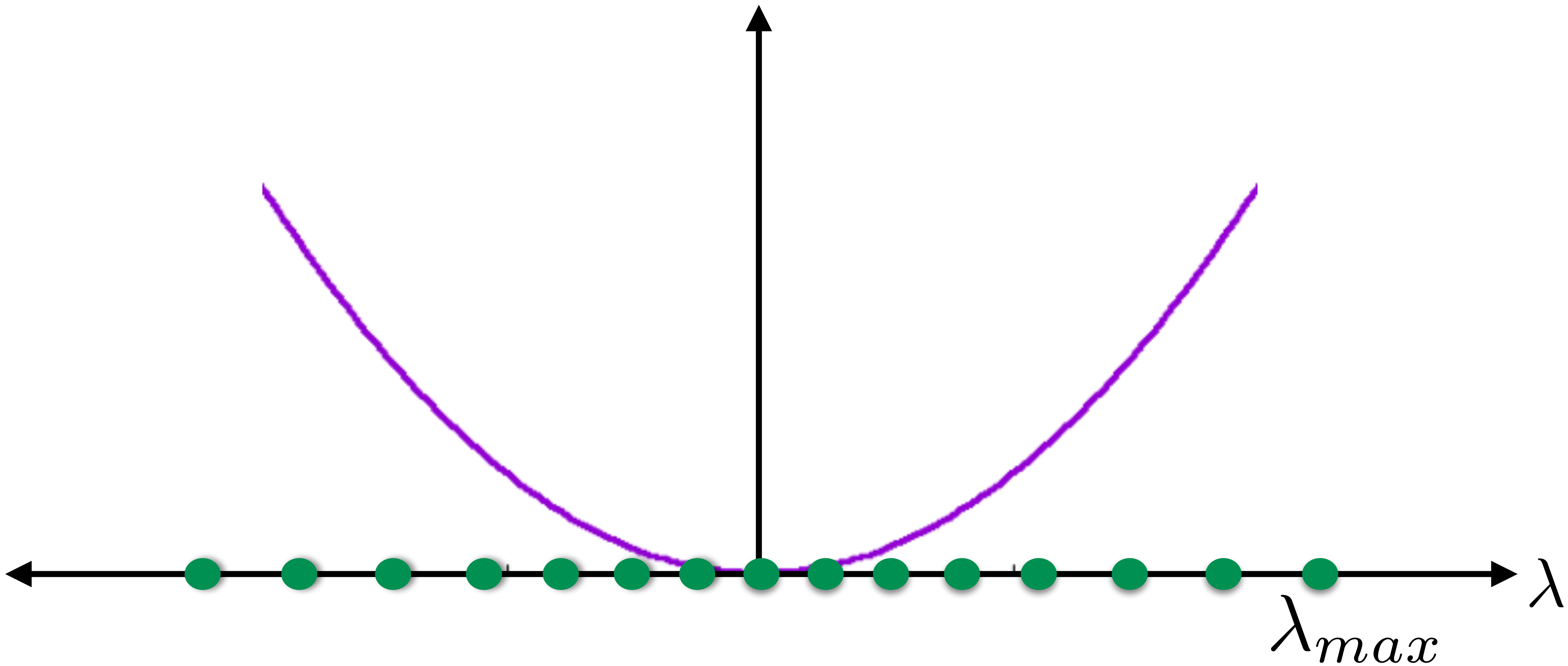
Tracy-Widom distribution is ubiquitous

- Ulam problem of longest increasing subsequence
- Kardar-Parisi-Zhang equation in (1+1) dimension
- Height fluctuations in stochastic growth models in KPZ class
- Maximum displacement in non-intersecting Brownian bridges
- Mesoscopic fluctuations of the spectrum in quantum dots
- EVS in non-interacting fermions confined in a harmonic potential
- Fluctuations in Financial performances
- Observed in Liquid crystal experiments
- and in coupled lasers

"Equivalence Principle",
M. Buchanan,
Nature Phys.
10, 543 (2014)

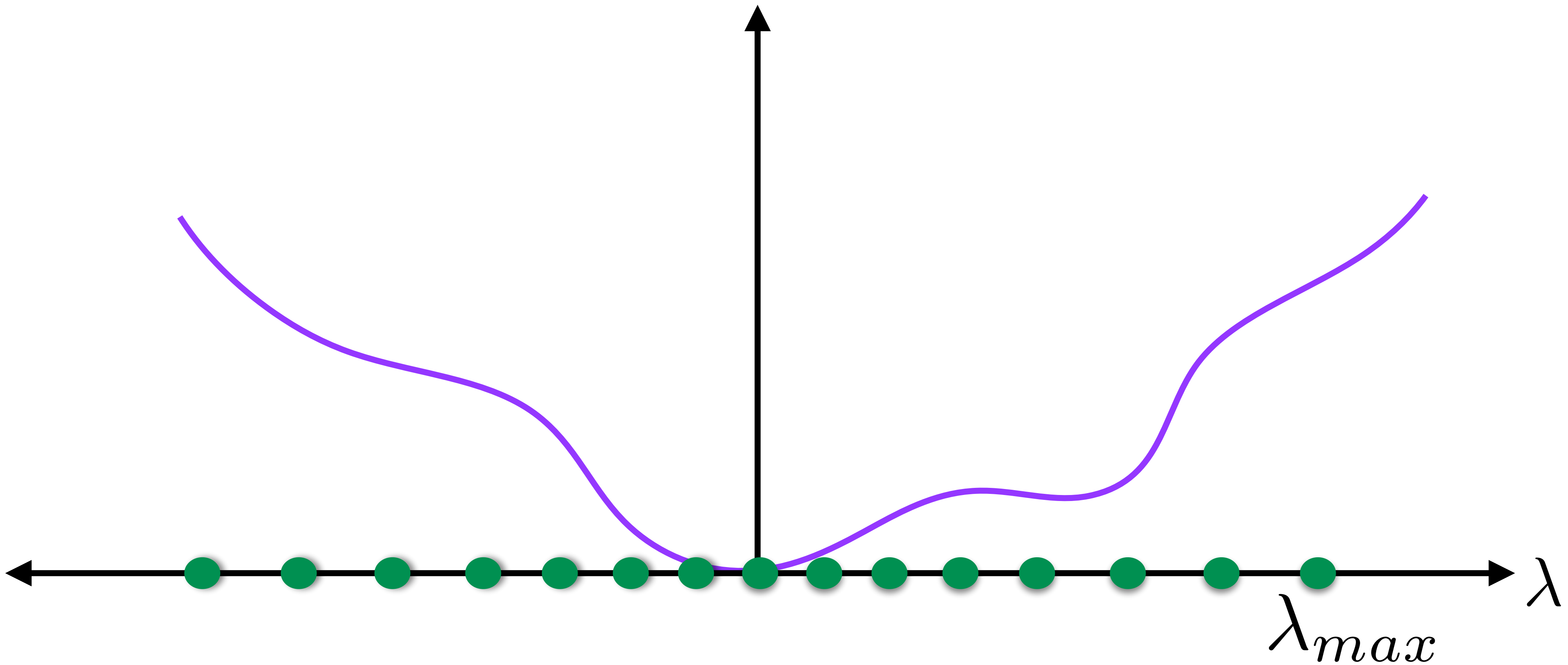
"At the far ends of a new universal law",
N. Wolchover,
Quanta Magazine
(october, 2014)

Another Universality:



$$P(\lambda_1, \lambda_2, ..., \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2}\left(N\sum_{i=1}^{N}\lambda_i^2 - \sum_{j\neq k}\log|\lambda_j - \lambda_k|\right)\right]$$

Another Universality:



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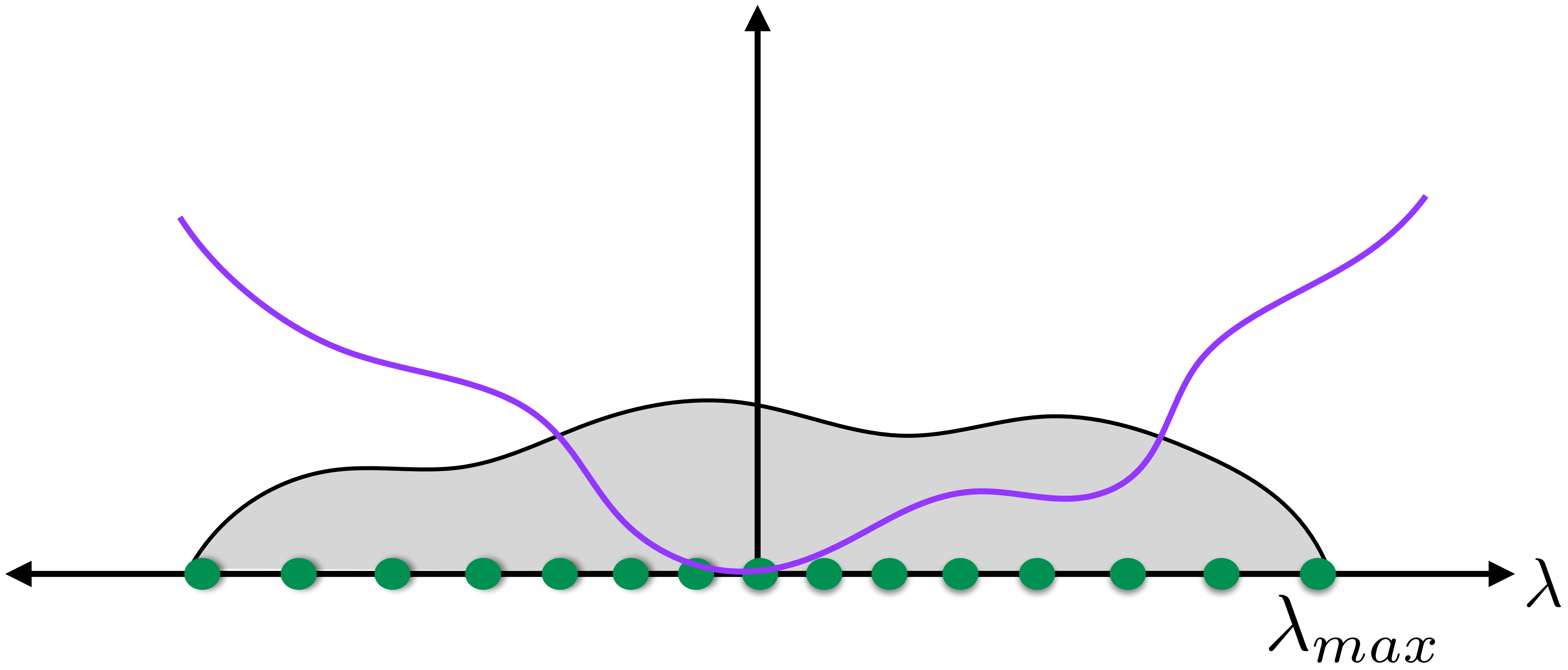

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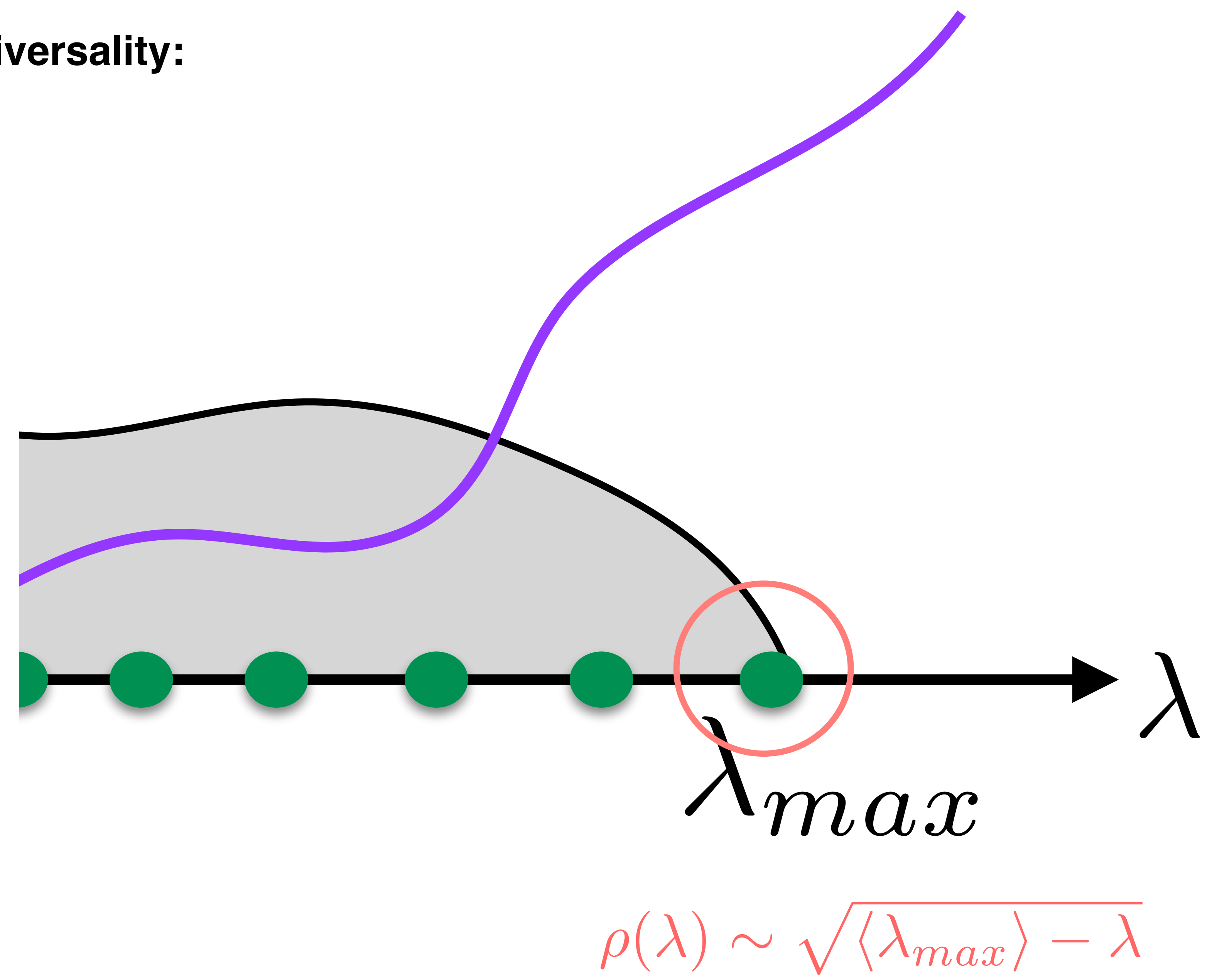

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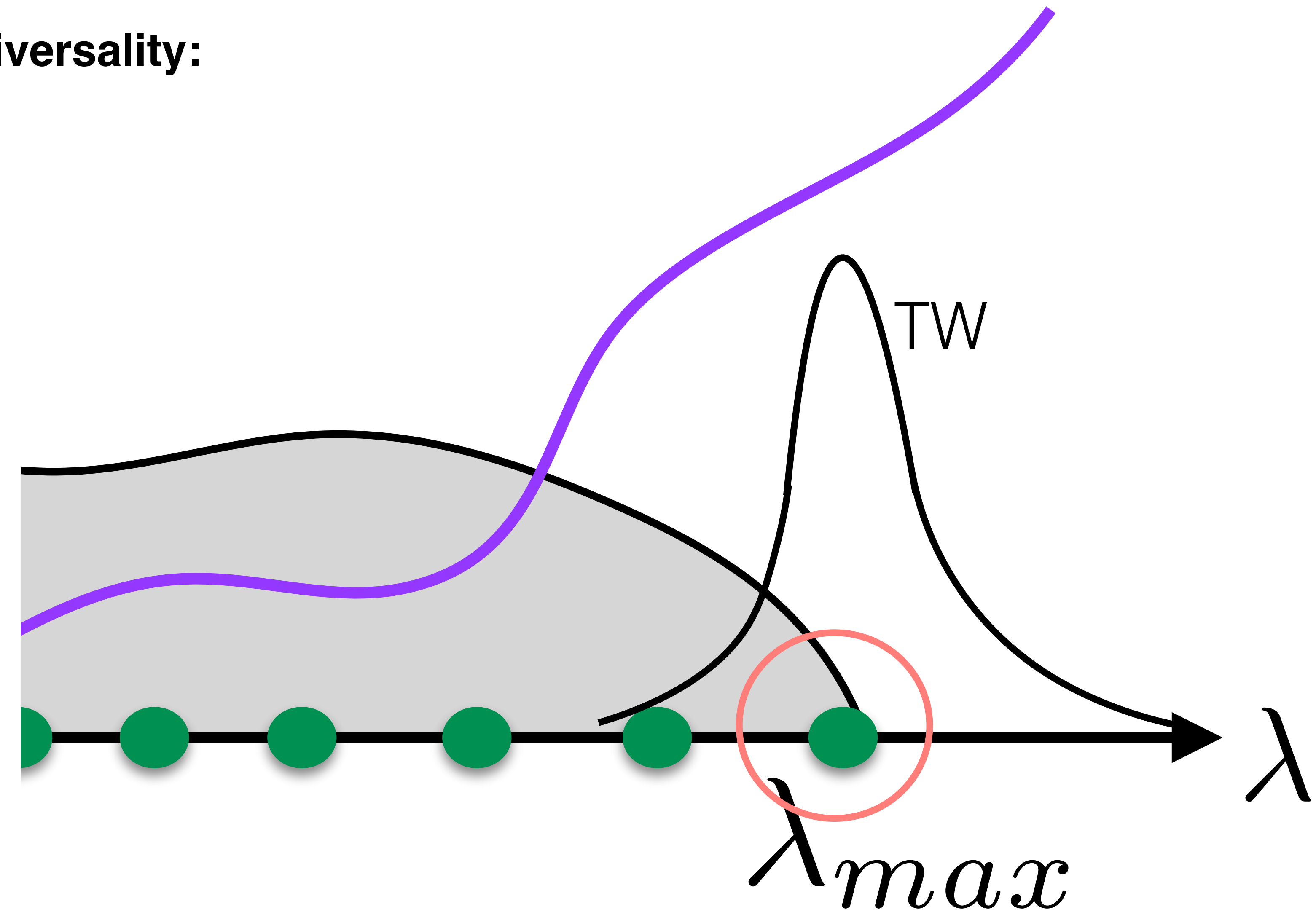


$$\rho(\lambda) \sim \sqrt{\langle \lambda_{max} \rangle - \lambda}$$

It has been shown to be universal with respect to the shape of the confining potential, as long as the average density vanishes at the upper edge as a square root.

Question:

Is the Tracy-Widom distribution for λ_{max} , robust with respect to the type of interaction between the charges?

1d Coulomb gas:

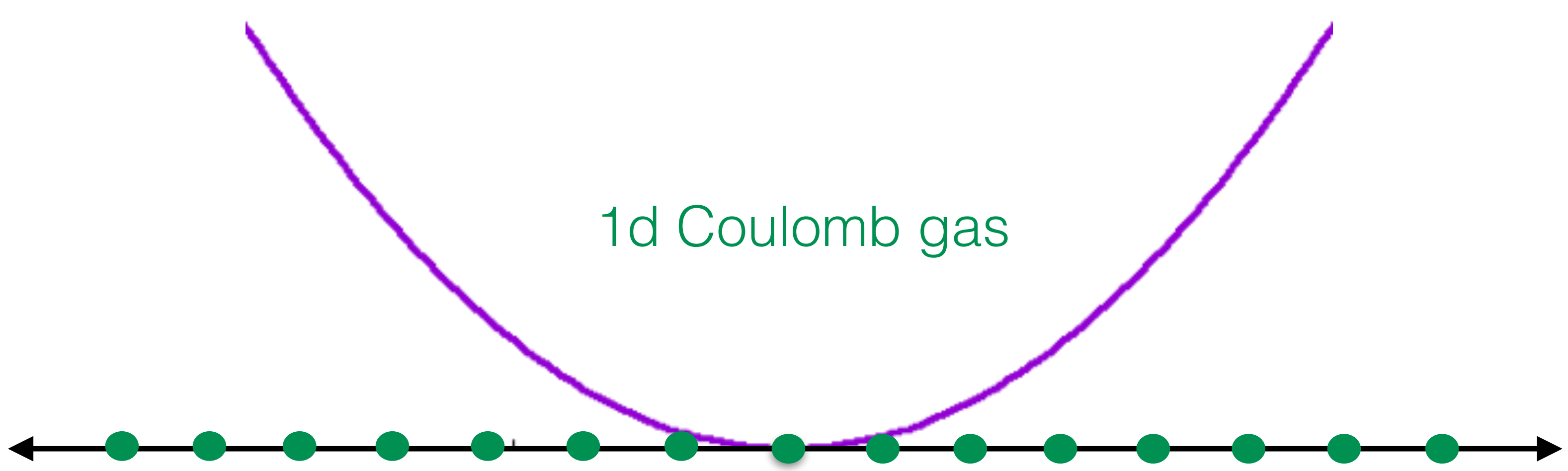
$$P(x_1, x_2, ..., x_N) = \frac{1}{Z_N} \exp\left[-\left(\frac{N^2}{2}\sum_{i=1}^{N} x_i^2 - \alpha N \sum_{i \neq j} |x_i - x_j|\right)\right]$$

Linear

Dyson's Log gas:

$$P(\lambda_1, \lambda_2, ..., \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2}\left(N\sum_{i=1}^{N} \lambda_i^2 - \sum_{j \neq k} \log|\lambda_j - \lambda_k|\right)\right]$$

Logarithmic



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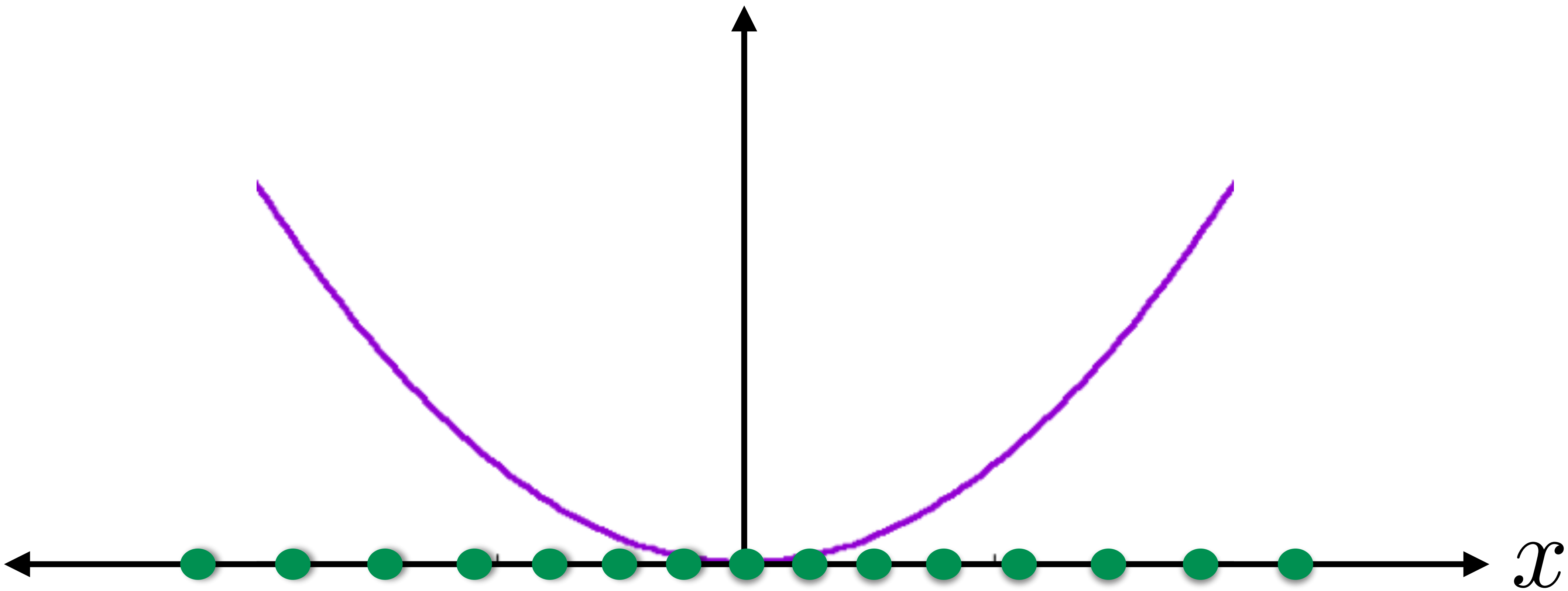
Is the Tracy-Widom distribution for x_{max}, robust with respect to the type of interaction between the charges?

No

AD, AK, SNM, SS, GS, PRL 119, 060601, (2017)

1d Coulomb gas:

$$P(x_1, x_2, ..., x_N) = \frac{1}{Z_N} \exp\left[-\left(\frac{N^2}{2}\sum_{i=1}^{N} x_i^2 - \alpha N \sum_{i\neq j} |x_i - x_j|\right)\right]$$

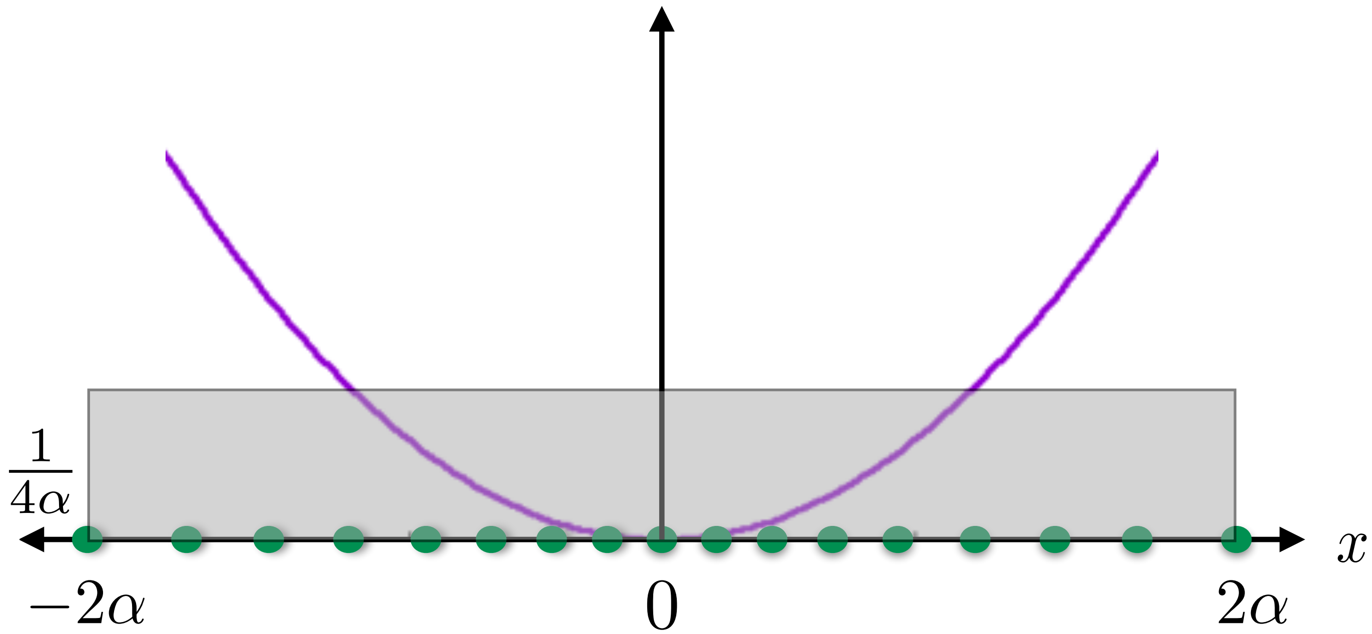


Average charge density:

In 1d Coulomb gas:

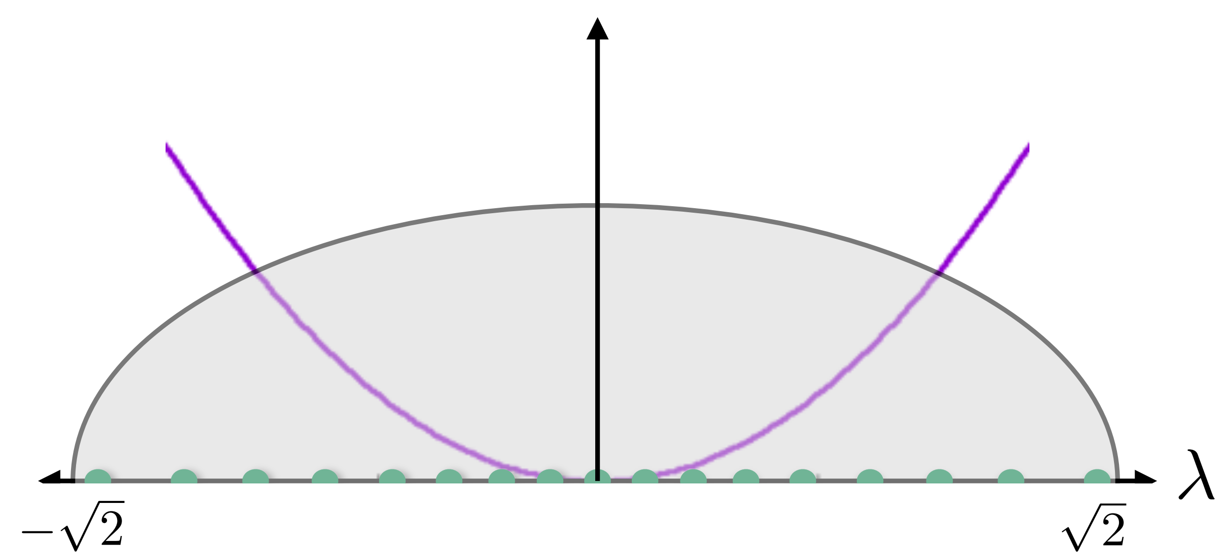
$$\rho_N(x) \xrightarrow[N\to\infty]{} \rho(x) = \frac{1}{4\alpha}$$

$$|x| \leq 2\alpha$$



Remember in Log gas:

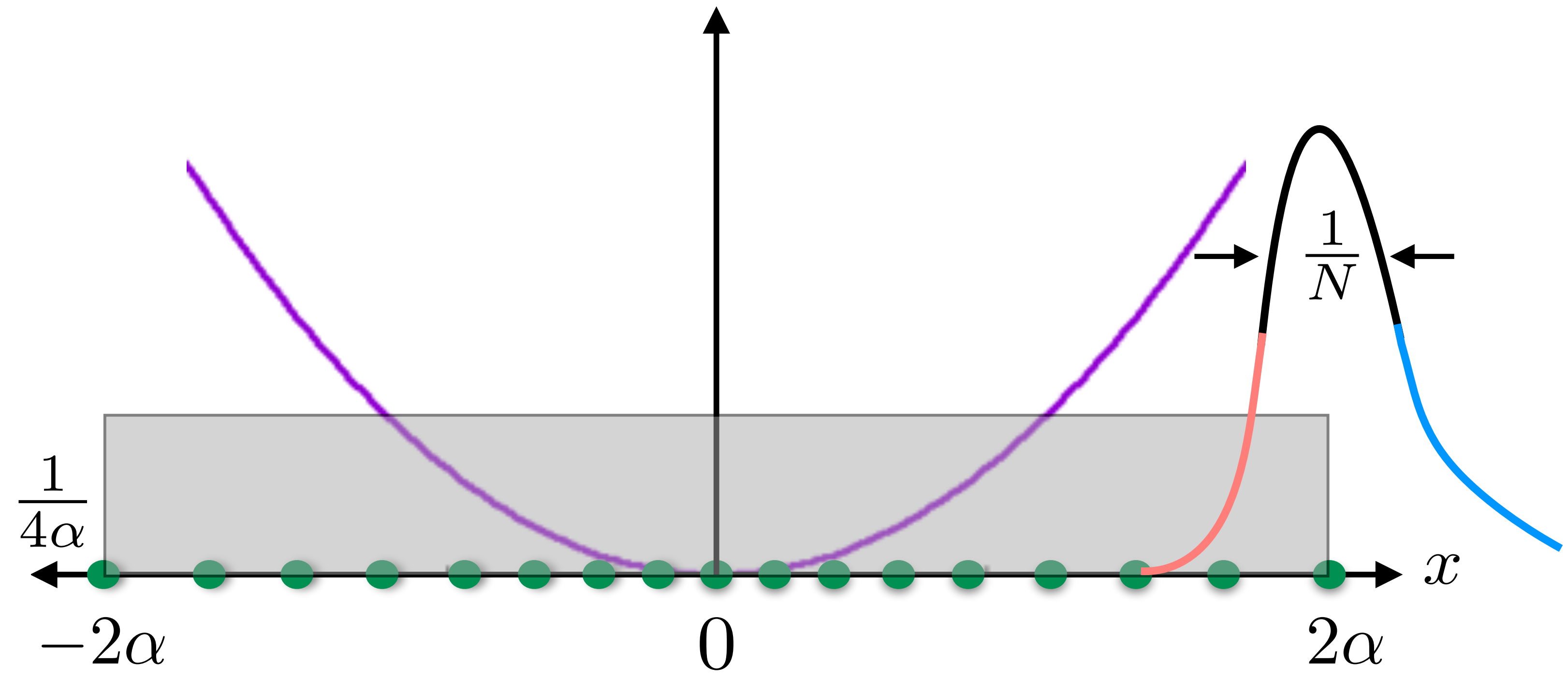
$$\rho_N(\lambda) \xrightarrow[N\to\infty]{} \rho(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}$$



Distribution of x_{max} : Typical fluctuations

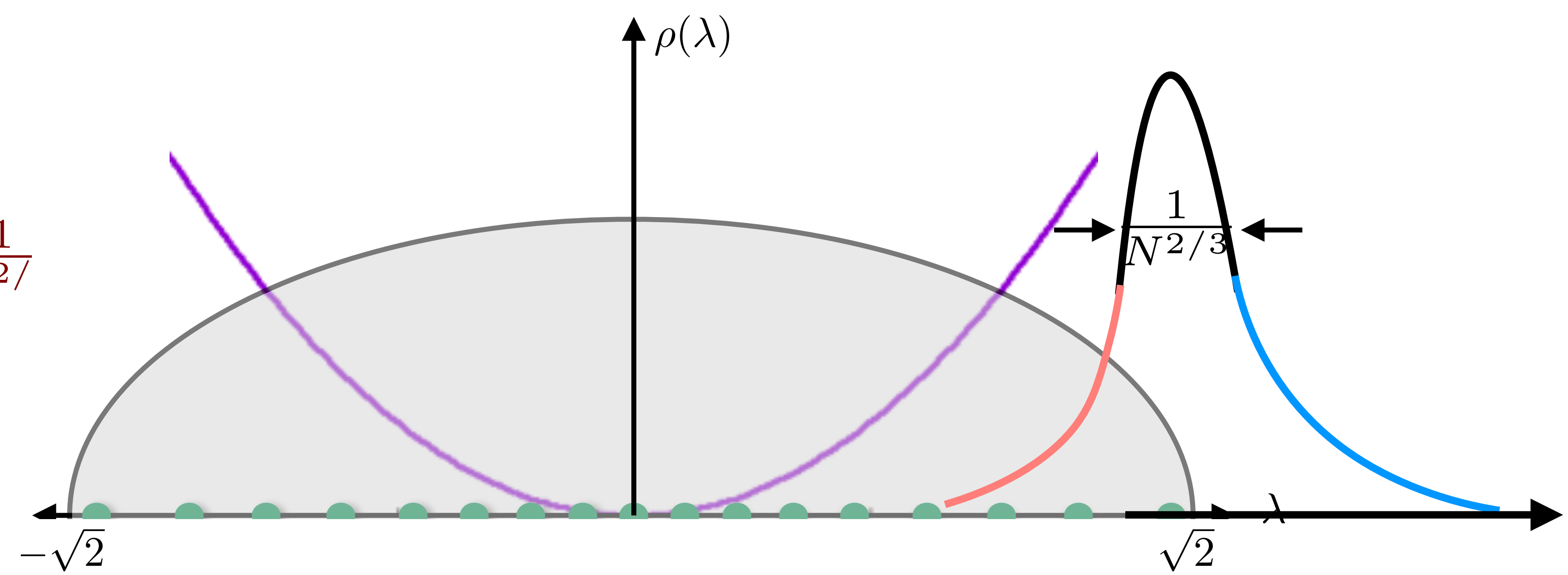
In 1d Coulomb gas:

Typical fluctuations around the mean are of order $\sim \frac{1}{N}$



In Log gas:

Typical fluctuations around the mean are of order $\sim \frac{1}{N^{2/}}$



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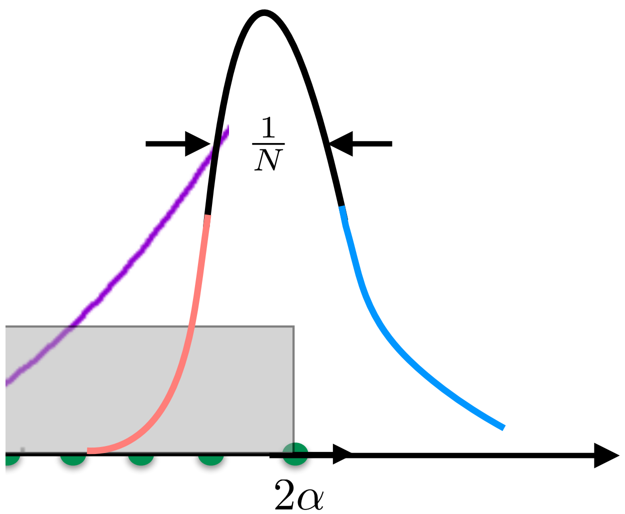
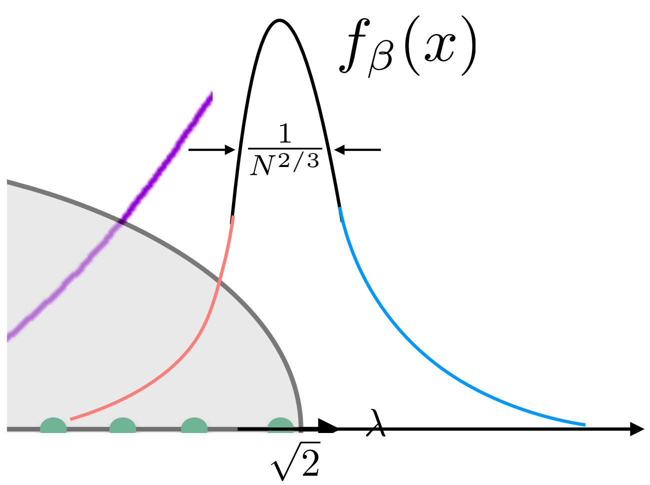


$$P_N(x_{max} = w) \xrightarrow[N\to\infty]{} N\ f_\alpha[N(w-2\alpha)+2\alpha]$$

$$f_\alpha(x) = \frac{dF_\alpha(x)}{dx}, \quad \text{where,}$$

$$\frac{dF_\alpha(x)}{dx} = A(\alpha)e^{-x^2/2}F_\alpha(x+4\alpha)$$

In Log gas: Typical fluctuations $\sim \frac{1}{N^{2/3}}$



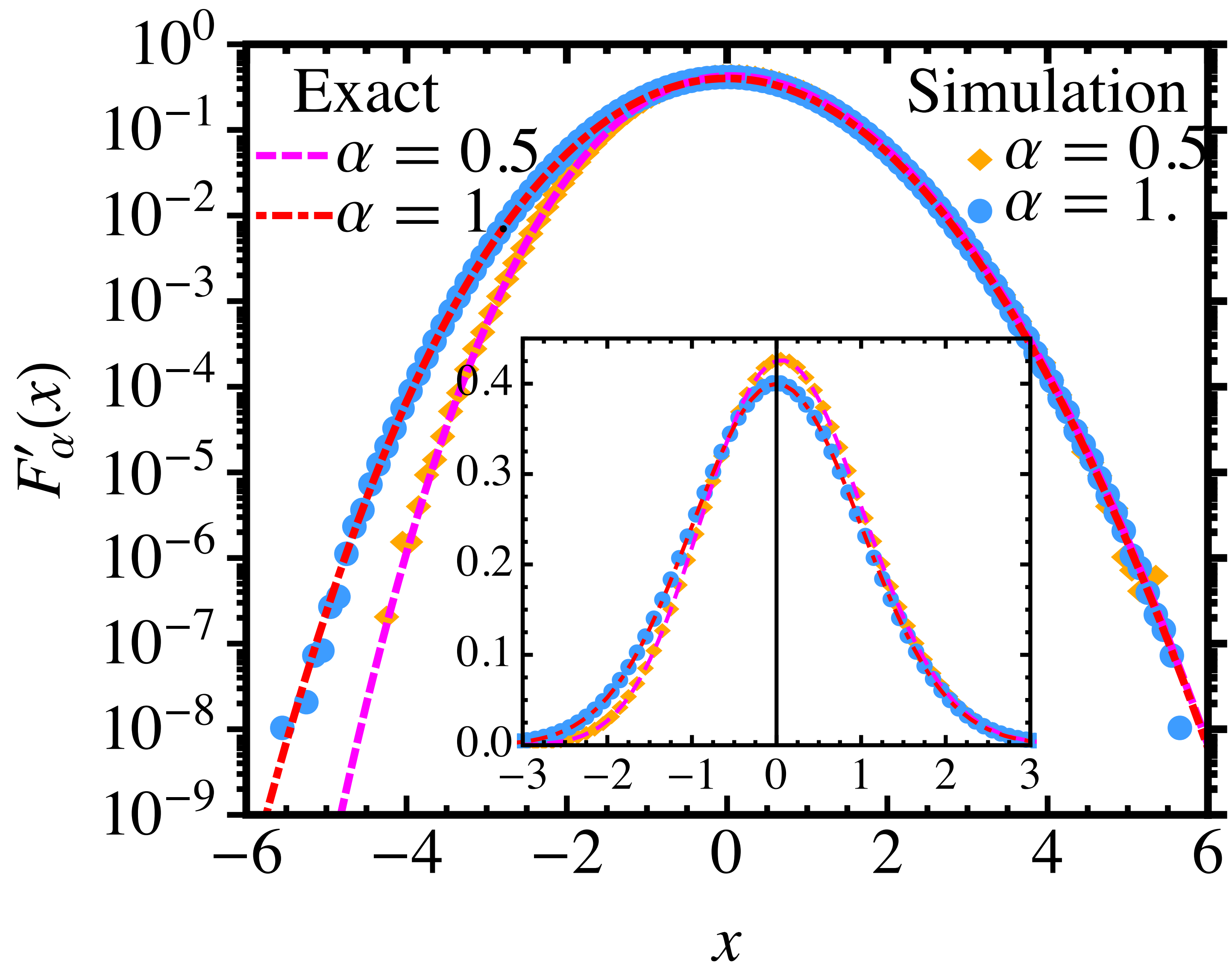
$$P_N(\lambda_{max} = w) \xrightarrow[N\to\infty]{} \sqrt{2}N^{2/3}\ f_\beta\left(\sqrt{2}N^{2/3}(w-\sqrt{2})\right)$$

$$f_\beta(z) \leftarrow Tracy-Widom\ distribution$$

$$f_\beta(z) = \frac{dF_\beta(z)}{dz}$$

Expressed in terms of the solutions of Painlevé II equation

Limiting distribution $f_\alpha(x)$



Lines :
Numerical solutions of the non-local eigenvalue equation

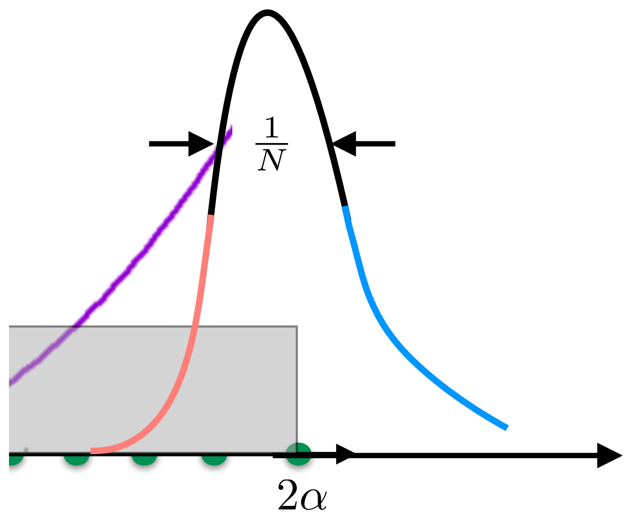
Points :
Monte Carlo simulation

Asymptotic tails

$$F'_\alpha(x) \approx \begin{cases} \exp\left[-|x|^3/24\alpha + O(x^2)\right] & \text{as } x \to -\infty \\ \exp\left[-x^2/2 + O(x)\right] & \text{as } x \to \infty \, . \end{cases}$$

Distribution of x_{max} : Typical fluctuations

In 1d Coulomb gas: Typical fluctuations $\sim \frac{1}{N}$



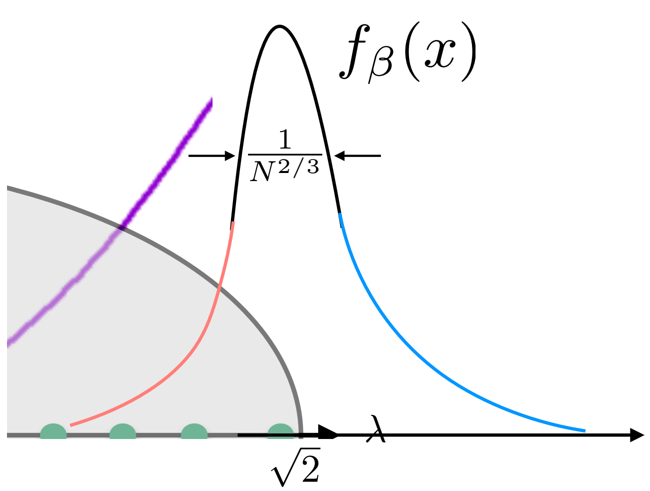
$$P_N(x_{max} = w) \xrightarrow[N\to\infty]{} N \; f_\alpha[N(w-2\alpha)+2\alpha]$$

$$f_\alpha(x) = \frac{dF_\alpha(x)}{dx}, \quad \text{where,}$$

$$\frac{dF_\alpha(x)}{dx} = A(\alpha)e^{-x^2/2}F_\alpha(x+4\alpha)$$

Atypical fluctuations ??

In Log gas: Typical fluctuations $\sim \frac{1}{N^{2/3}}$



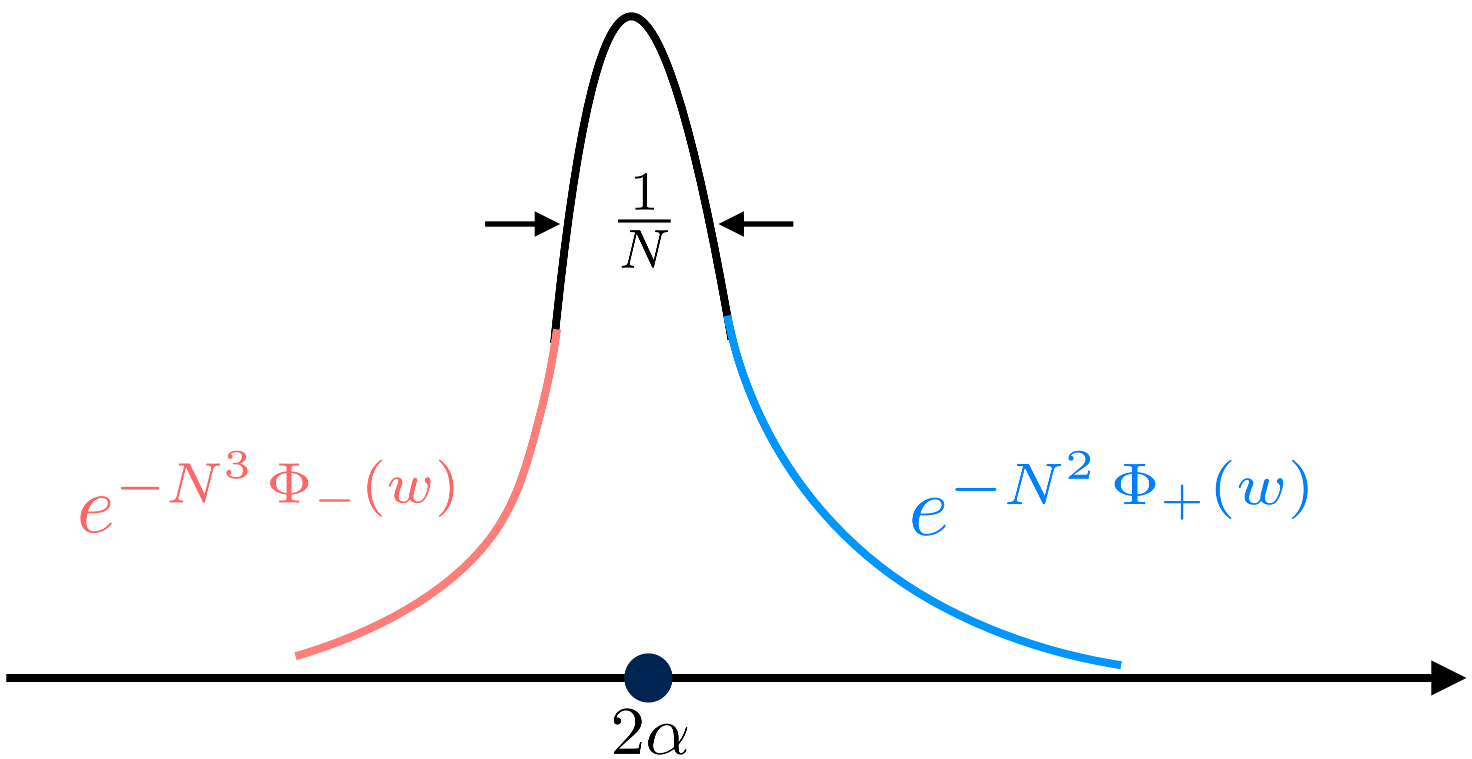
$$P_N(\lambda_{max} = w) \xrightarrow[N\to\infty]{} \sqrt{2}N^{2/3} \; f_\beta\left(\sqrt{2}N^{2/3}(w-\sqrt{2})\right)$$

$$f_\beta(z) \leftarrow Tracy - Widom \ distribution$$

$$f_\beta(z) = \frac{dF_\beta(z)}{dz}$$

Expressed in terms of the solutions of Painlevé II equation

Large deviations of 1d Coulomb gas:



$$P_N(x_{max}=w) \approx \begin{cases} e^{-N^3\,\Phi_-(w)+O(N^2)}, & 0<2\alpha-w\sim O(1) \\ N f_\alpha\left[N(w-2\alpha)+2\alpha\right], & |2\alpha-w|\sim O(1/N) \\ e^{-N^2\,\Phi_+(w)+O(N)}, & 0<w-2\alpha\sim O(1) \end{cases}$$

AD, AK, SNM, SS, GS, PRL 119, 060601, (2017)

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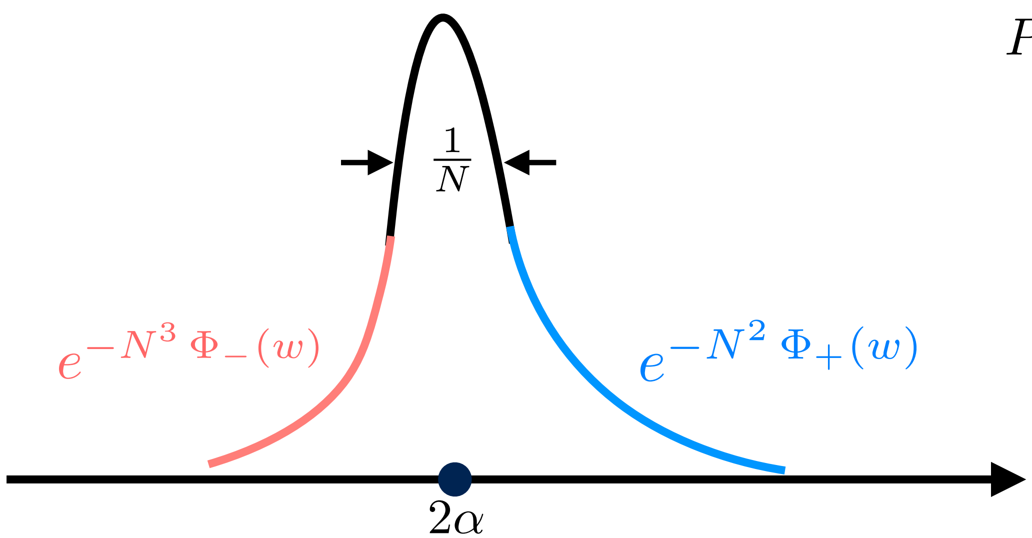

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$$f_\alpha(x) = \frac{dF_\alpha(x)}{dx}$$

$$\frac{d\,F_\alpha(x)}{d\,x} = A(\alpha)\,e^{-\frac{x^2}{2}}\,F_\alpha(x+4\alpha)$$

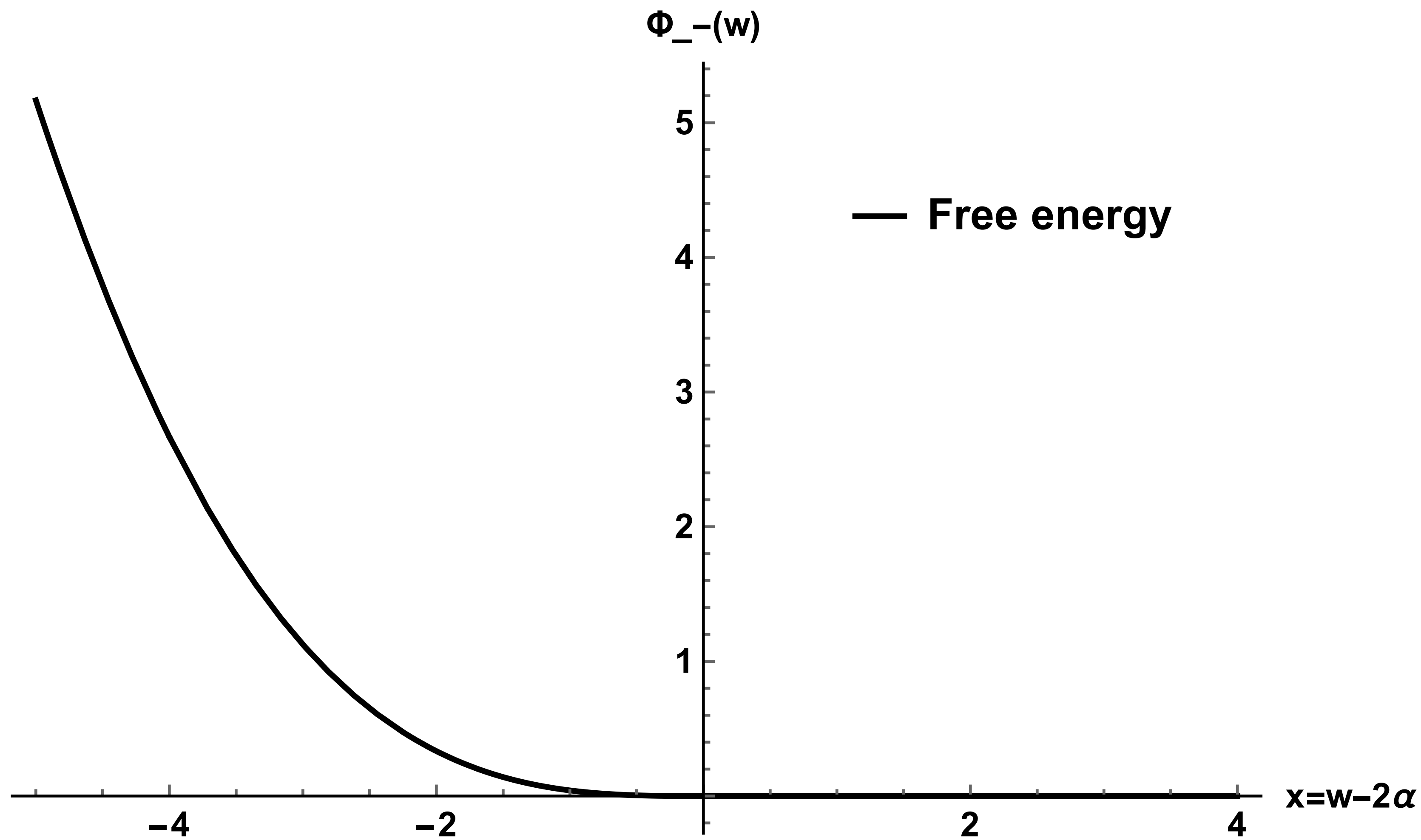
$$\Phi_-(w) = \begin{cases} \frac{(2\alpha-w)^3}{24\alpha}\,, & -2\alpha\le w\le 2\alpha \\ \frac{w^2}{2}+\frac{2}{3}\alpha^2\,, & w\le -2\alpha\,. \end{cases}$$

$$\Phi_+(w) = \frac{(w-2\alpha)^2}{2}\,, \quad w>2\alpha$$

AD, AK, SNM, SS, GS, PRL 119, 060601, (2017)

Large deviations of 1d Coulomb gas: 3rd order phase transition

$$\lim_{N\to\infty} -\frac{1}{\beta N^3}\log Q_N(\lambda_{max}\le w) = \begin{cases}\Phi_-(w)\sim \frac{1}{24\alpha}(2\alpha - w)^3 & as\ w\to 2\alpha_- \\ 0 & as\ w\to 2\alpha_+\end{cases}$$



Summary:

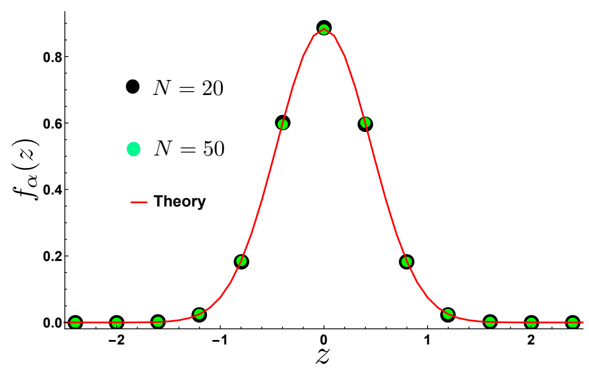
- With respect to changing the interaction among the particles the TW distribution is not universal

$$\frac{d\,F_\alpha(x)}{d\,x} = A(\alpha)\,e^{-\frac{x^2}{2}}\,F_\alpha(x+4\alpha)$$

$$f_\alpha(x) = \frac{dF_\alpha(x)}{dx}$$

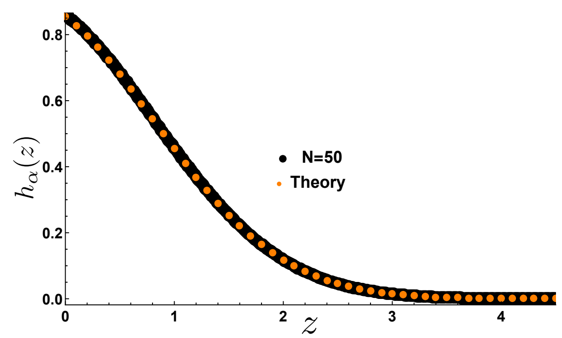
- Has similar large deviation tails in 1dC
- The function $F_\alpha(x)$ also describes other quantities.
- Index:

$$N_+ = \sum_{i=1}^{n} \theta(x_i)$$



- Gap:

$$g = x_N - x_{N-1}$$



- Evolution? higher dimension? Connection to quantum systems?
- General interactions ?

Thank you