Integrable Field Theories

from

4d Chern-Simons Theory

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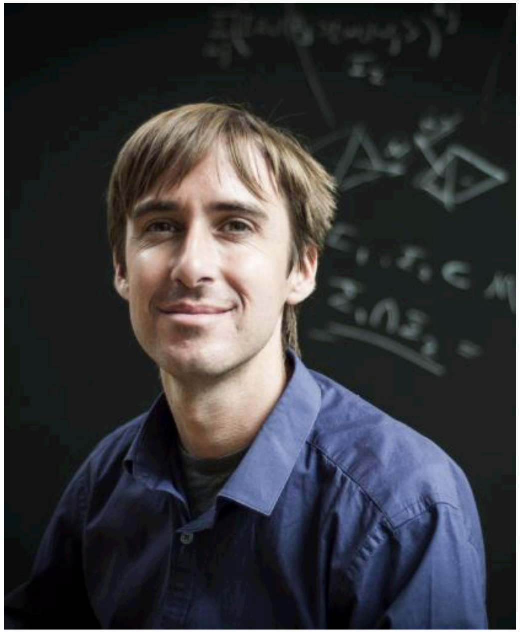
ICTS

Based on collaboration
with Kevin Costello and Edward Witten



Based on collaboration
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Part I arXiv:1709.09993

Part II arXiv:1802.01579

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Part III to appear

Part IV to appear

Based on collaboration
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Part I arXiv:1709.09993

Part II arXiv:1802.01579

integrable
lattice models

classical

Part III to appear

Part IV to appear

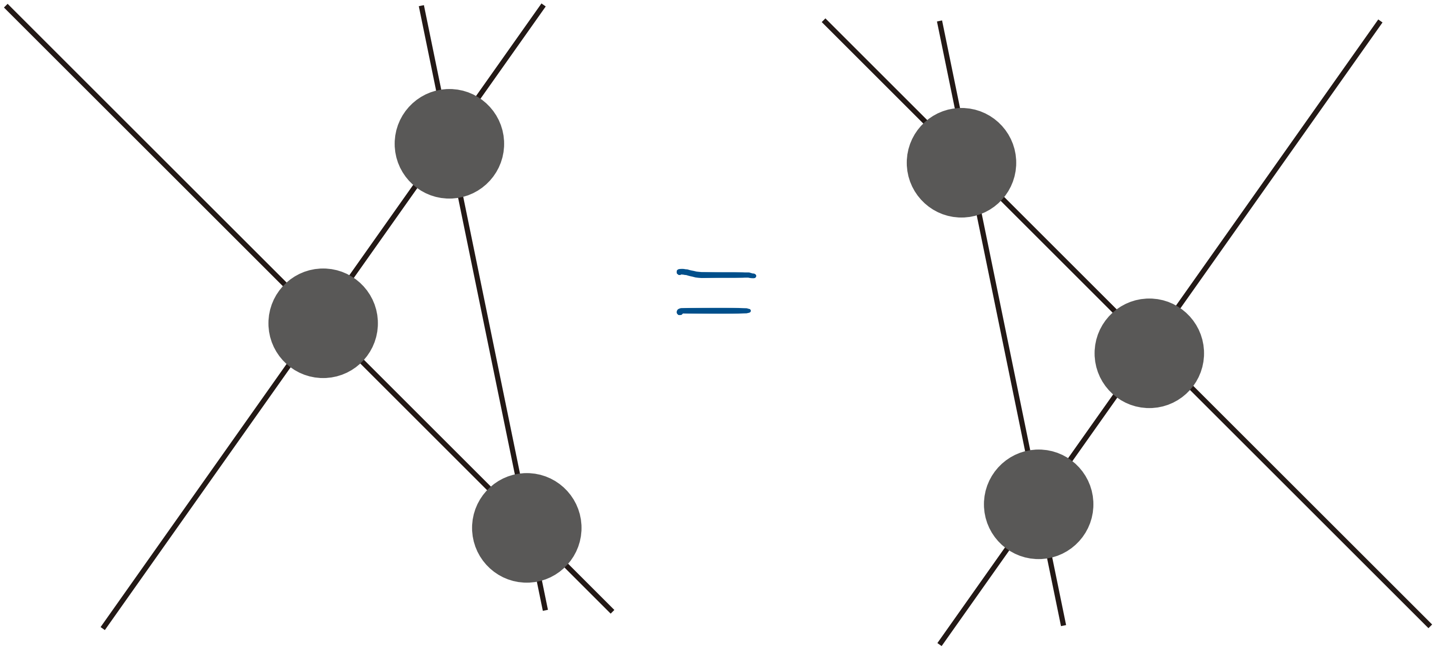
quantum

integrable
field theories

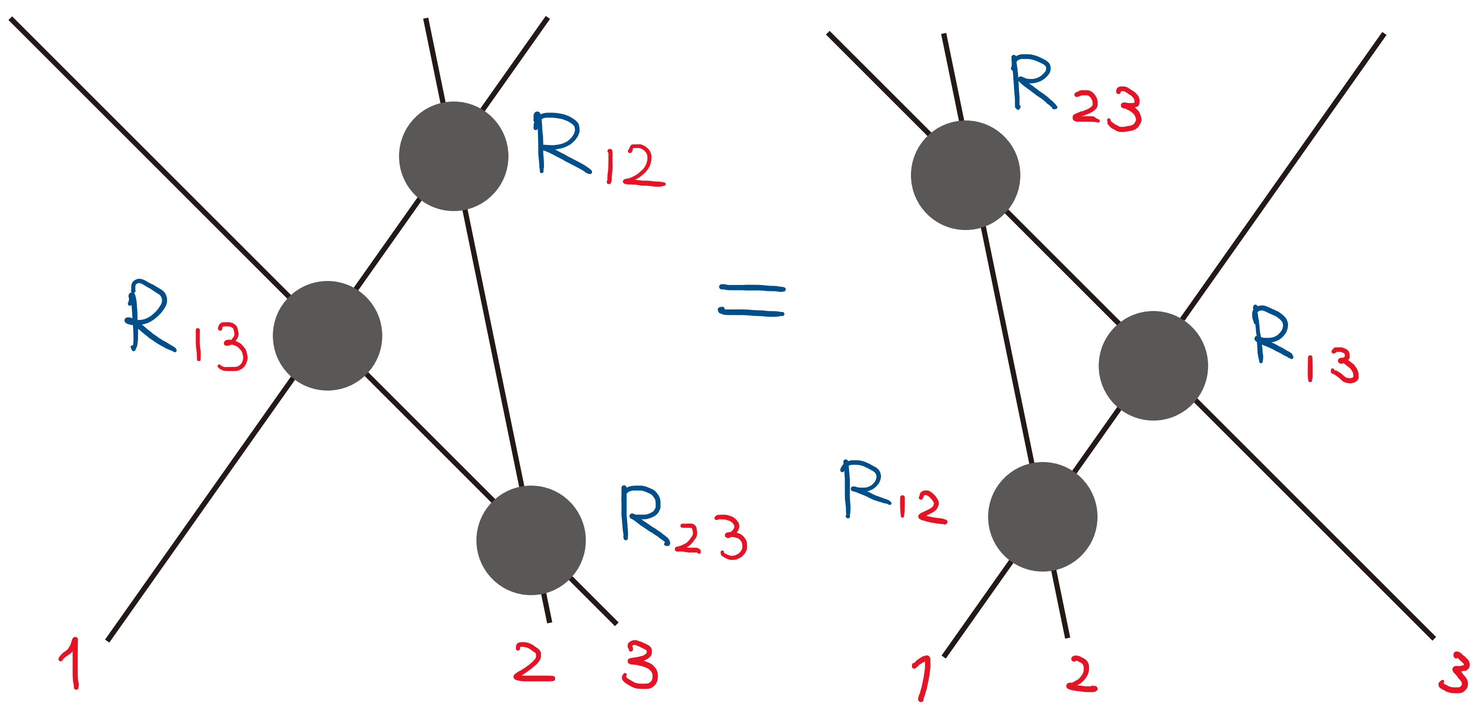
Integrable Lattice Models

(Part I and II)

Integrability in lattice models:
characterized by Yang-Baxter equation

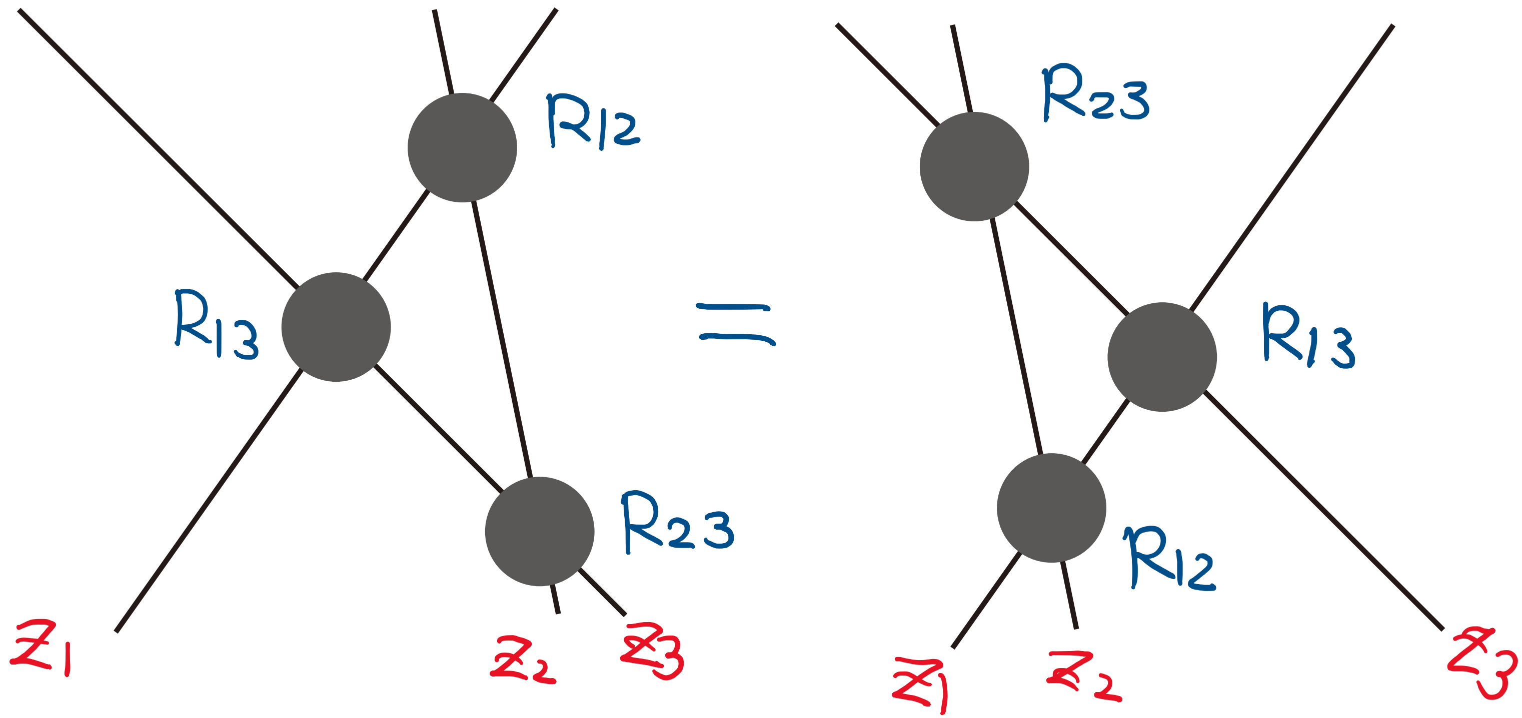


Integrability in lattice models:
characterized by Yang-Baxter equation



$$R_{23} R_{13} R_{12} = R_{12} R_{13} R_{23}$$

Integrability in lattice models:
characterized by Yang-Baxter equation
with spectral parameters

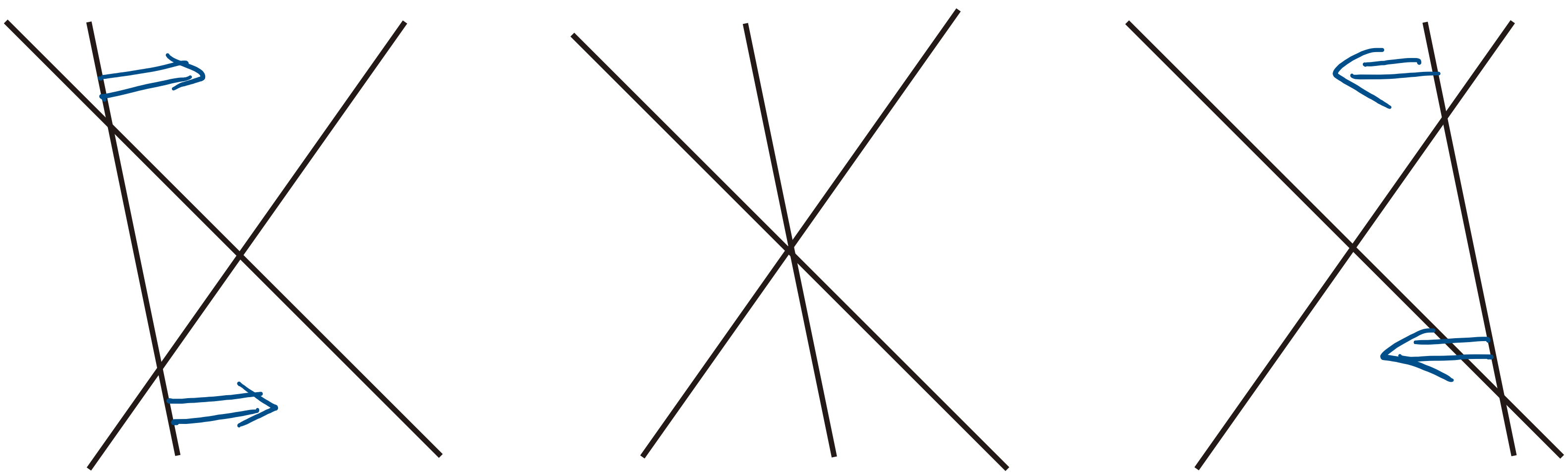


$$R_{ij} = R_{ij}(z_i - z_j)$$

spectral parameter

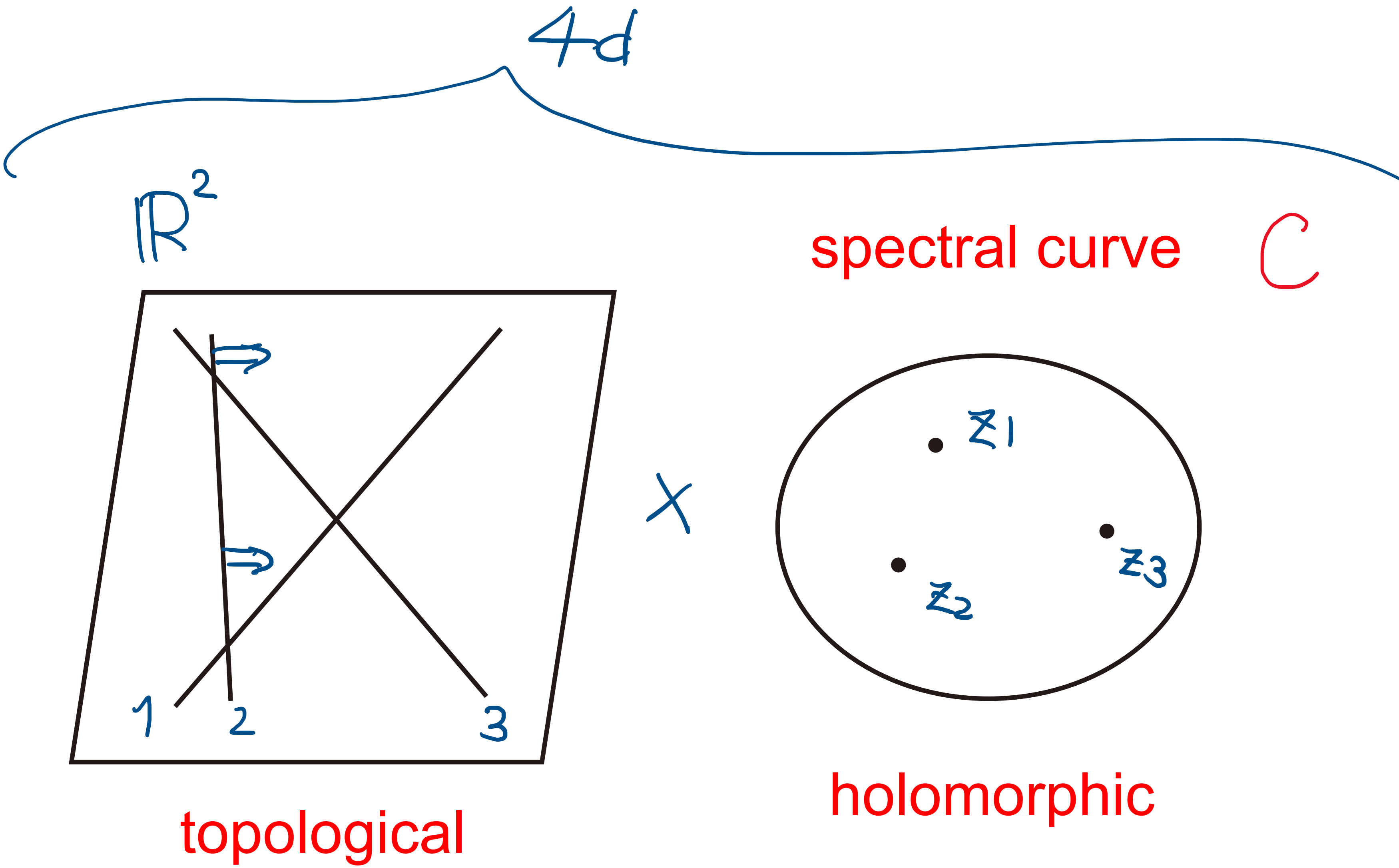
integrability as topological invariance?

integrability as topological invariance?



singular

4d = 2d (topological) + 2d (holomorphic)



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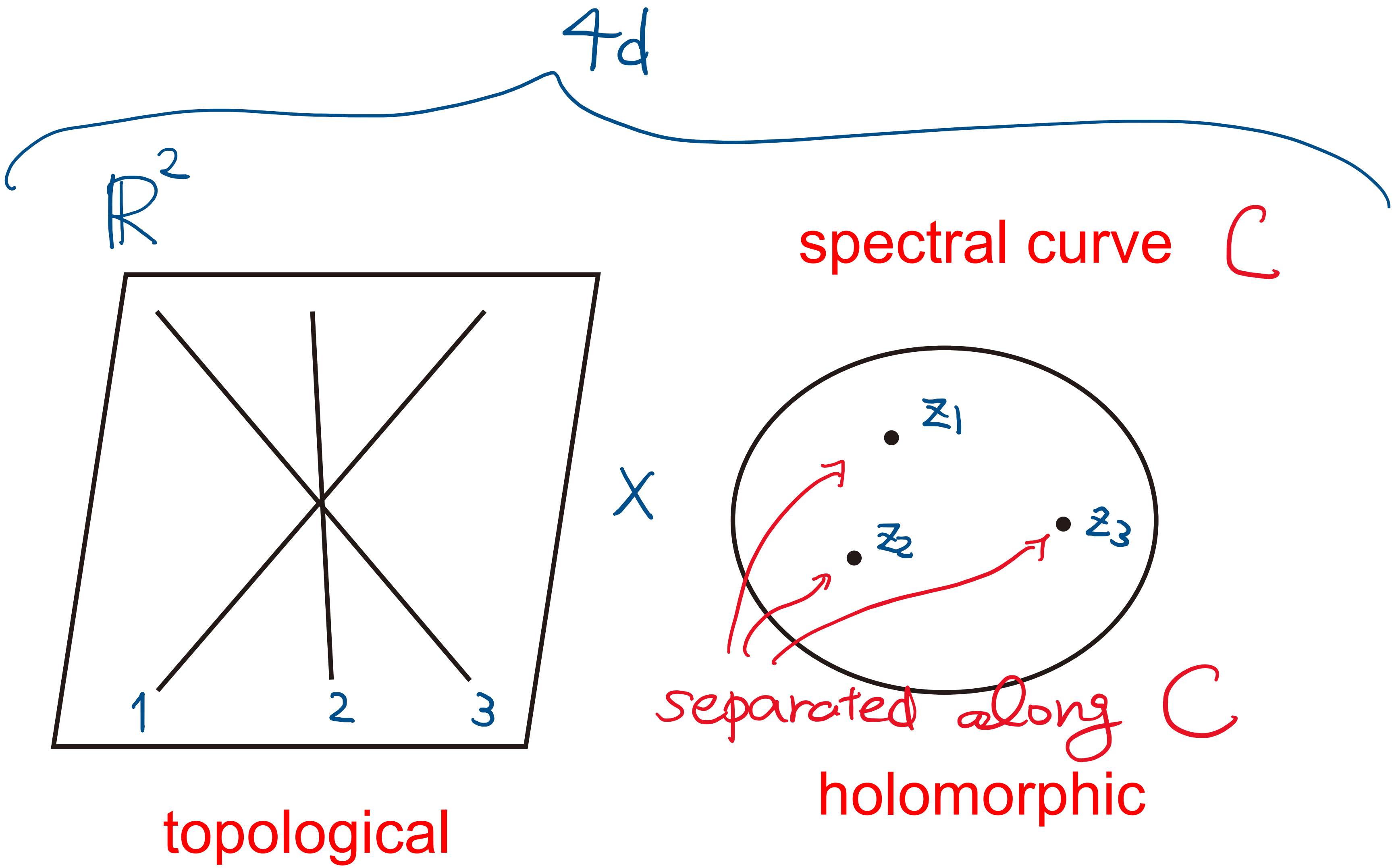

“4d Chern-Simons” by [Costello] (‘13)

$$\mathcal{L} = \frac{1}{\hbar} \int_{\mathbb{R}^2 \times C} dz \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$\{t, x\}$ $\{z, \bar{z}\}$

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$$A = A_t \, dt + A_x \, dx + A_{\bar{z}} \, d\bar{z} + \cancel{A_z \, dz}$$

depends on all $t, x, z, \bar{z}$

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$$\{t, x\} \quad \{z, \bar{z}\}$$

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“T-dual” to ordinary 3d Chern-Simons

[Vafa-Y] (to appear)

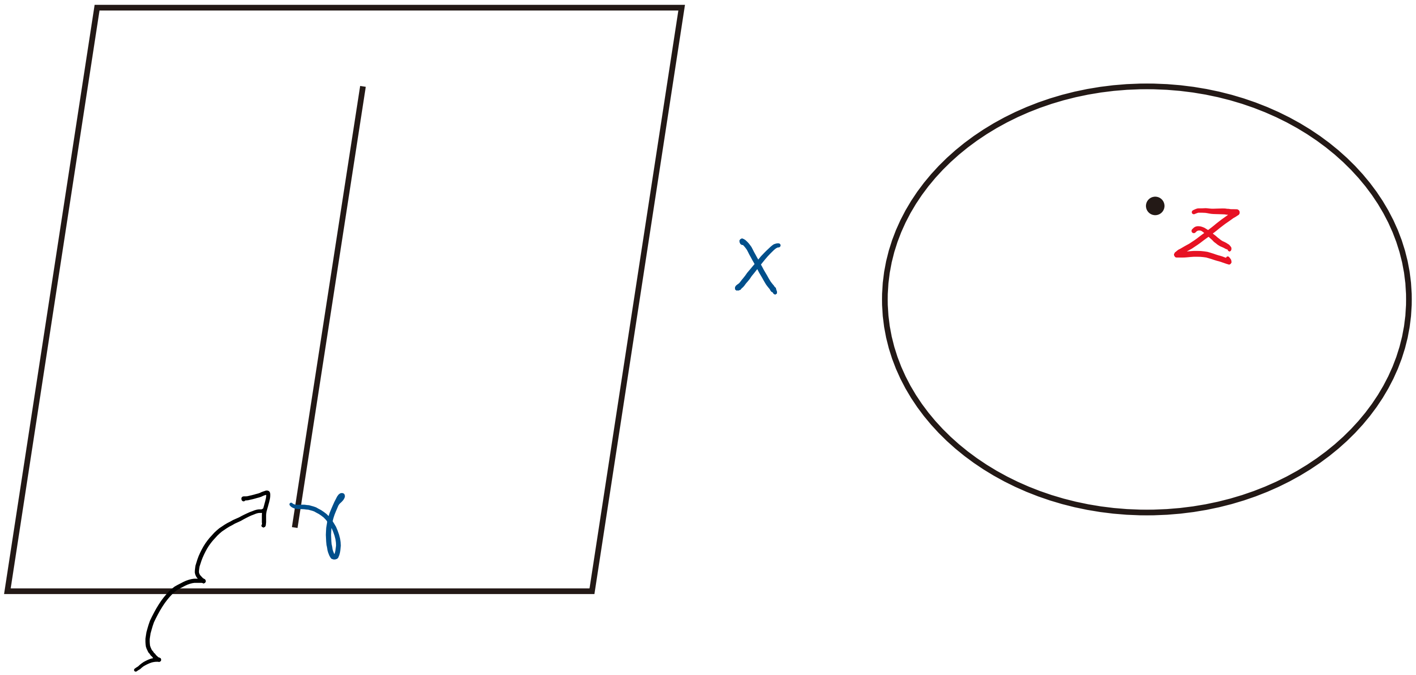
“4d Chern-Simons” by [Costello] (‘13)

$$\mathcal{L} = \frac{1}{\hbar} \int_{\underset{\{t,x\}}{\mathbb{R}^2} \times \underset{\{z,\bar{z}\}}{C}} dz \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

Perturbative expansion in $\hbar$
around isolated classical solution
(for standard, i.e. “non-dynamical” YBE)

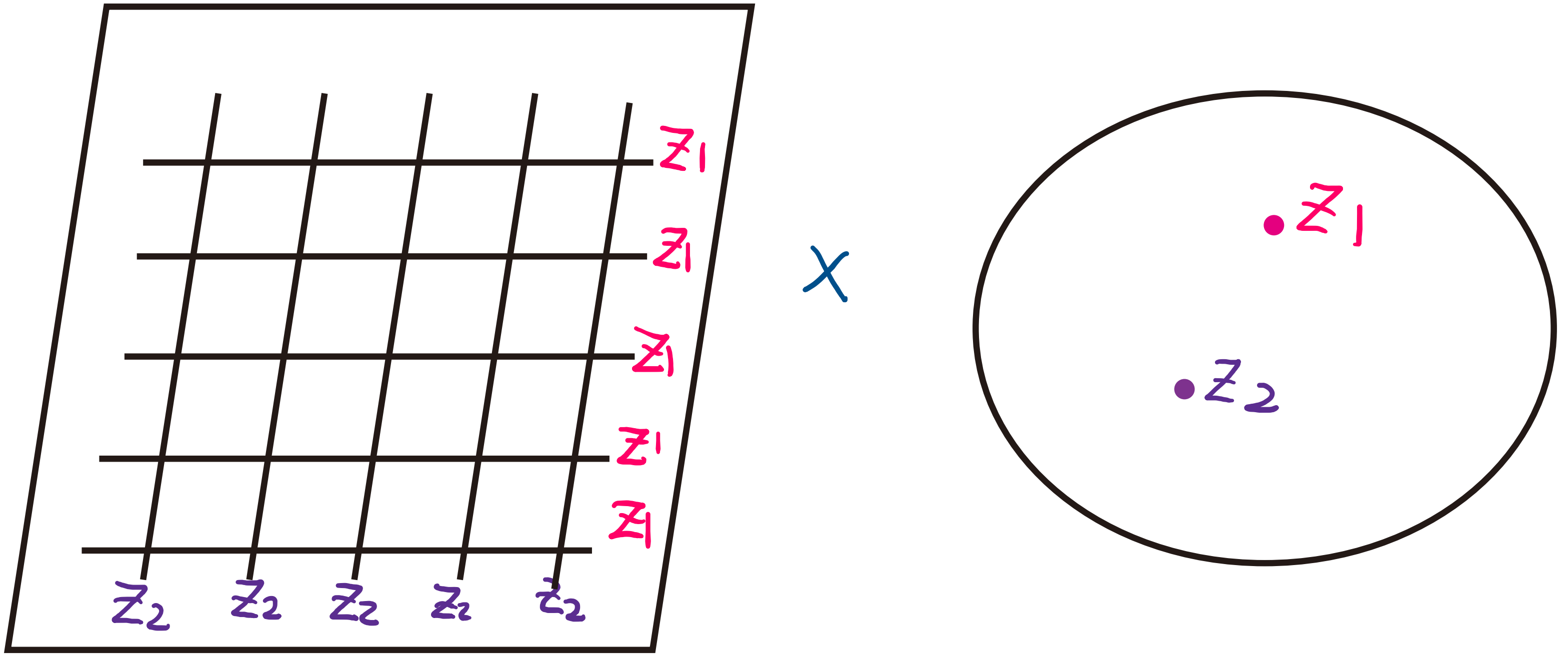
e.g. $A = 0$ for $C = \mathbb{C}$

statistical lattice from Wilson lines



$$W_\gamma(z) = P \exp \int_{\gamma \times \{z\}} A$$

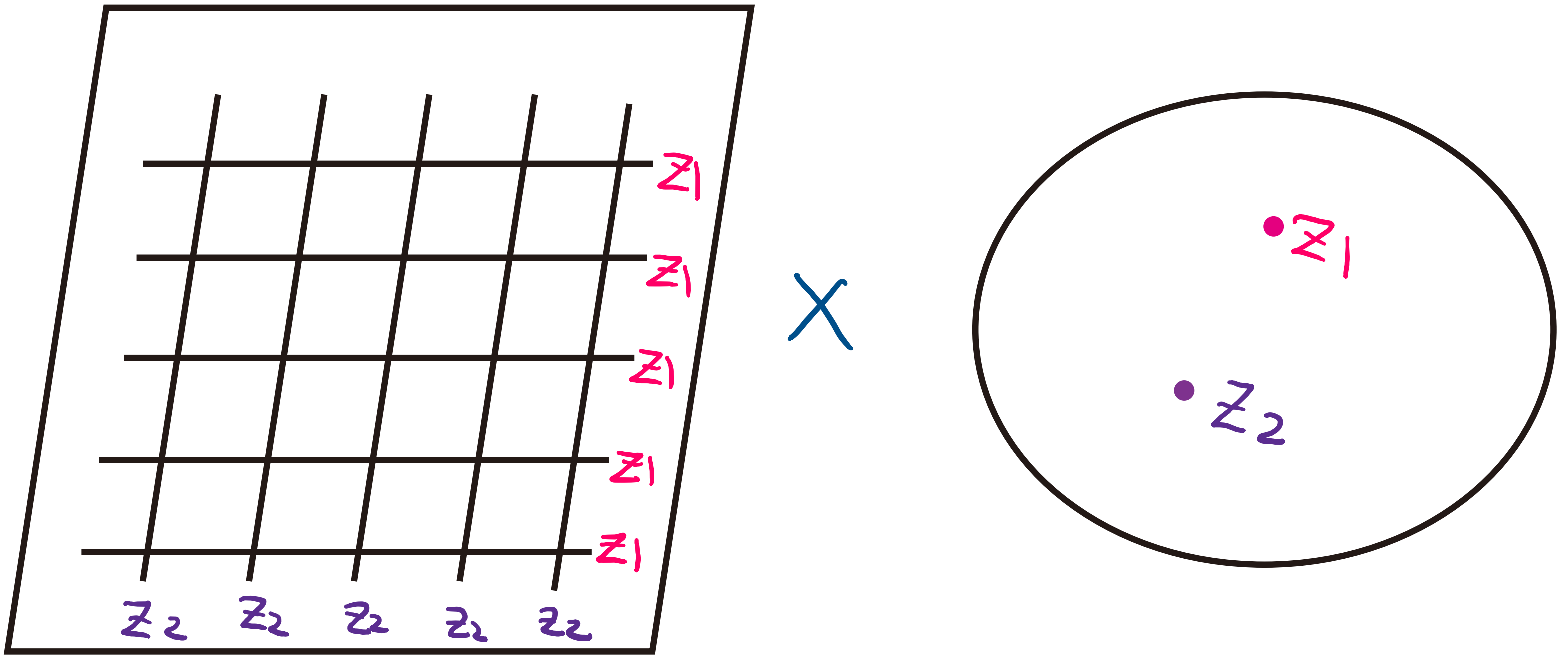
statistical lattice from Wilson lines



Integrable Field Theories

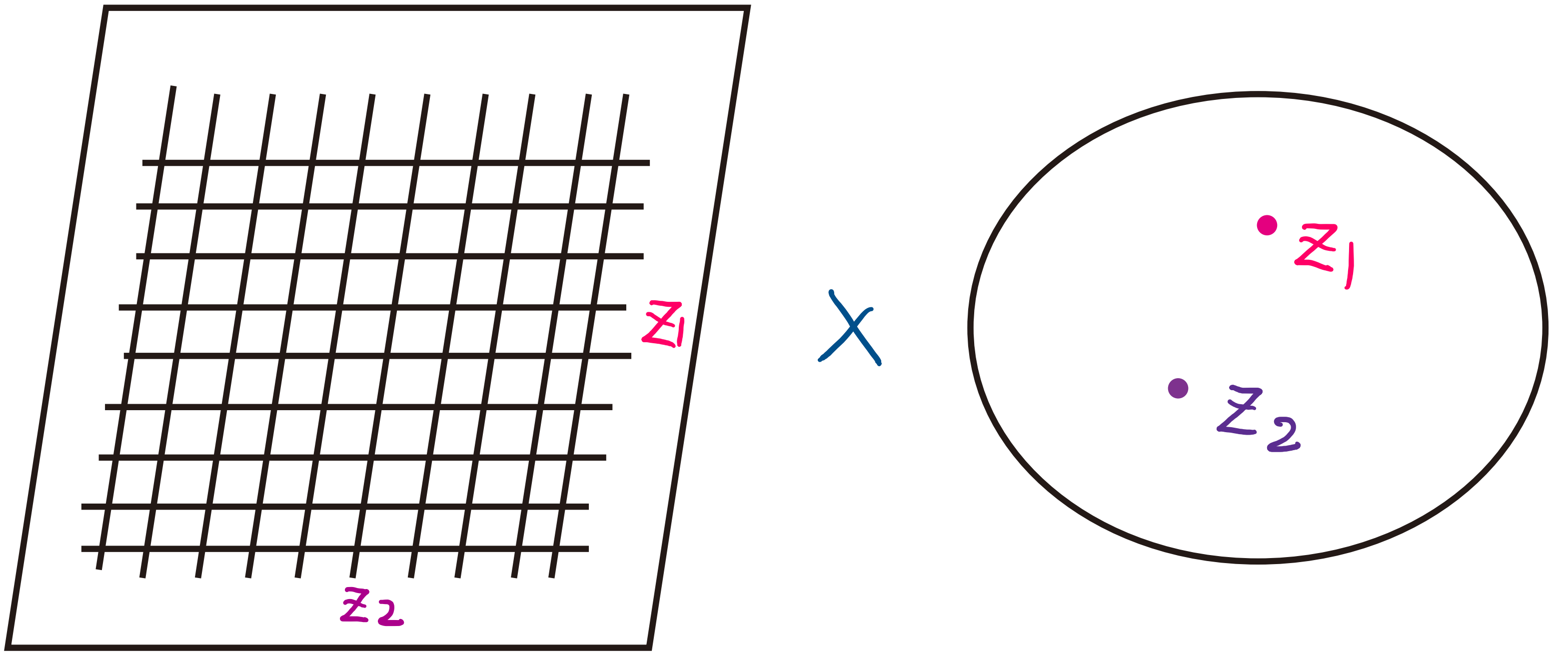
(Part III and IV)

thermodynamic limit



lattice model from
Wilson lines

thermodynamic limit



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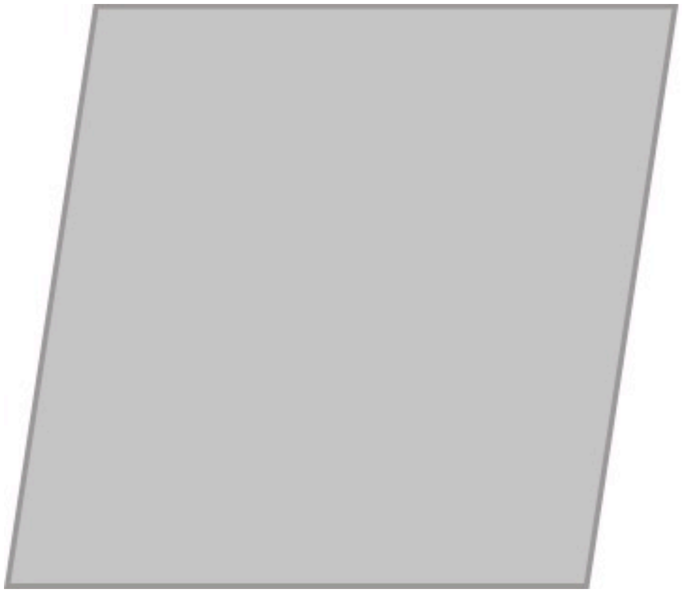
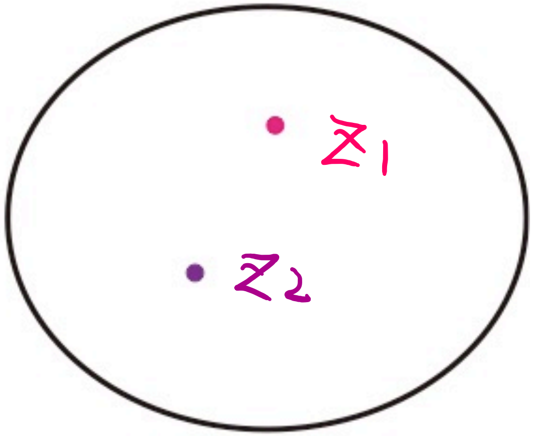
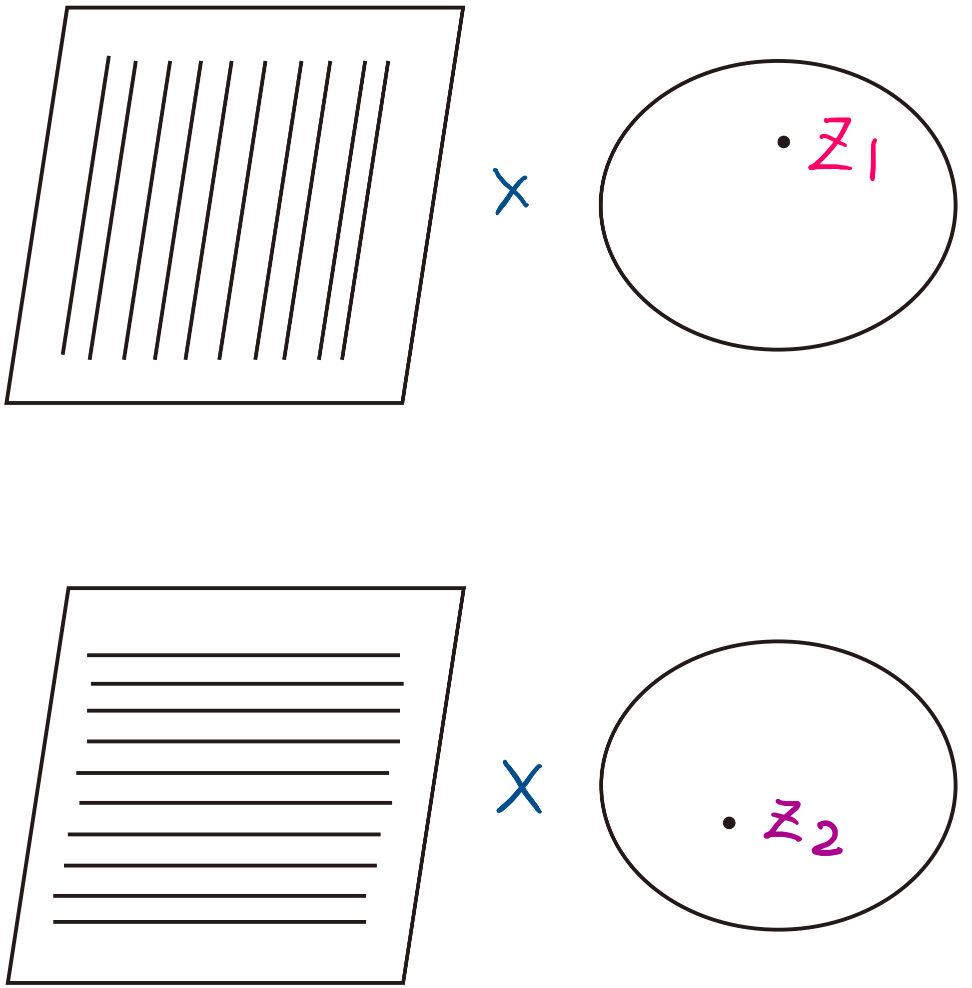




2d field theory from
surface defects @ $z = z_1$ & $z = z_2$

coupled 4d-2d system

two defects: vertical and horizontal



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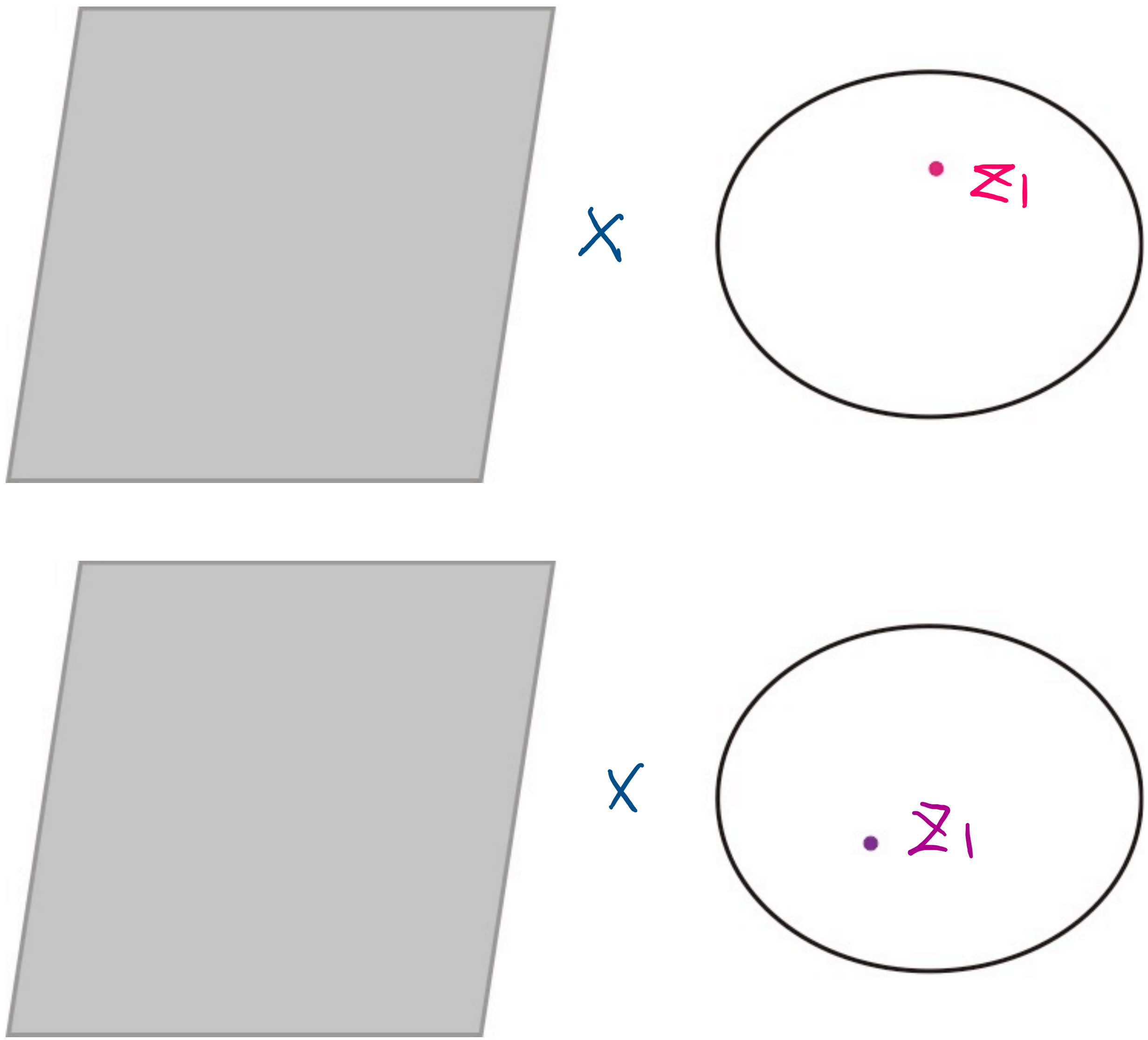
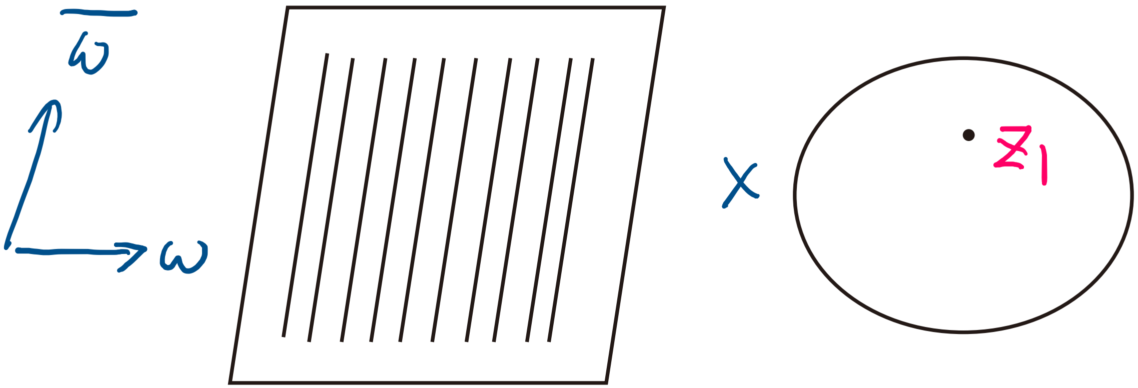
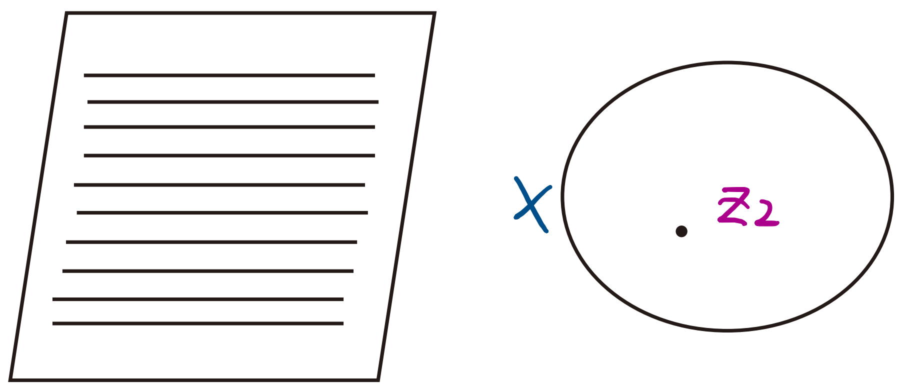


two defects: chiral and anti-chiral

$(w = t + x, \ \bar{w} = t - x)$

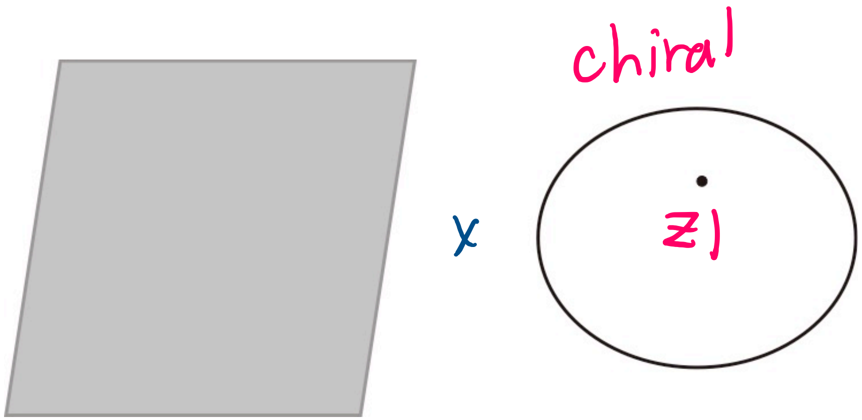


$$\int d\bar{w}\, A_{\bar{w}}$$



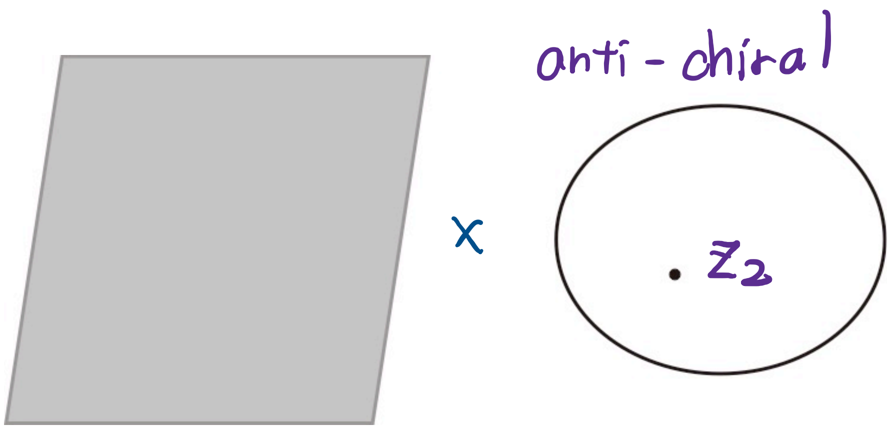
$$\int dw\, A_w$$

two defects: chiral and anti-chiral



$$\mathcal{L}_{\text{chiral}} \supset \frac{1}{\hbar}\, \underset{\sim\sim}{J_{\omega}}\, A_{\bar{\omega}}$$

2d current



$$\mathcal{L}_{\text{anti-chiral}} \supset \frac{1}{\hbar}\, \underset{\sim\sim}{J_{\bar{\omega}}}\, A_{\omega}$$

Why Integrable?

(Part III)

Lax operator (1-form on $\mathbb{R}^2$)

$$\mathcal{L}(z) = A_w(z)\,dw + A_{\bar{w}}(z)\,d\bar{w}$$

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$$\mathcal{L}(z) = A_w(z)\,dw + A_{\bar{w}}(z)\,d\bar{w}$$

Flat connection

$$d\mathcal{L}(z) + \frac{1}{2}\mathcal{L}(z)\wedge\mathcal{L}(z) \propto F_{w\bar{w}} = 0$$

4d e.o.m.

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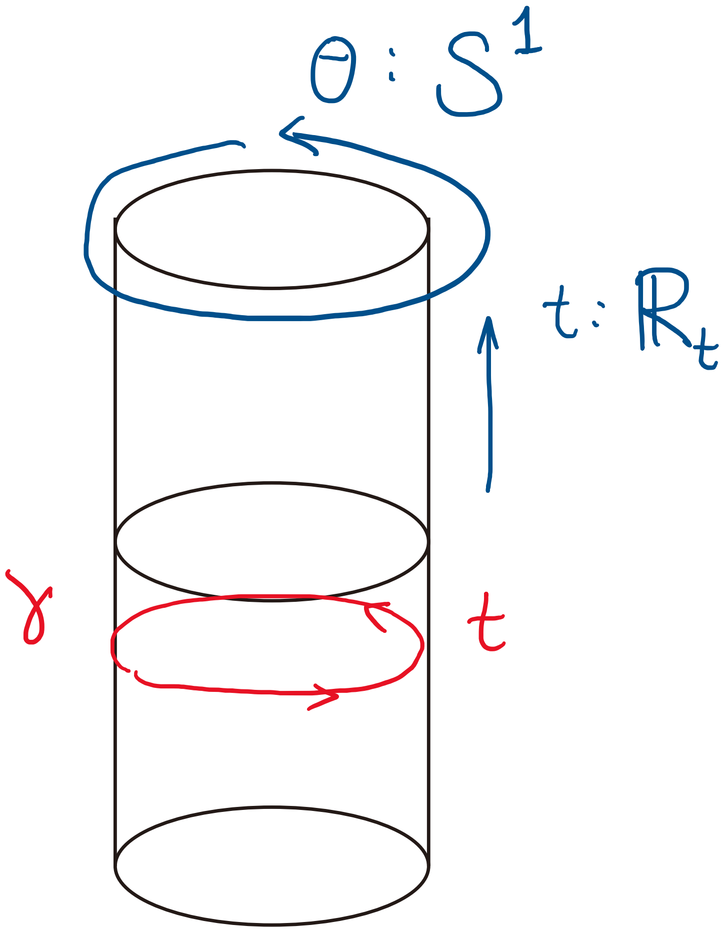
4d e.o.m.

infinitely-many conserved charges

$$W(z) = \mathrm{Tr}\, P\exp\int \mathcal{L}(z) = \exp\left(\sum_n \frac{Q_n}{z^n}\right)$$

$\partial_t W(z) = 0$, namely $\partial_t Q_n = 0$

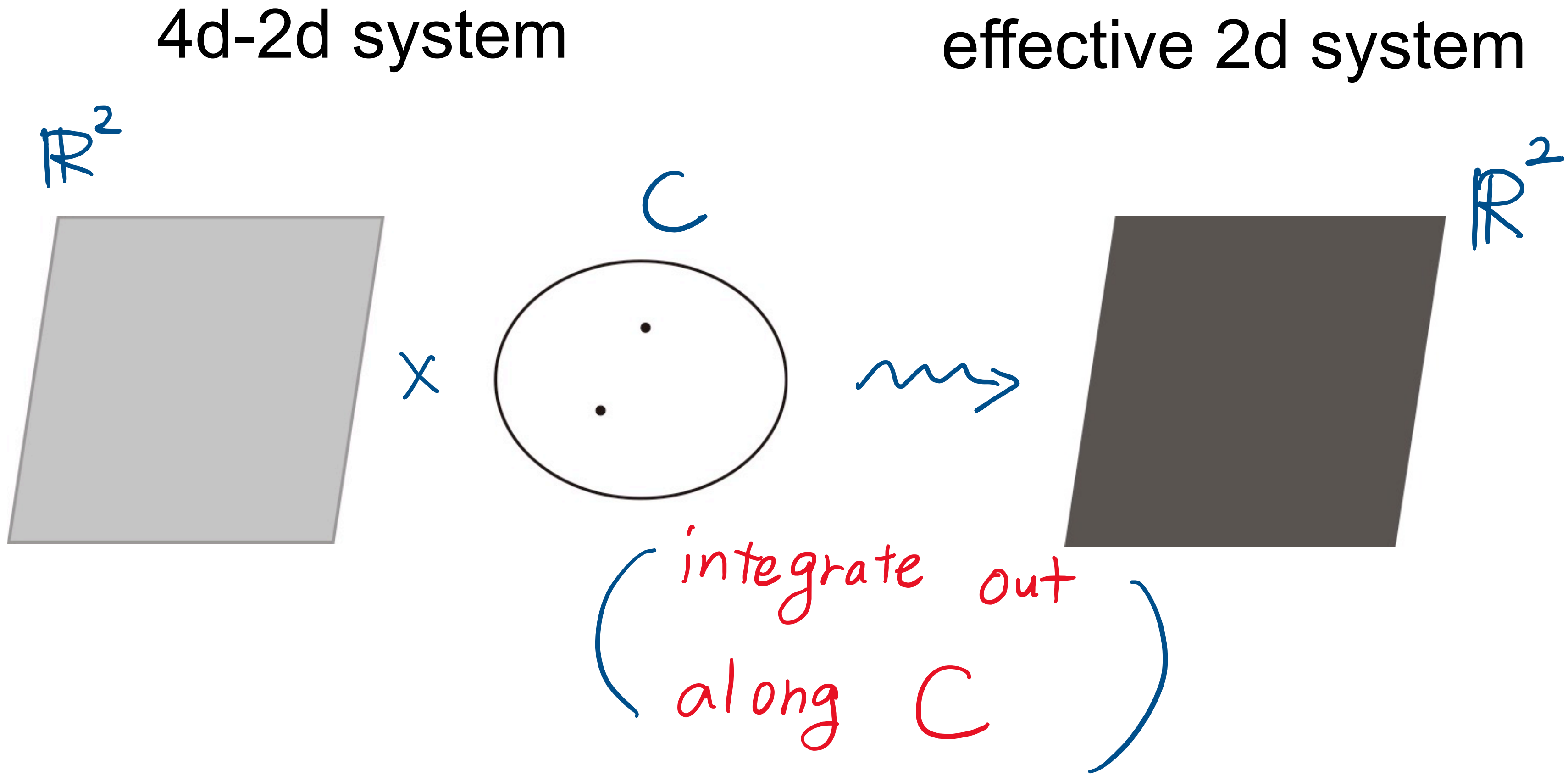


$$W_\gamma = \mathrm{Tr}\, P \exp \int_\gamma A$$

Lax operator = 4d Wilson line!

Effective 2d Theory

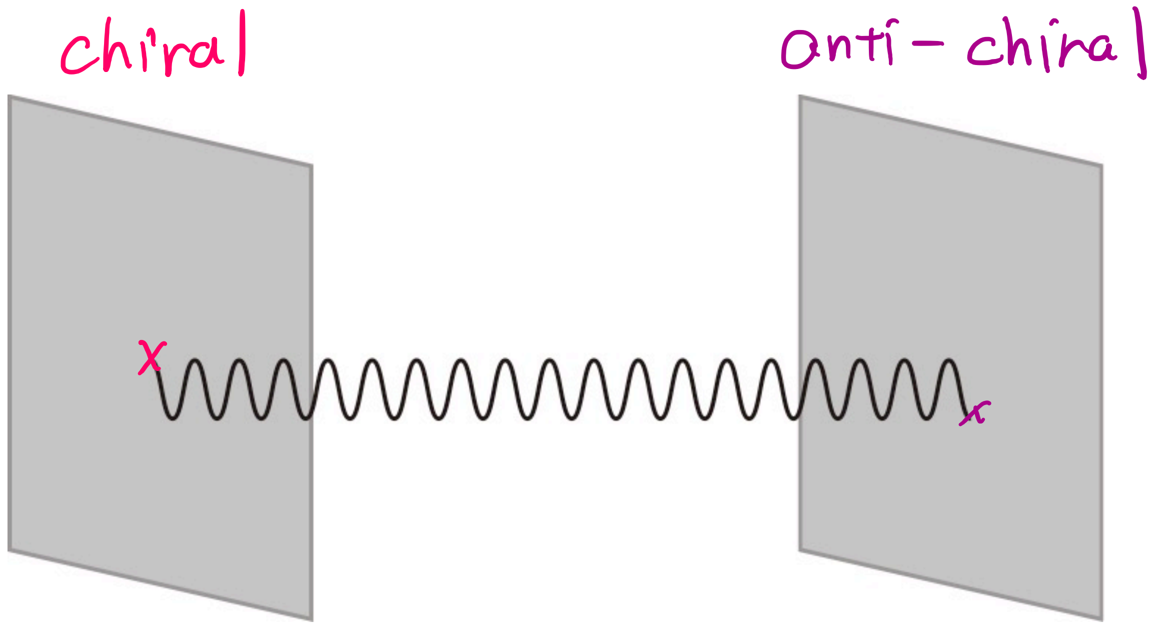
(Part III)


4d-2d system
effective 2d system
$\mathbb{R}^2$
$\times$
C
$\mathbb{R}^2$
integrate out along C

No 4d zero modes: we have perturbative expansion around an isolated solution of equation of motion (e.g. $A=0$ for $C=\mathbb{C}$)

All zero modes comes from 2d surface defects

The interaction comes from exchange of 4d gauge bosons



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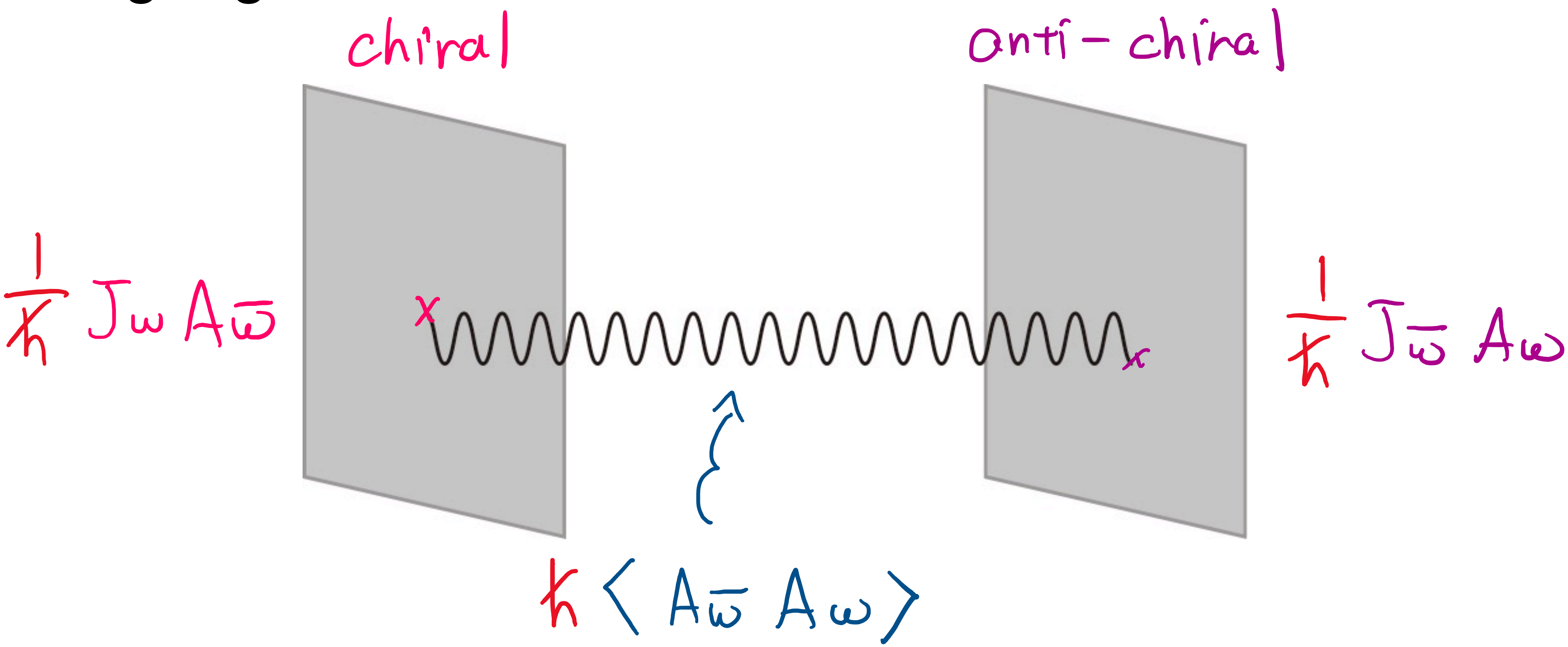
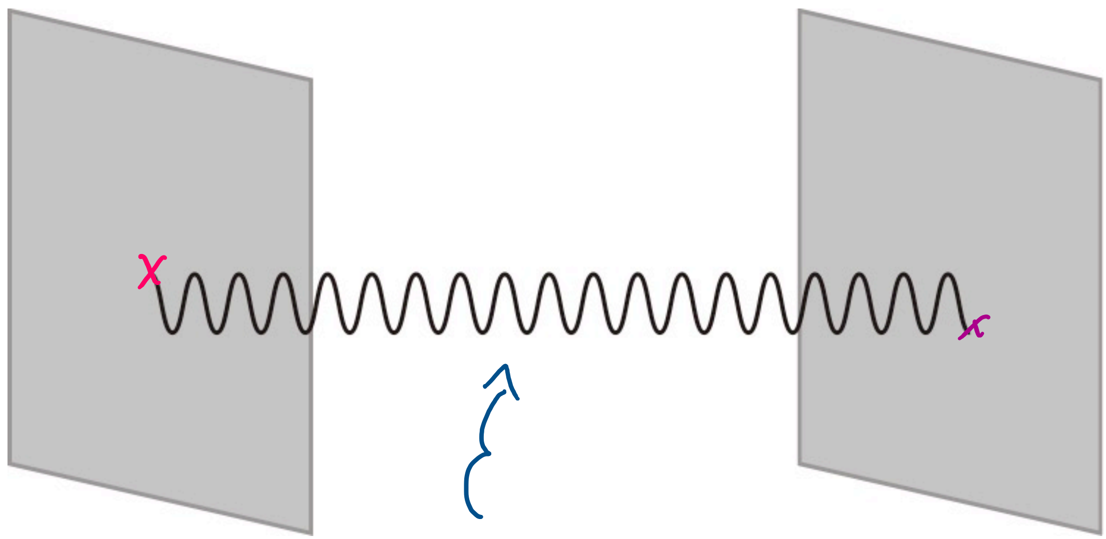


The interaction comes from exchange of 4d gauge bosons

chiral $\qquad\qquad\qquad\qquad$ anti-chiral

$\frac{1}{\hbar} J_w A_{\bar{w}}$



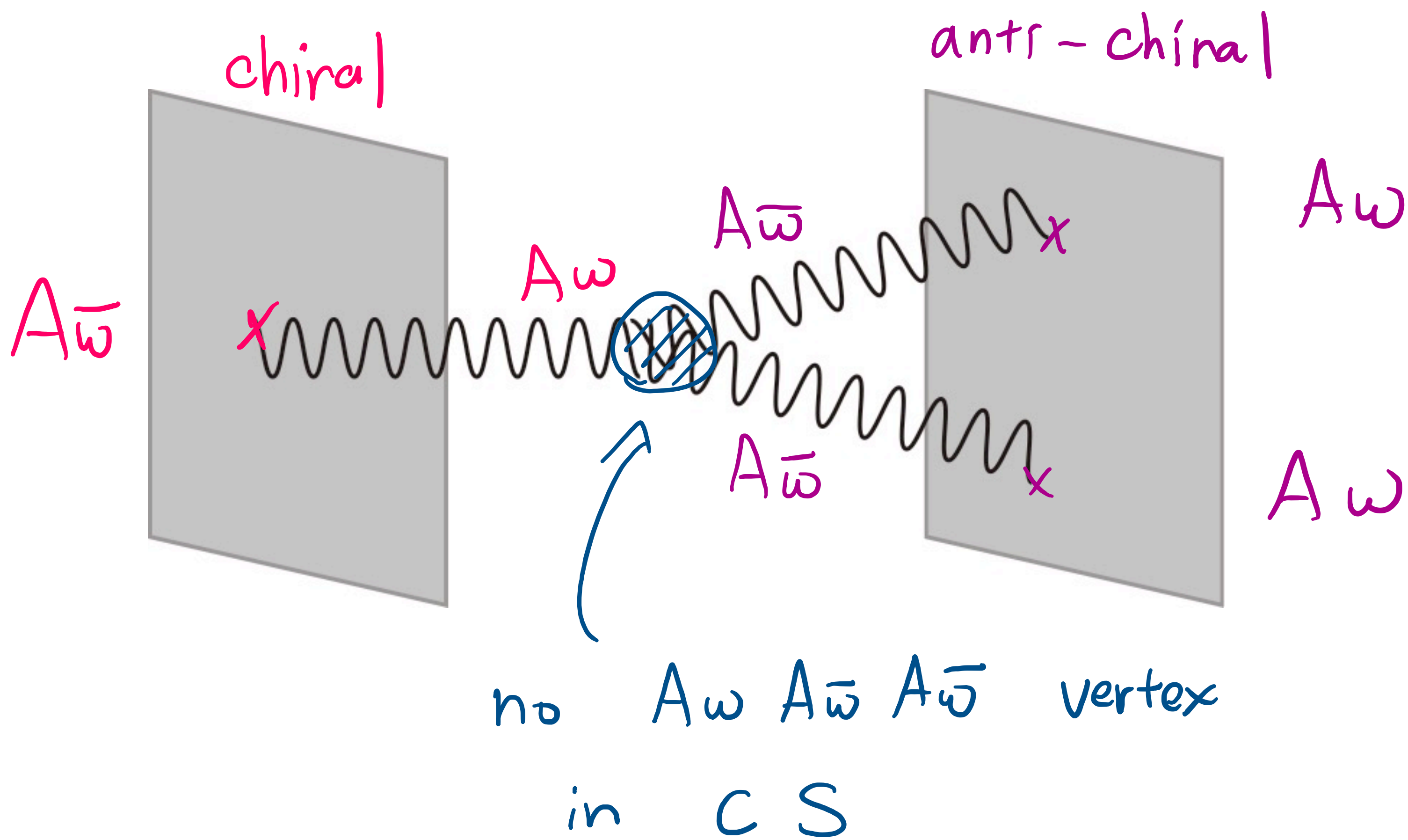
$\frac{1}{\hbar} J_{\bar{w}} A_w$

$\hbar \langle A_{\bar{w}} A_w \rangle$

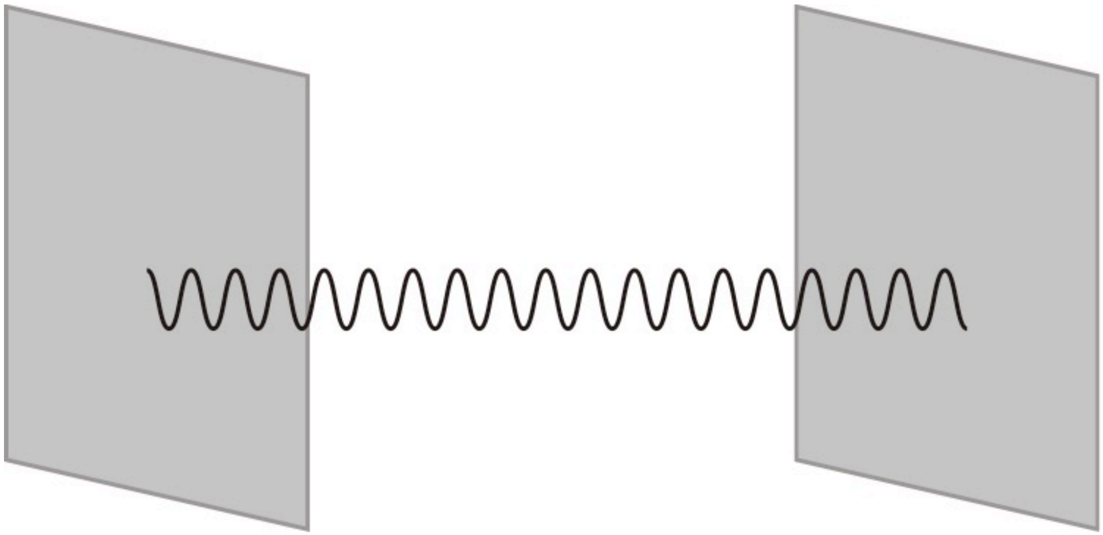
$$\left[\left(\frac{1}{\hbar}\right)^2 \hbar = \frac{1}{\hbar} \right]$$

only this diagram on the left contributes at tree-level namely $\mathcal{O}\left(\frac{1}{\hbar}\right)$

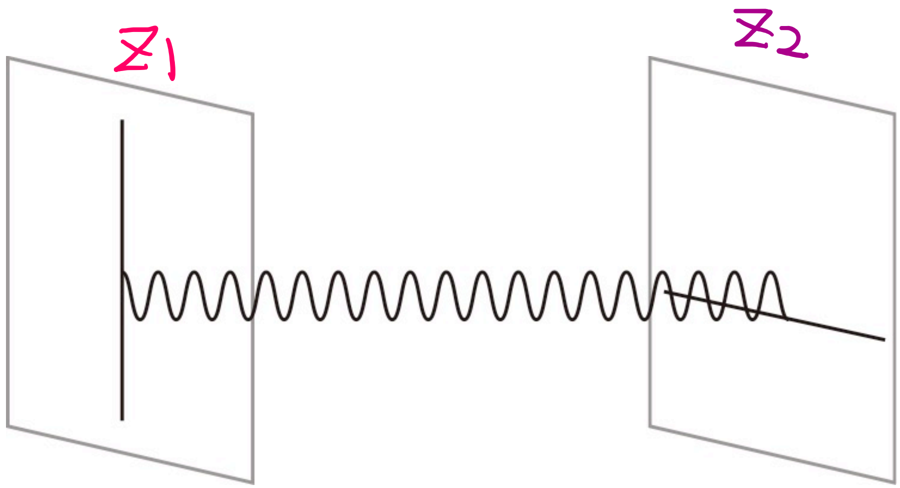
For example, no such diagram:



Let’s now compute this diagram



The computation is the same as in the computation of leading-order term of R-matrix in Part I



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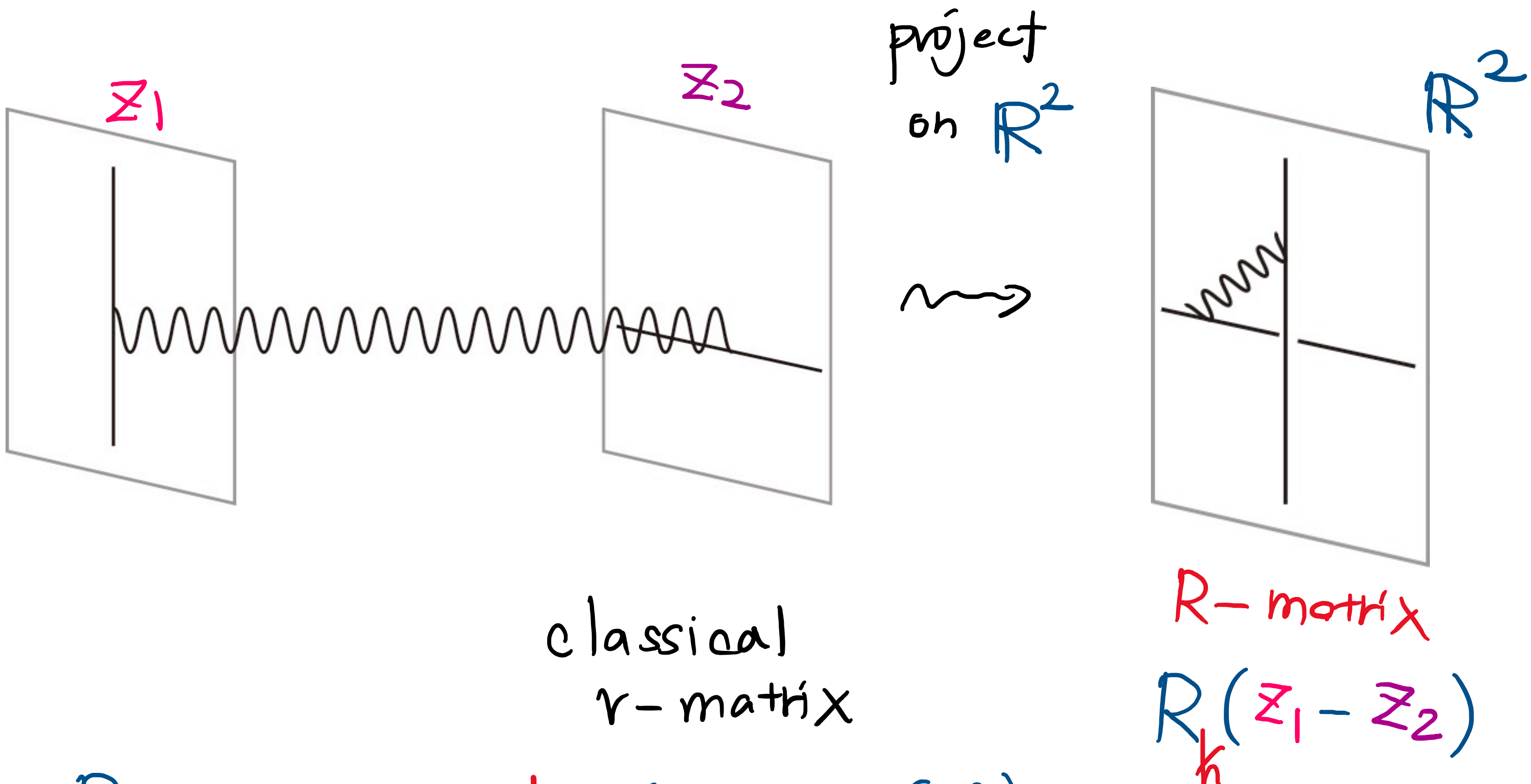
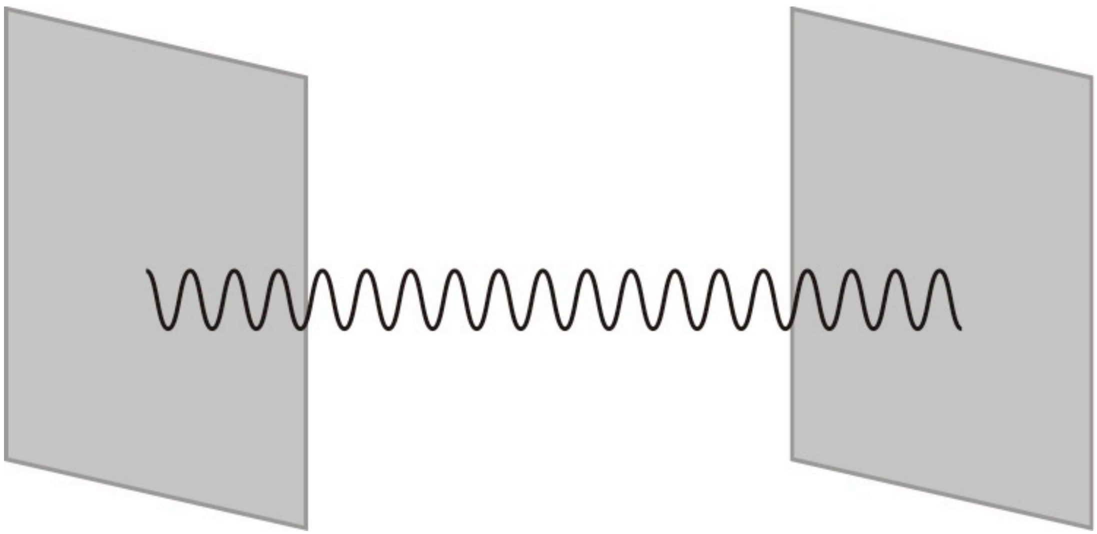


$$R_\hbar(z) = \mathrm{Id} + \hbar\, r(z) + O(\hbar^2)$$

We thus have the classical r-matrix



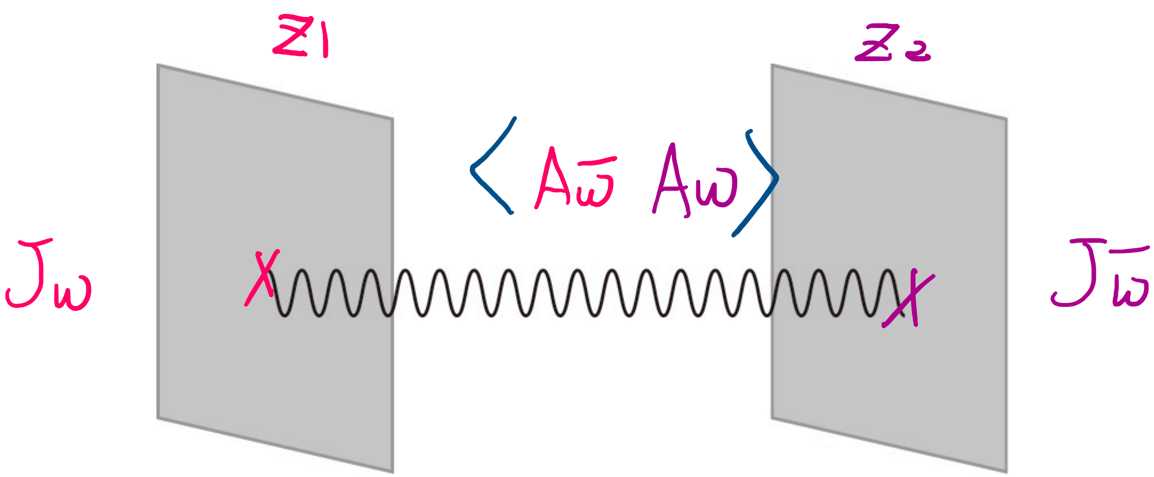
$$= r(z)$$

$$R_\hbar(z) = \mathrm{Id} + \hbar\, r(z) + \mathcal{O}(\hbar^2)$$

We obtained the effective 2d theory:

$$\mathcal{L}_{2d\ \mathrm{eff}} = \mathcal{L}_{2d\ \mathrm{chiral}}(z_1) + \mathcal{L}_{2d\ \mathrm{anti\text{-}chiral}}(z_2)$$

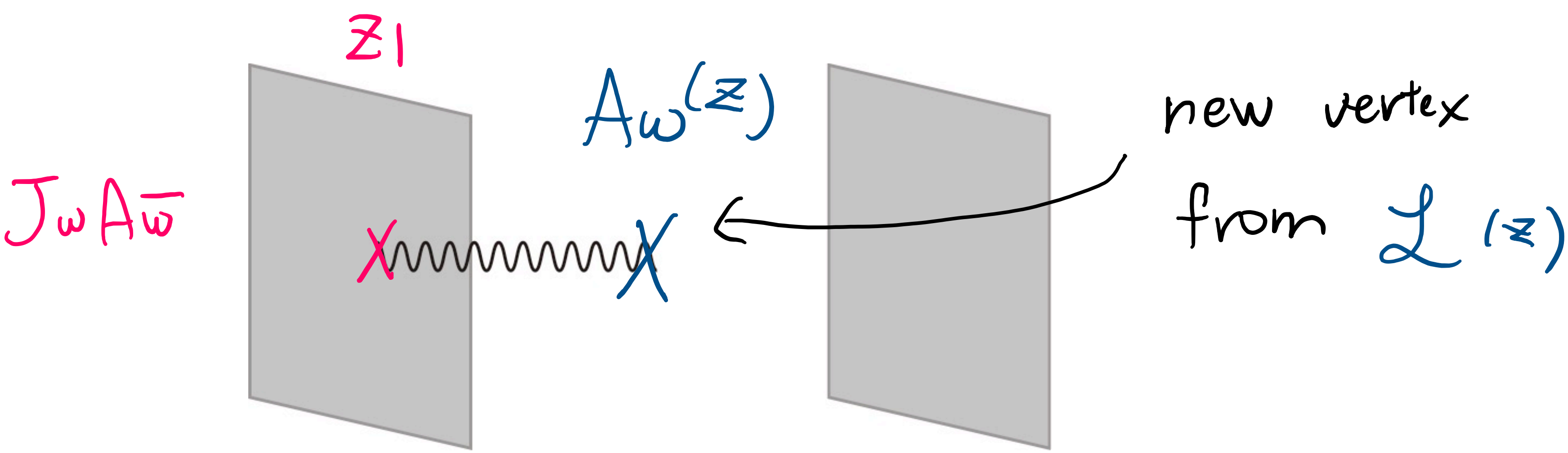
$$+ \; r^{ab}(z_1 - z_2)\, J^a_{w}(z_1)\, J^b_{\bar{w}}(z_2)$$



Similarly, we can compute Lax matrix for the effective 2d theory:

$$\mathcal{L}(z) = A_w(z)\, dw + A_{\bar{w}}(z)\, d\bar{w}$$

$$\mathcal{L}(z) = r_{ab}(z - z_1)\, J^b_{\bar{w}}(z_1) + r_{ab}(z_2 - z)\, J^b_{\bar{w}}(z_2)$$



For the rational case, we have

$$r_{ab}(z) = \frac{c_{ab}}{z} \leftarrow \text{Casimir element}$$

and we reproduce the standard formula

$$\mathcal{L}(z) = \frac{\tilde{j} + z \ast \tilde{j}}{z^2 - 1}$$

where

$$\tilde{j} = J_w(z_1)\, dw + J_{\bar{w}}(z_2)\, d\bar{w}$$

and we choose $z_1 = 1, \; z_2 = -1$

Examples and Generalizations

Simple example: chiral/anti-chiral free fermions

$\psi, \bar{\psi}$

$$\mathcal{L} = \psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}$$

$$+ r_{ab}(z_1 - z_2) \left(\psi t^a \psi (z_1) \right) \left(\bar{\psi} t^b \bar{\psi} (z_2) \right)$$

Reproduce Gross-Neveu and Thirring models

$G = SO(N)$ $\qquad$ $G = SU(N)$

The framework generalize in several directions:

1. trigonometric/elliptic cases

spectral curve

$C = \mathbb{C}$: rational

$\mathbb{C}^\times$: trigonometric

E : elliptic

2. More general defects

e.g. curved beta-gamma system

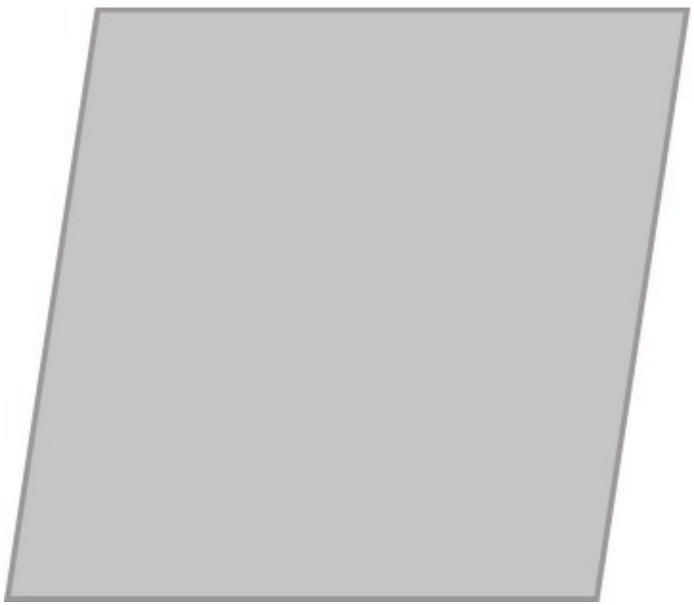
$$\mathcal{L}_{defect} = \beta D_A \gamma$$

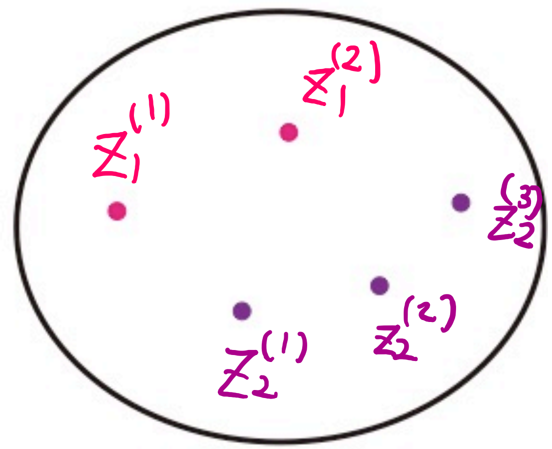
from which we obtain sigma models

Also non-chiral defects, e.g. free boson ϕ

$$\mathcal{L}_{defect} = D_A \phi D_A \phi$$

3. multiple defects





$$\mathcal{L} \supset \sum_{i,j} r_{ab}\left(z_1^{(i)} - z_2^{(j)}\right) J_w^a\left(z_1^{(i)}\right) J_w^b\left(z_2^{(j)}\right)$$

Quantum Integrability

(Part IV)

Classical integrability from Lax operator is in general broken by quantum effects

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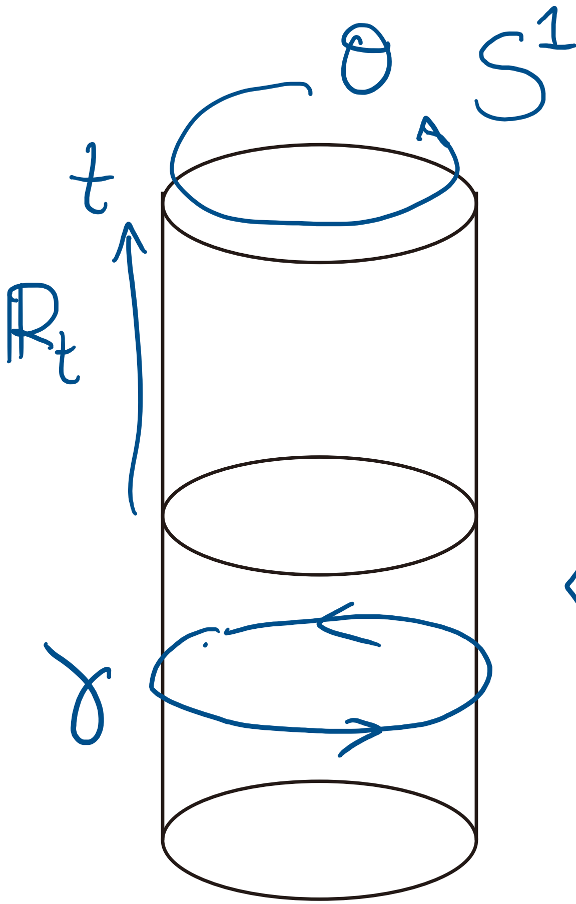
in general broken by quantum effects

For consistency we need anomaly cancellation
for the full 4d-2d system; this can be achieved
by modifying the CS level appropriately.
Today I will not discuss this.

Instead let me explain how to see
quantum integrability in our framework,
assuming cancellation of anomalies

Recall: Lax operator = 4d Wilson line

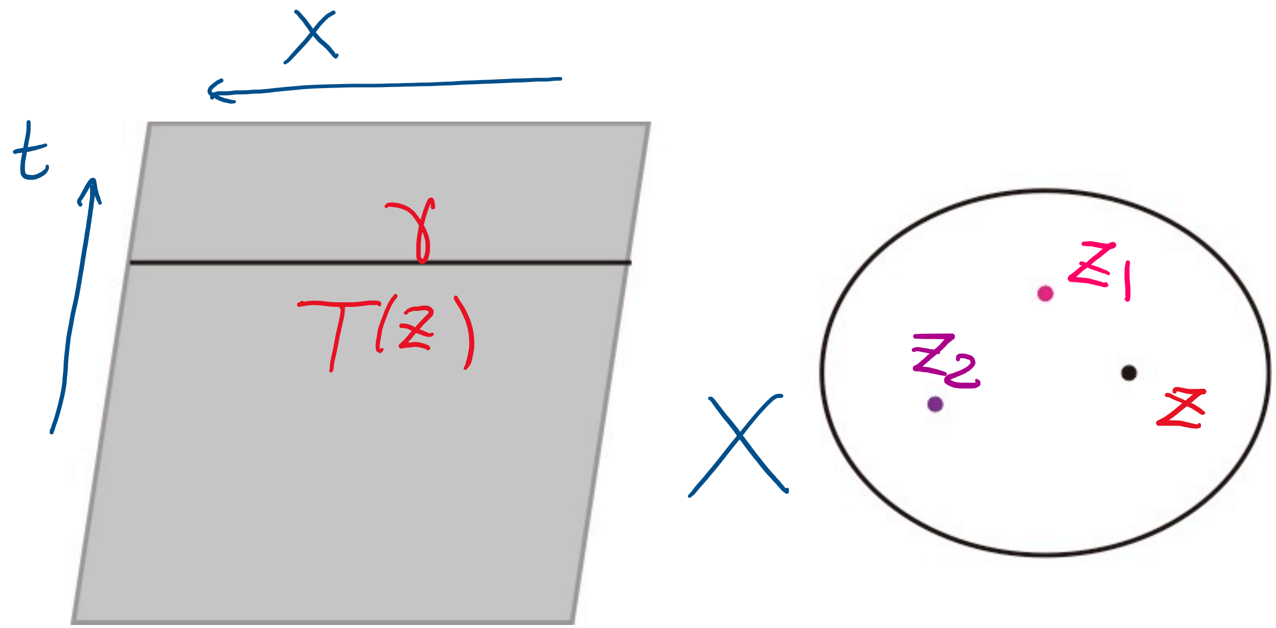


Wilson line

$$\left\langle \mathrm{Tr}\, P \exp \int_\gamma A \right\rangle$$

instead of $\mathbb{R} \times S^1$

let's consider $\mathbb{R}^2$



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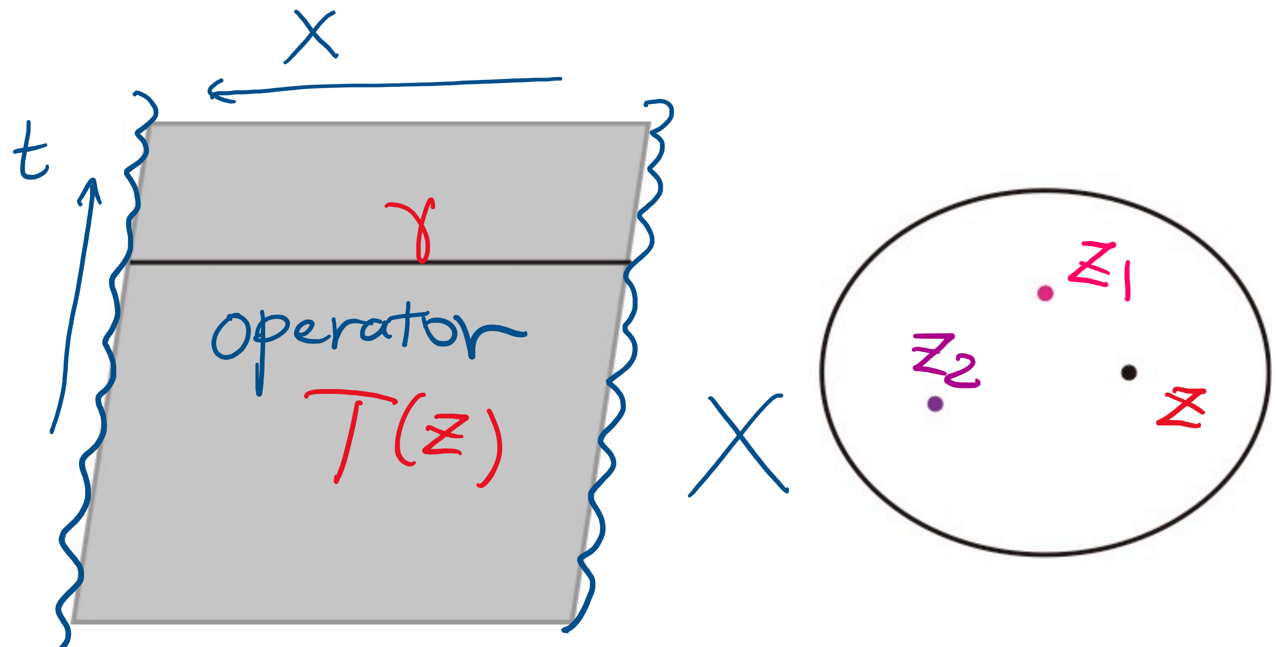
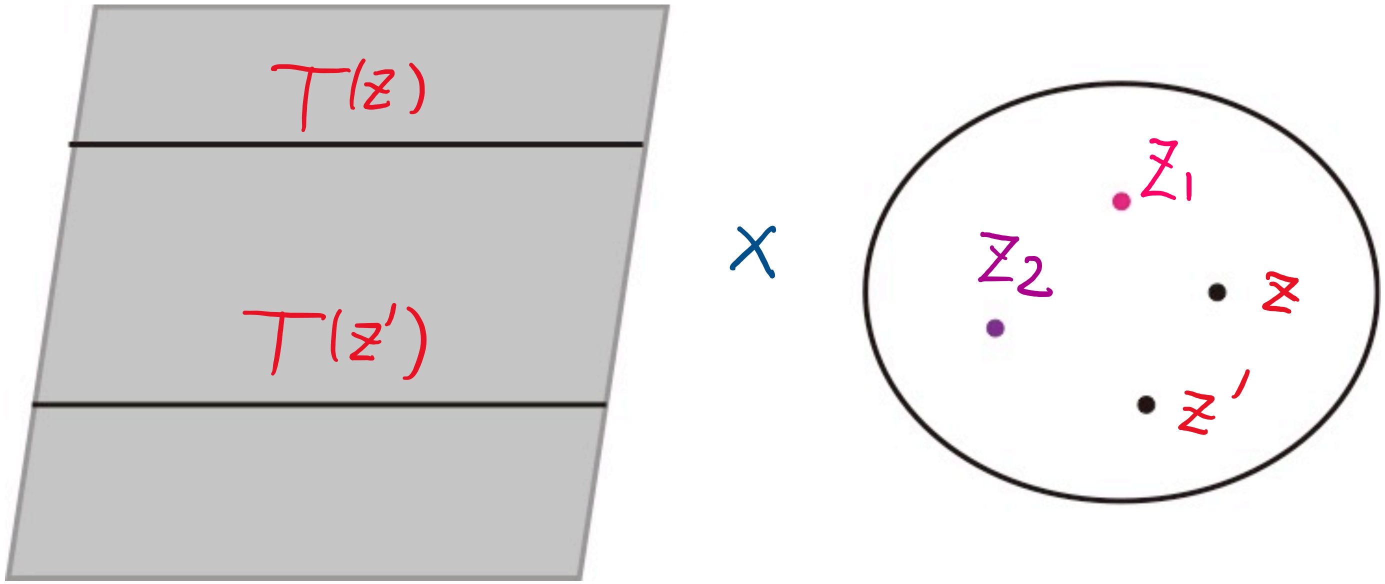
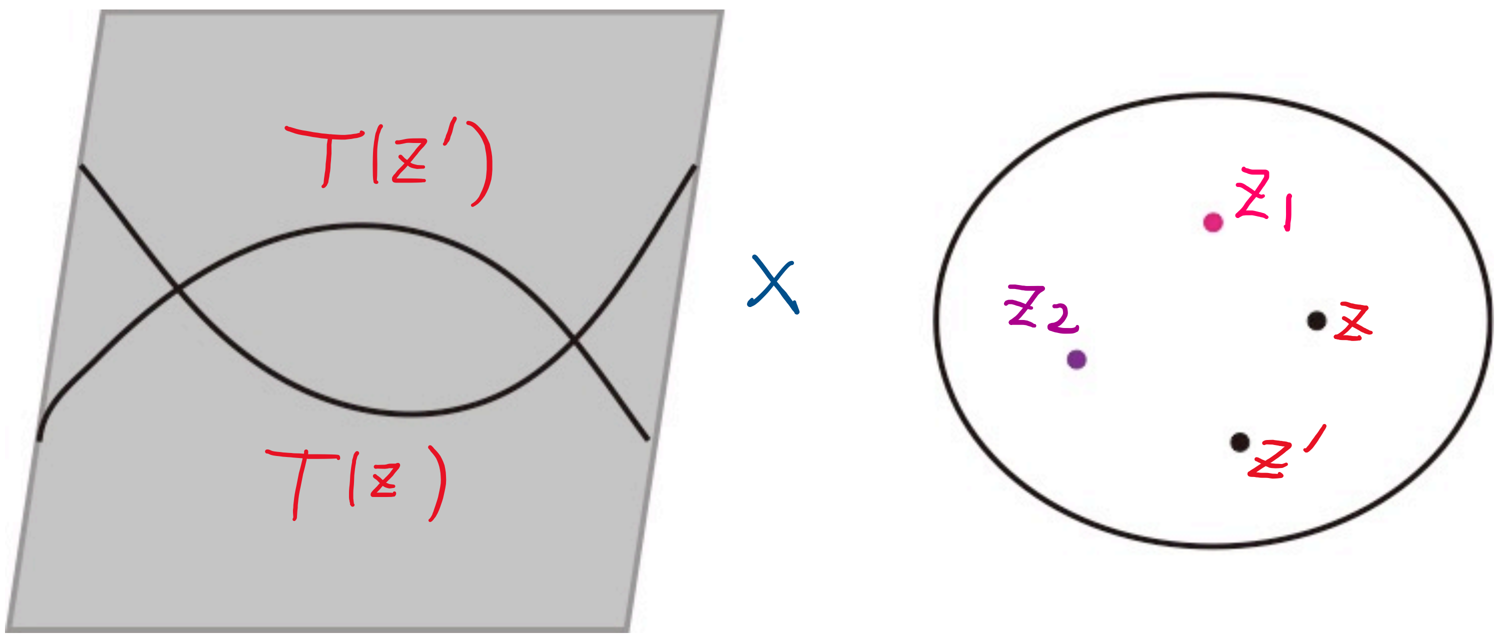


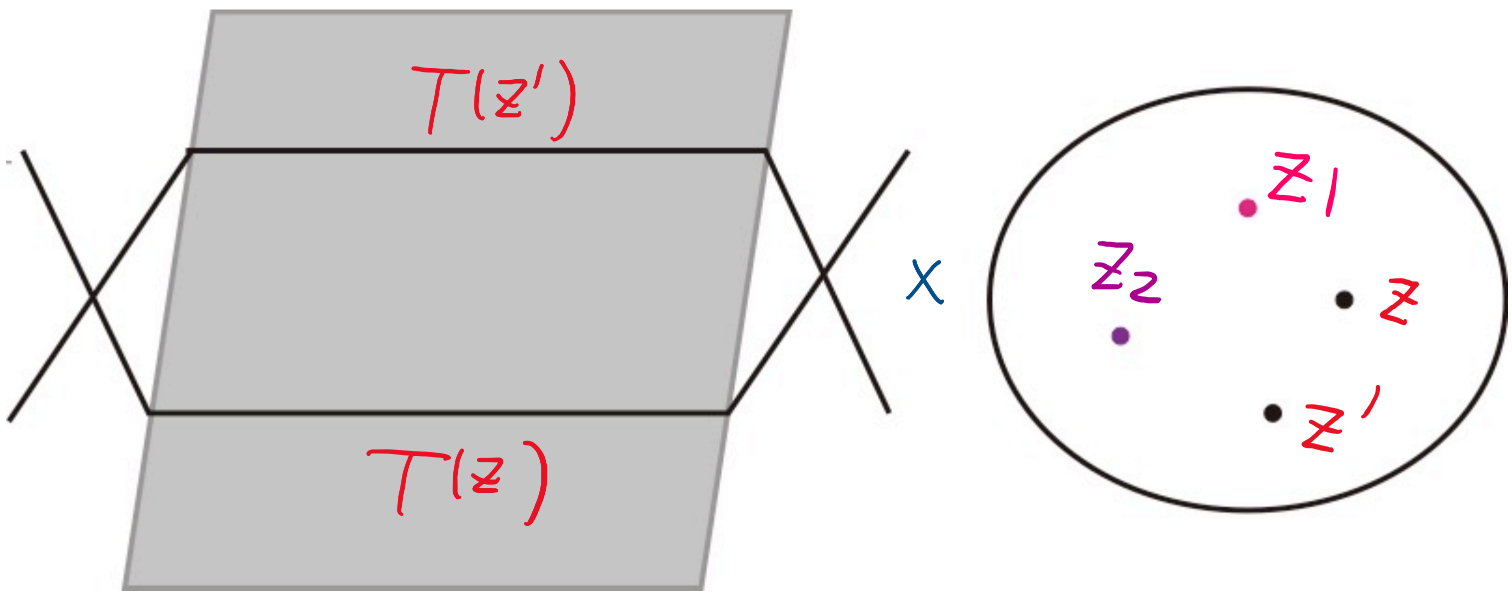
$\langle \text{out} |$ @ $x = \infty$ $\qquad$ $| \text{in} \rangle$ @ $x = -\infty$

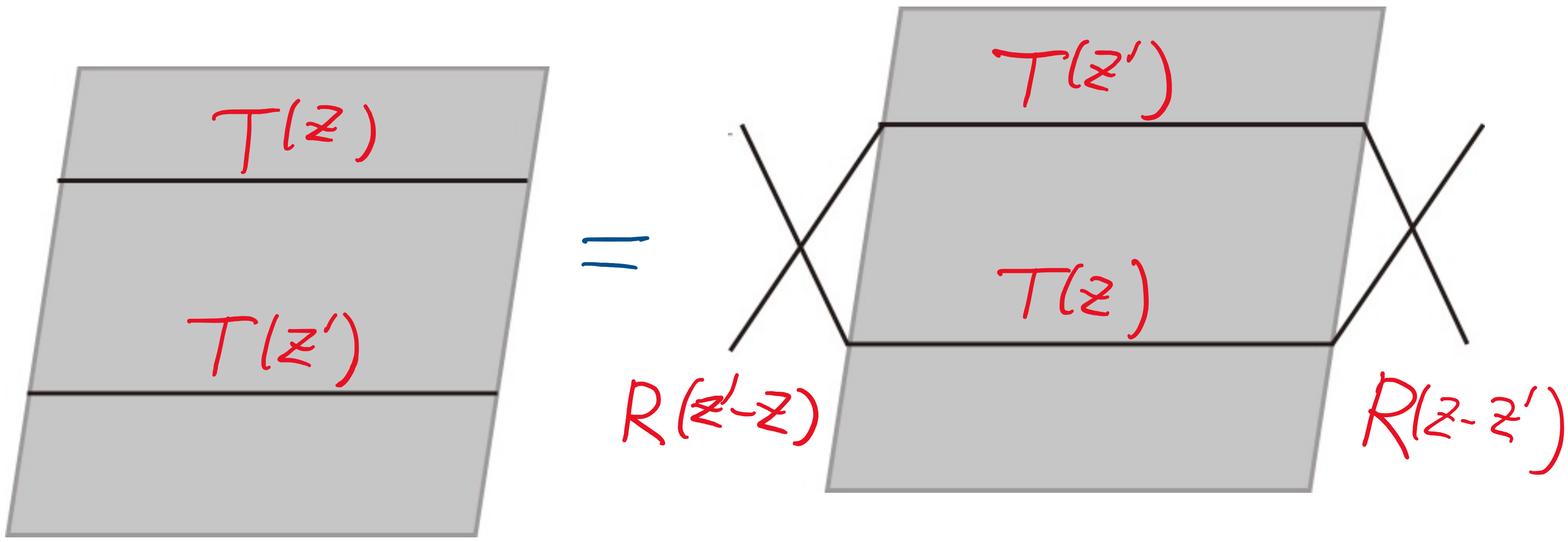
$$\langle \text{out} | \; T(z) = P \exp \int_\gamma A(z) \; | \text{in} \rangle$$


T(z)
T(z')
x
z₁
z₂
z
z'

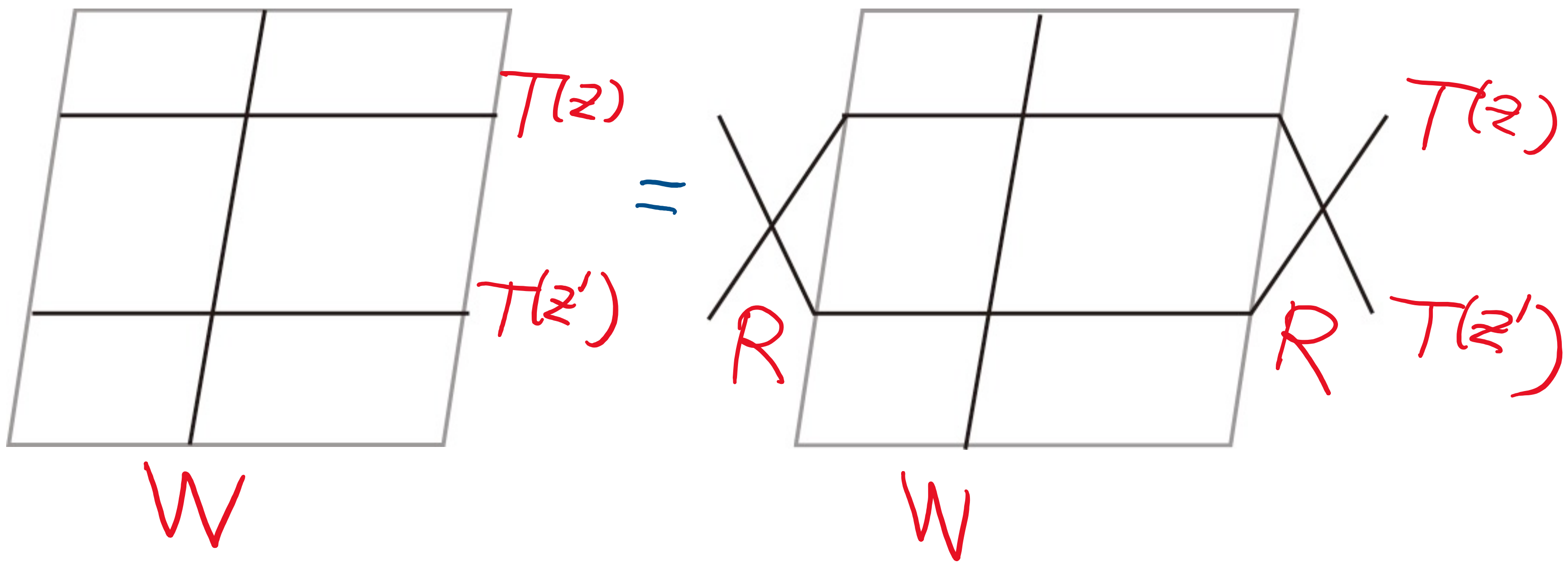


T(z')
T(z)
×
z1
z2
z
z'


T(z')
T(z)
X
z1
z2
z
z'



RTT relation: definition of the Yangian
(and their trigonometric/elliptic counterparts),
and ensures quantum integrability



This can be thought of the “continuum limit” of the RTT relation for discrete lattice models, discussed in Part II

Our 4d framework says more, about e.g.

- Local conserved charges
- Renormalization group flow
- S-matrix factorization

Part IV

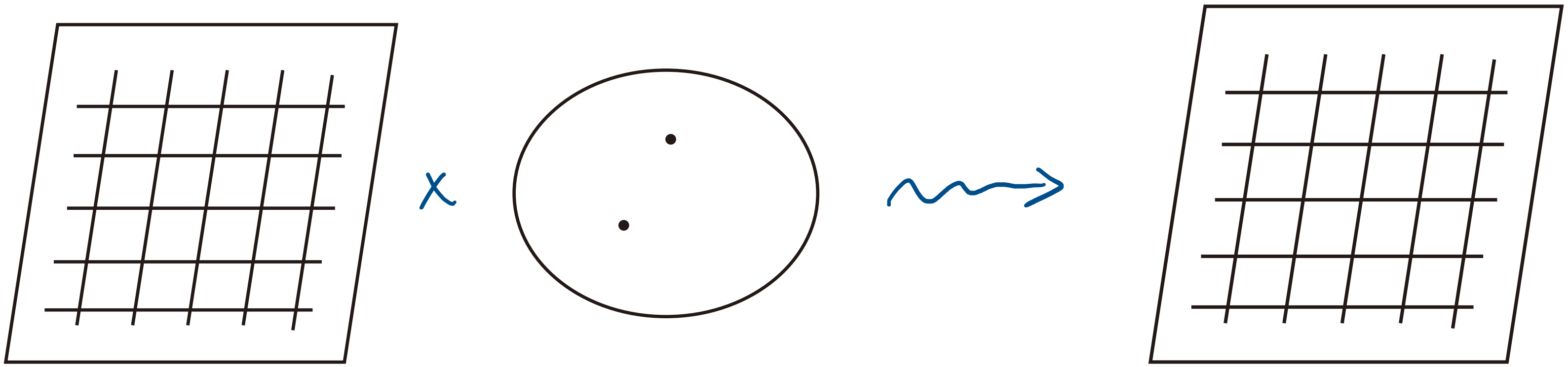
- Higher genus spectral curves

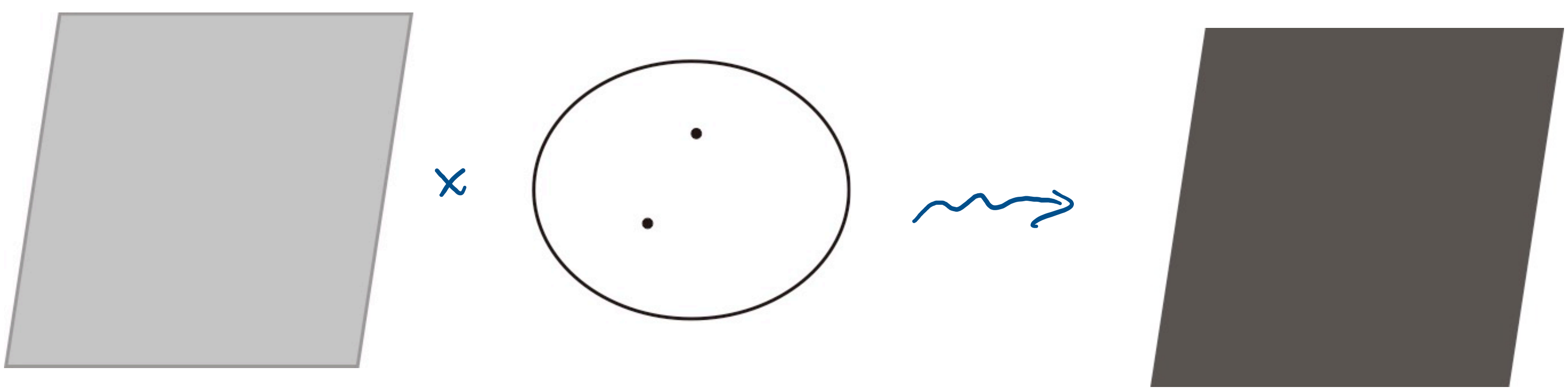
Part III

⋮

Summary

Part I & II





Part III & IV