The City College of New York

Odd surface waves in two dimensional incompressible fluids

Or

How I stopped being frustrated by odd and started loving it!!!

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SG, A. Abanov Phys. Rev. Fluids 2, 094101 (2017)

A. Abanov, T. Can, SG
SciPost Phys. 5, 010 (2018)

A. Abanov, T. Can, SG Unpublished

Tankut Can

Alexander Abanov

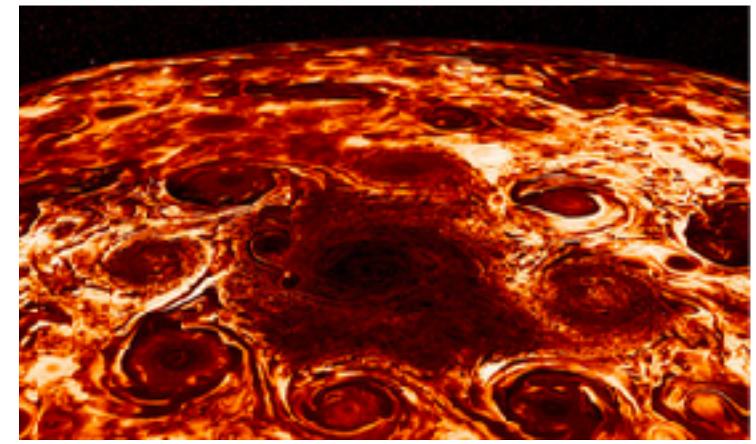
Classical fluids with broken parity in two dimensions

Remark: Parity is violated by microscopic constituents (eg fluid of chiral/spinning objects)

Chiral vortex fluids

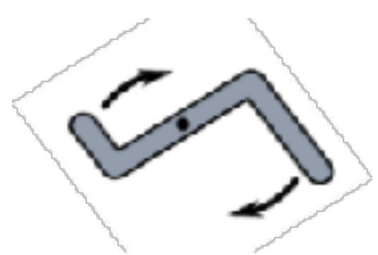
Wiegmann, Abanov 2014

Jupiter atomshere (NASA)



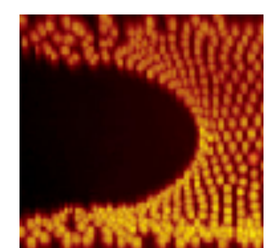
Active matter (motors, collection of spinning tops, spinning magnetic colloids)

Banerjee et al 2017, William Irvine (unpublished)



Vitelli lab

Irvine Lab

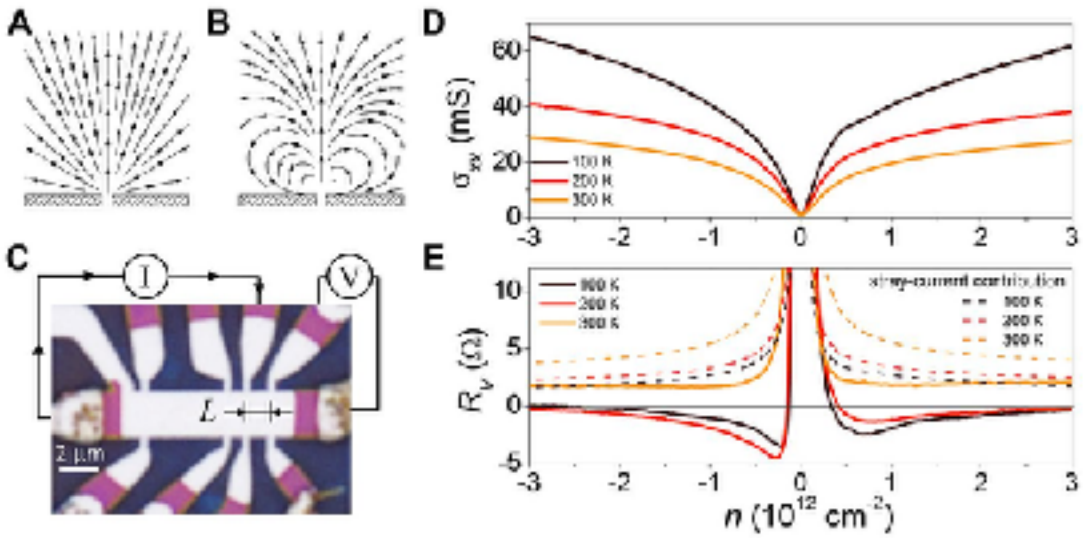


In 3D-Plasma in magnetic field (Landau, Lifshitz, v10), Superfluid Helium

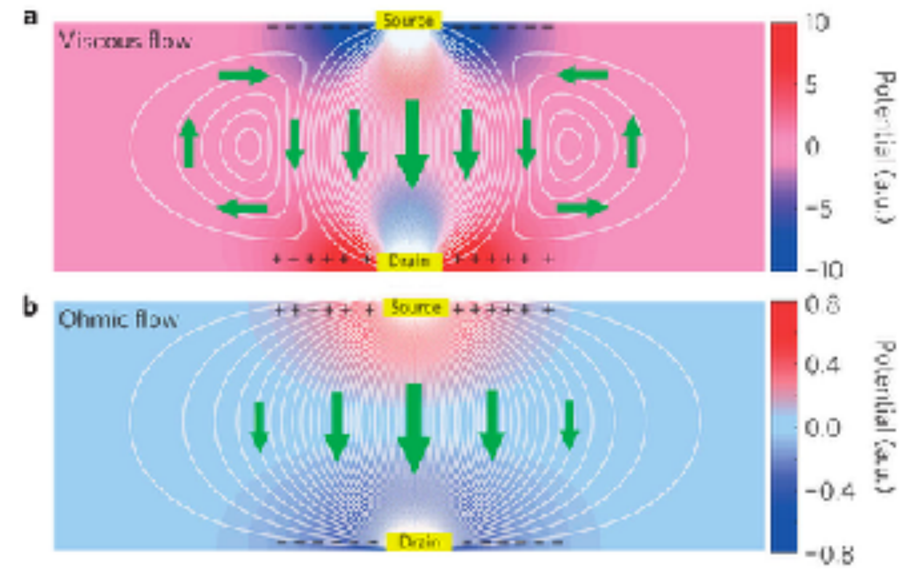
Hydrodynamic flows in electronic systems

$$l_{ee} \ll (W_{sample}, l_{mf})$$

Viscous back flows in graphene due to electron fluid vortices



Experiment (Geim et al)



Theory (Levitov, Falkovich)

Hydrodynamic electronic flows with broken parity

Science REPORTS

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Evidence for hydrodynamic electron flow in $PdCoO_2$

Philip J. W. Moll,[1,2,3] Pallavi Kushwaha,[3] Nabhanila Nandi,[3] Burkhard Schmidt,[3] Andrew P. Mackenzie[3,4]

[1]Laboratory for Solid State Physics, ETH Zurich, CH-8093 Zurich, Switzerland. [2]Department of Physics, University of California, Berkeley, CA 94720, USA. [3]Max Planck Institute for Chemical Physics of Solids, 01187 Dresden, Germany. [4]Scottish Universities Physics Alliance, School of Physics and Astronomy, University of St. Andrews, St. Andrews KY16 9SS, UK.

Measuring Hall Viscosity of Graphene's Electron Fluid

A. I. Berdyugin[1], S. G. Xu[1,2], F. M. D. Pellegrino[3,4], R. Krishna Kumar[1,2], A. Principi[1], I. Torre[5], M. Ben Shalom[1,2], T. Taniguchi[6], K. Watanabe[6], I. V. Grigorieva[1], M. Polini[7,1], A. K. Geim[1,2], D. A. Bandurin[1]

This talk

Question : What are the observable effects of a particular parity breaking term (odd/Hall viscosity) in two dimensional classical (chiral active) fluids?

Goal : Build library of broken parity classical hydro phenomena in 2D

Application : Quantum Hall hydrodynamics

Hydrodynamics with broken parity

Avron 1998

Parity breaking is compatible with isotropy only in 2D $(x, y) \to (-x, y)$

Incompressible Navier-Stokes Equation with odd viscosity $\rho = 1$

$$\partial_i v_i = 0 \qquad \partial_t v_i + v_j \partial_j v_i = -\partial_i p + \nu_e \Delta v_i + \nu_o \Delta v_i^*$$

$$v_i^* = \epsilon_{ij} v_j, \; v_x^* = v_y, \; v_y^* = -v_x$$

ν_o Kinematic odd viscosity (can be any sign in contrast to ν_e)

Navier Stokes in terms of stress tensor $\partial_t v_i + v_j \partial_j v_i = \partial_j T_{ij}$

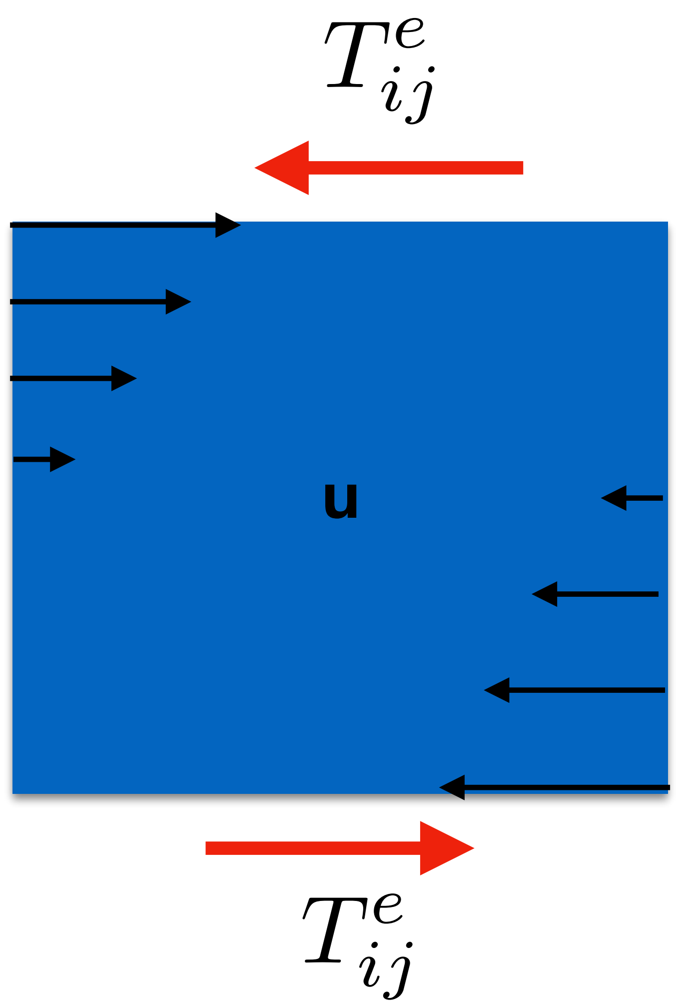
$$T_{ij} = -p\delta_{ij} + \underbrace{\nu_e(\partial_i v_j + \partial_j v_i)}_{\text{Viscous Shear Tensor}} + \underbrace{\nu_o(\partial_i^* v_j + \partial_i v_j^*)}_{\text{“Lorentz” Shear Tensor}}$$

Remark: Not the most general parity violating stress tensor in 2D (there are other symmetry allowed terms), Ignore temperature effects.

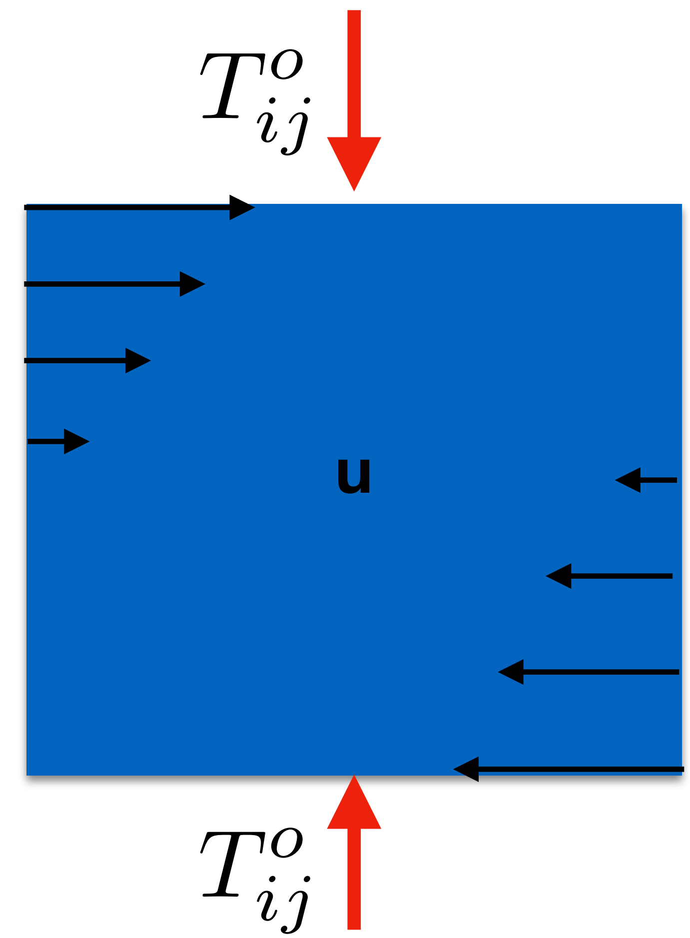
Shear (even) v Odd stresses

$$T^e_{ij} = \nu_e(\partial_i v_j + \partial_j v_i)$$

$$T^o_{ij} = \nu_o(\partial^*_i v_j + \partial_i v^*_j)$$



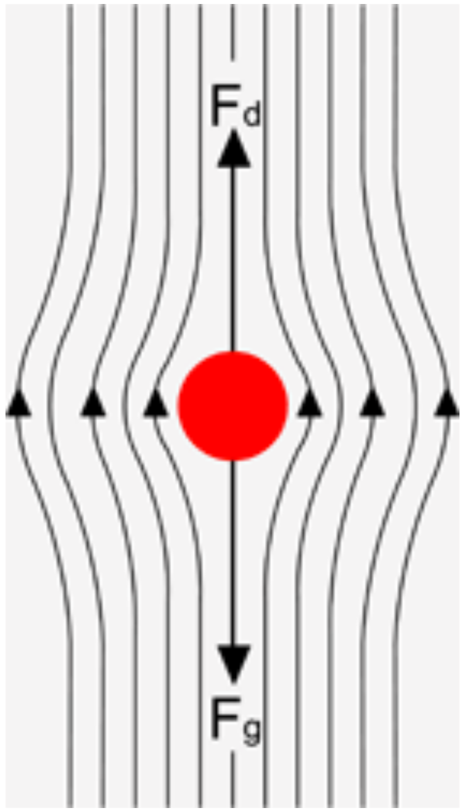
Dissipative



non-dissipative (dispersive)

Boundary conditions

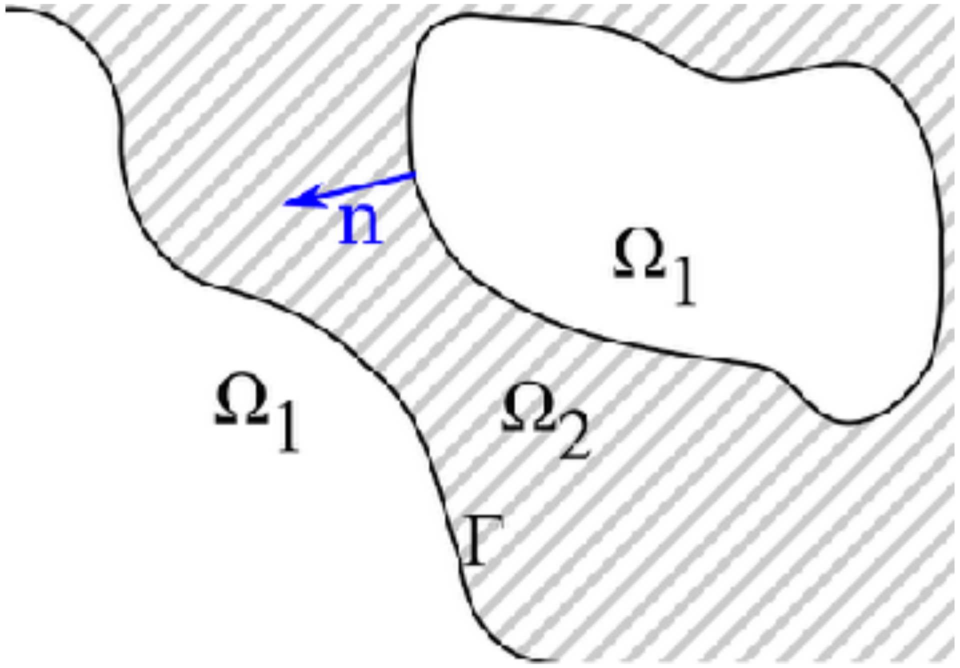
Flow dependent BC for
solid-liquid interface
(eg. no-slip, partial slip etc)



$$v_{x,y}|_{\Gamma} = constant$$

Viscosity independent BC
(Electronic hydro flows studied
were of this type)

Force matching conditions for
fluid-fluid interface
no-stress conditions



$$n_i(T^1_{ij} - T^2_{ij})|_{interface} = -\tau \kappa n_j$$

τ is the surface tension

BC depends on viscosities via T_{ij}

Odd viscosity with "no-slip" BC: Three exact statements

SG, A. Abanov Phys. Rev. Fluids 2, 094101 (2017)

If boundary conditions of an incompressible flow depend only on the flow itself, then

I. The flow **v**(**x**, t) does not depend on the value of odd viscosity ν_o.

II. The net force (lift force) **F** acting on a closed contour does not depend on the value of odd viscosity ν_o.

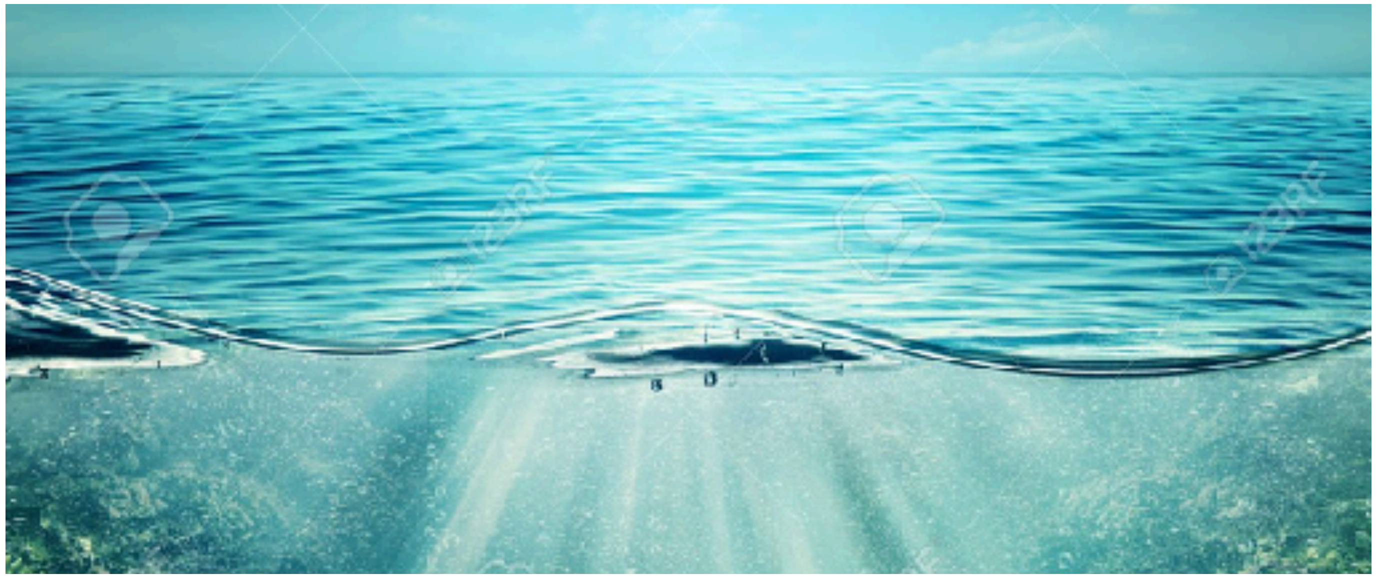
III. The net torque $\mathcal{T}_o$ acting on a closed contour which depends on the value of odd viscosity ν_o is given by:

$$\mathcal{T}^o = 2\nu_o \oint_\Gamma ds\, v_n = 2\nu^o \frac{d\mathcal{A}}{dt}$$

$\mathcal{A}$ is the area enclosed by the contour

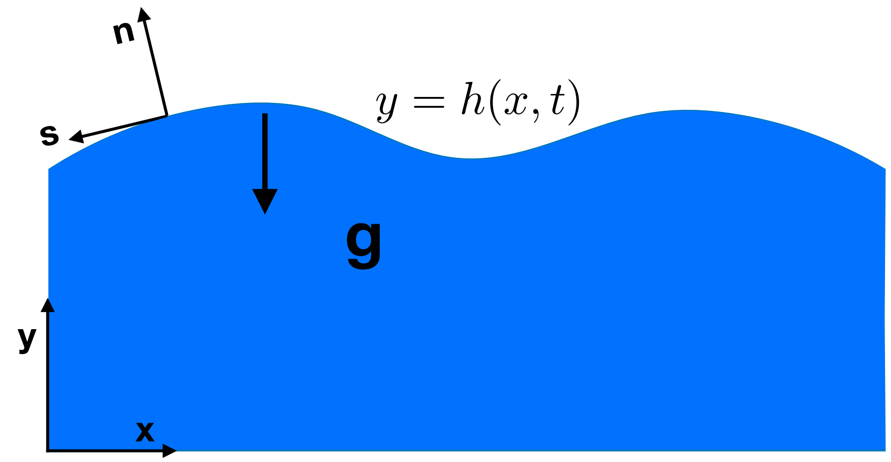
All three statements appeared before: Avron '98; Lapa, Hughes '14 under less general conditions.

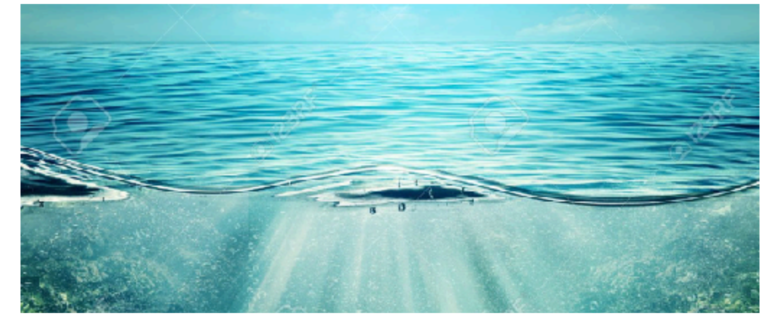
No stress boundary conditions: Odd surface waves



Surface dynamics of a two dimensional ocean with broken parity

$$\mathbf{n} = \frac{1}{\sqrt{1+(\partial_x h)^2}}(-\partial_x h, 1)$$





Equations of motion

$$\partial_i v_i = 0 \qquad D_t v_i = -\partial_i \tilde{p} + \nu_e \Delta v_i$$

$$D_t = \partial_t + v_i \partial_i \qquad \tilde{p} = p - \nu_o \omega + g y \qquad \omega = \partial_i v_i^*$$

Boundary conditions (two-fluid interface)

Dynamical boundary conditions

$$n_j T_{ij}\Big|_{y=h(x,t)} = 0$$

Kinematic boundary conditions

$$\partial_t h + v_x \partial_x h = v_y$$

$$v_y|_{y=-\infty} = 0$$

Solve the above system for $(v_x, v_y, h(x,t), \tilde{p})$

Surface gravity waves in inviscid ocean

$$\nu_e = 0, \nu_o = 0$$

Equations of motion $\quad v_i = \partial_i \phi \qquad \Delta\phi = 0 \qquad \partial_t \phi + \cancel{\frac{1}{2}(\nabla\phi)^2} = -p - gy$

Boundary conditions $\quad p|_{y=h} = 0 \qquad h_t + \cancel{h_x \partial_x \phi} = \partial_y \phi$

Any time dependent harmonic potential is a solution

Linearized bulk potential Solutions

$$v_x = A|k|e^{|k|y}e^{ikx-i\Omega t} \qquad v_y = -iAke^{|k|y}e^{ikx-i\Omega t} \qquad p = \Omega\frac{k}{|k|}Ae^{|k|y}e^{ikx-i\Omega t} - gy$$

Surface dynamics satisfying no stress BC

$$h(x,t) = \frac{k}{\Omega}Ae^{ikx-i\Omega t} \qquad \Omega = \pm\sqrt{g|k|}$$

Surface gravity waves with shear viscosity

Lamb 1932

$$\nu_e \neq 0, \nu_o = 0$$

Equations of motion

$$\partial_i v_i = 0 \qquad \partial_t v_i + \cancel{v_j \partial_j} v_i = -\partial_i (p + gy) + \nu_e \Delta v_i$$

Boundary conditions $(y = h(x,t))$

$$p = 2\nu_e \partial_y v_y \qquad \omega = 2\partial_x v_y \qquad \partial_t h + v_x \cancel{\partial_x} h = v_y$$

Cannot satisfy boundary conditions using harmonic functions (flow must be vortical)

Linearized bulk solutions

$$v_x = \left(A|k|e^{|k|y} + Bme^{my} \right) e^{ikx - i\Omega t}$$

$$v_y = -ik \left(Ae^{|k|y} + Be^{my} \right) e^{ikx - i\Omega t}$$

$$m^2 = k^2 - \frac{i\Omega}{\nu_e}$$

$$\omega = e^{ikx - i\Omega t} \left(\frac{i\Omega}{\nu_e} B e^{my} \right),$$

$$p = \Omega \frac{k}{|k|} A e^{|k|y} e^{ikx - i\Omega t} - gy$$

Vorticity decay brings in boundary layer thickness scale $\quad \delta \sim m^{-1} \sim \sqrt{\nu_e}$

Surface dynamics satisfying no stress BC

$$h(x,t) = \frac{k}{\Omega} (A + B) \, e^{ikx - i\Omega t}$$

$$\Omega \approx \pm \sqrt{g|k|} - 2i\nu_e k^2$$

Linear "odd" gravity waves with shear viscosity

$$\nu_e \neq 0, \nu_o \neq 0$$

Equations of motion

Same as Lamb's case except $\tilde{p} = p - \nu_o \omega + gy$

Boundary conditions

$(y = h(x,t))$

$$\tilde{p} = 2\nu_e \partial_y v_y - 2\nu_o \partial_x v_y \qquad \omega = 2\partial_x v_y + 2\frac{\nu_o}{\nu_e}\partial_y v_y$$

No change in bulk solutions

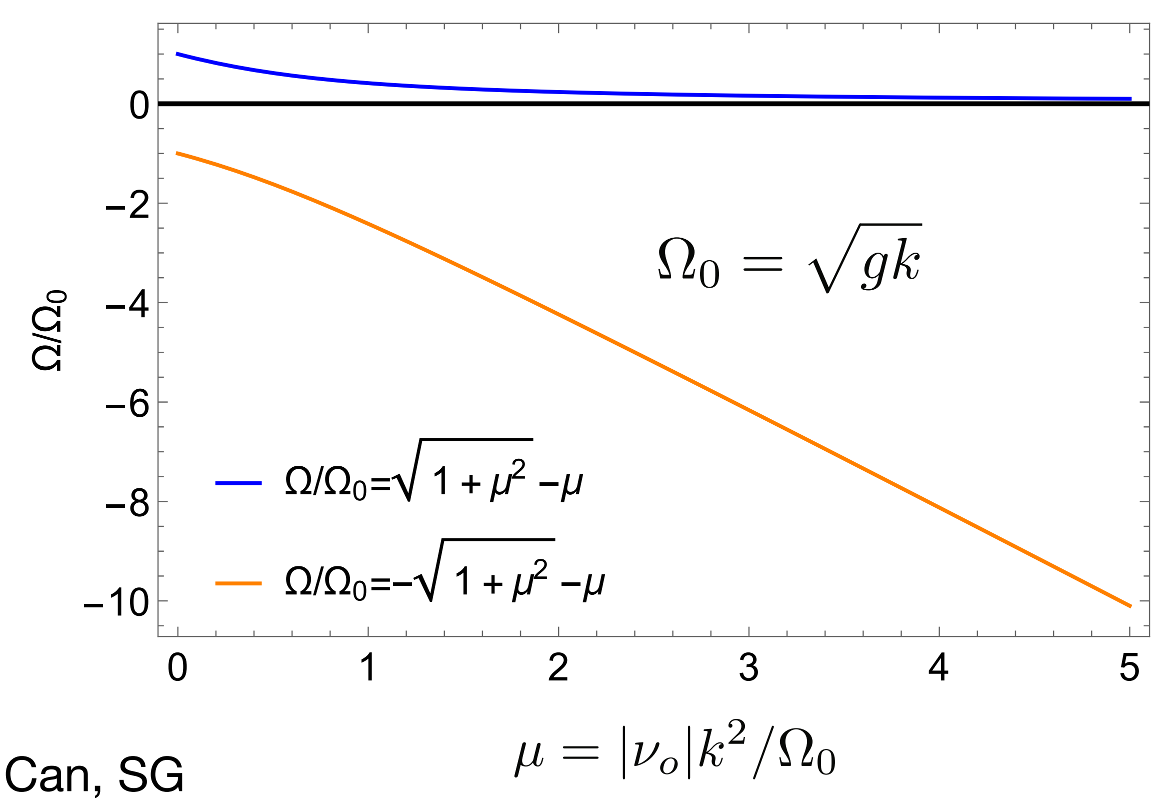
Vorticity decay brings in boundary layer thickness scale $\delta = m^{-1} = \sqrt{\dfrac{\nu_e}{\nu_o}}$

Surface dynamics satisfying no stress BC

$$\Omega \approx \pm\sqrt{g|k|} - 2i\nu_e k^2 - 2\nu_o k|k|$$

$$g \to 0, \nu_e \to 0 \qquad \Omega = 0, -2\nu_o k|k|$$

Chiral propagation $\quad v_k = \dfrac{\partial \Omega}{\partial k} \approx -4\nu_o|k|$



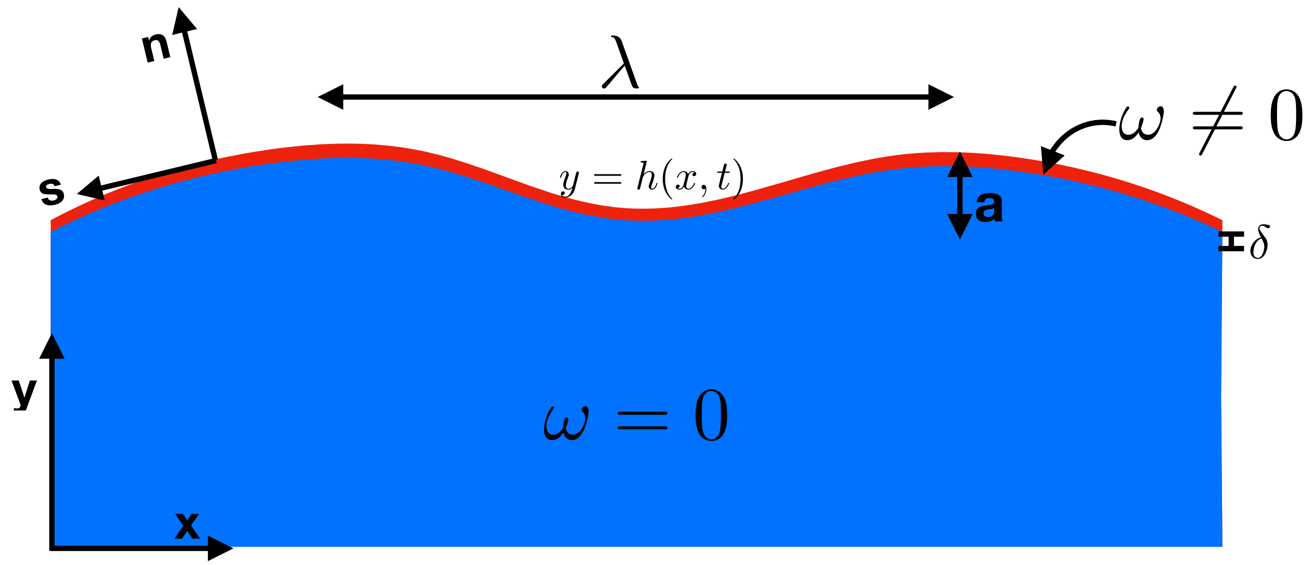
A. Abanov, T. Can, SG
arXiv:1801.10150

Weakly non-linear odd surface waves

Loguet-Higgins, 1953, Ruvinsky, 1981, Zakharov 2004

$$\nu_e \to 0, g = 0$$

A. Abanov, T. Can, SG
arXiv:1801.10150



Non-linear surface dynamics integrating out boundary layer

$$\partial_t \phi + \frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}(\partial_y \phi)^2 + \tilde{p}\Big|_{y=h(x,t)} = 0 \qquad \tilde{p} = 2\nu_o \partial_x \partial_y \phi \qquad h_t = \partial_y \phi - h_x \partial_x \phi$$

One dimensional Hamiltonian system

$$H = \int dx \left\{ h\left[\frac{1}{2}(\tilde{\phi}_x)^2 - \frac{1}{2}(\tilde{\phi}_x^H)^2\right] - \frac{1}{2}\tilde{\phi}\tilde{\phi}_x^H \right\}$$

$$\tilde{f}^H = i(\tilde{f}_+ - \tilde{f}_-) \qquad \tilde{f}(x,t) = f(x, h(x,t), t)$$

$$\{\tilde{\phi}, h'\} = \delta(x - x'),$$
$$\{\tilde{\phi}, \tilde{\phi}'\} = -2\nu_o \delta'(x - x'),$$
$$\{h, h'\} = 0$$

$$\tilde{u}_t + \tilde{u}\tilde{u}_x - \tilde{u}^H \tilde{u}_x^H + 2\nu_o \tilde{u}_{xx}^H = -2\nu_o \cancel{\left[h\tilde{u} + (h\tilde{u}^H)^H\right]_{xxx}},$$

$$h_t + \tilde{u}^H = -\cancel{\left[h\tilde{u} + (h\tilde{u}^H)^H\right]_x}$$

$$\tilde{u} = \tilde{\phi}_x$$

Chiral Burgers equation

$$\tilde{u}_t + \tilde{u}\tilde{u}_x - \tilde{u}^H \tilde{u}^H_x + 2\nu_o \tilde{u}^H_{xx} = 0$$
$$h_t + \tilde{u}^H = 0$$

$$\tilde{u} = u + \bar{u}$$
$$\tilde{u}^H = -i(u - \bar{u})$$

A. Abanov, T. Can, SG arXiv:1801.10150

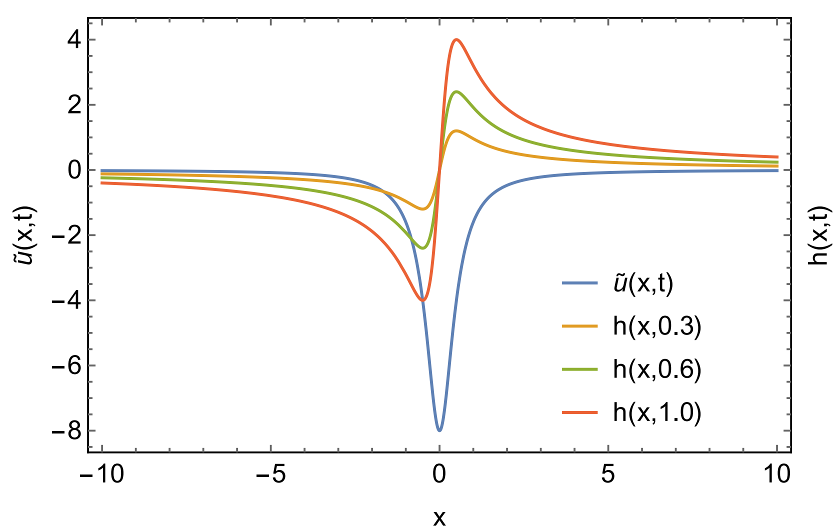
$$u_t + 2uu_x - 2i\nu_o u_{xx} = 0$$

u(x,t) is only analytic in the upper half plane and hence represent chiral non-linear dynamics

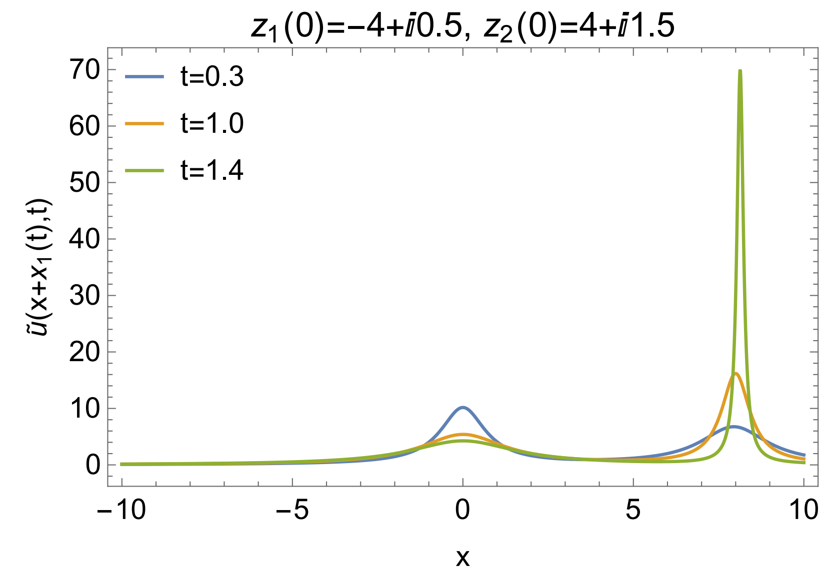
Without odd viscosity it has finite time singularities /wave breaking (Kusnetsov, Spector, Zakharov 1994)

Arbitrary soliton configurations have finite time singularities

One Soliton



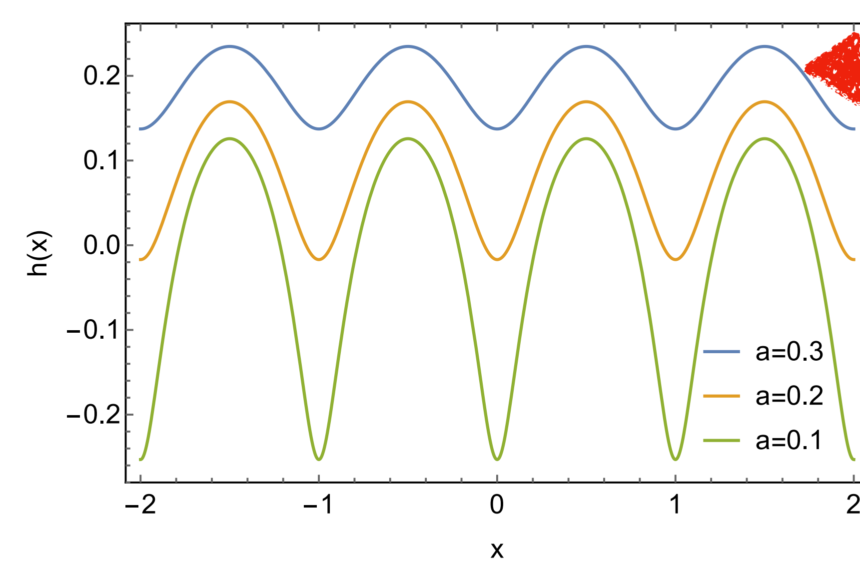
Two Soliton



long lived periodic soliton configuration specific to odd viscosity

$$\tilde{u} = -2\nu_o k\left(1 - \frac{\sinh(ka)}{\cosh(ka) - \cos(k(x - Ut))}\right)$$

$$h = \frac{1}{k}\ln\left[\cosh(ka) - \cos(k(x - Ut))\right]$$



Linear Lamb limit

Mapping to chiral-Schrödinger

A. Abanov, T. Can, SG arXiv:1801.10150

$$u_t + 2uu_x - 2i\nu_o u_{xx} = 0 \xrightarrow{u = 2i\nu_o \psi_x/\psi} i\psi_t = 2\nu_o \psi_{xx}$$

Quantum particle of 2m=1 on a line with $\nu_o = -\hbar/2$

analogies to quantum Hall

General family of equations (soliton solutions)

$$u_t + 2\lambda u u_x - 2(1-\lambda) u^H u_x^H + 2\nu_o u_{xx}^H = 0$$

$$u(x,t) = -2i\nu_o \sum_{j=1}^{N} \frac{1}{x - z_j} + c.c. \qquad \dot{z}_j = 4i\nu_o \left(\sum_{k=1, k\neq j}^{N} \frac{1}{z_k - z_j} + (1-2\lambda) \sum_{k=1}^{N} \frac{1}{\bar{z}_k - z_j} \right).$$

$\lambda = 1$: Benjamin-Davis-Ono, (scattering states of solitons)

$\lambda = 0$: Modified Benjamin-Davis-Ono, (bound states of solitons)

$\lambda = \frac{1}{2}$: Chiral Burgers

Towards quantum Hall fluids

Compressible hydrodynamics with odd viscosity in 2D

$$\rho = const. \quad \partial_i v_i = 0$$

$$\omega = \omega_c - \frac{eB}{c}\rho - \nu_o \Delta \ln \rho$$

Incompressible hydro with Hall viscosity

quantum Hall hydro with odd viscosity

quantum Hall hydro with odd viscosity

$$\partial_t \rho + \partial_i(\rho v_i) = 0 \qquad D_t v_i = \frac{1}{\rho}\partial_j T_{ij} - \frac{eB}{c} v_i^* \qquad p = f(\rho)$$

$$T_{ij} = -p\delta_{ij} + \nu_o \rho(\partial_i^* v_j + \partial_i v_j^*)$$

What makes it quantum Hall-esque

$$\partial_t(\underline{\omega - \omega_c + \nu_o \Delta \log \rho}) + \partial_i((\underline{\omega - \omega_c + \nu_o \Delta \log \rho})v_i) = 0$$

quantum Hall constraint is subset of initializations that respect

$$(\omega - \omega_c + \nu_o \Delta \log \rho) \propto \rho$$

Incompressibility violation at the boundary of quantum Hall fluids

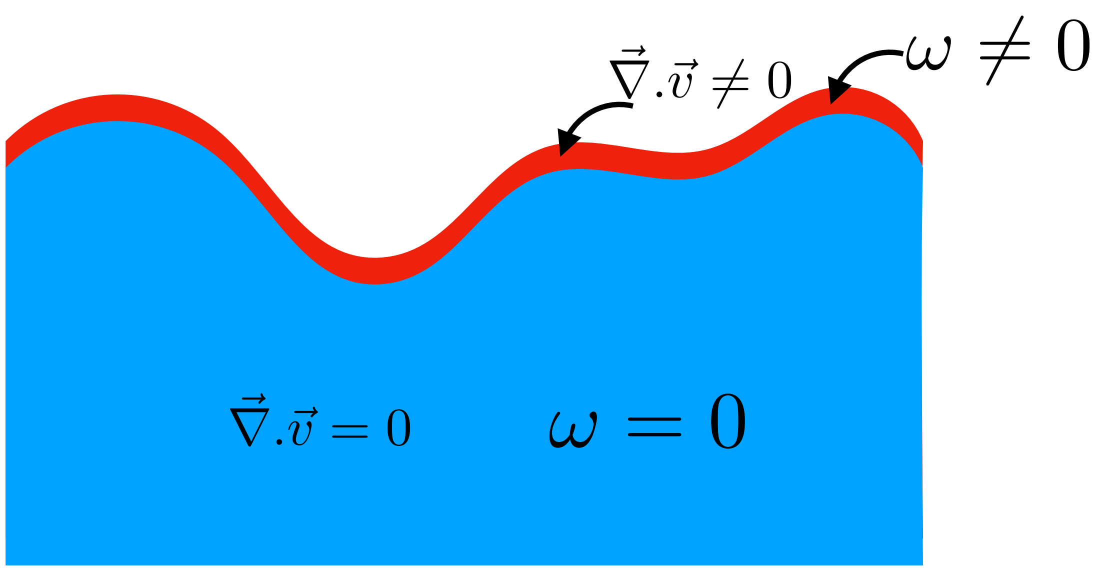
$$T_{xy}\Big|_{h(x,t)} = \nu_o(\partial_y v_y - \partial_x v_x) = 0 \qquad \partial_x v_x + \partial_y v_y \neq 0$$

If tangent boundary condition is satisfied, incompressibility must be violated !!

Incompressibility violation generates vorticity via odd viscosity

$$\partial_t \omega = -(\nu_o \Delta + B)(\partial_i v_i)$$

'Compressibility' boundary layer with sound modes + incompressible bulk



Compressibility layer scaling $\qquad \omega \sim v_s \qquad \delta \sim \frac{1}{v_s} \qquad v_s^2 = \frac{dp}{d\rho}$

Replacement rule in the large sound velocity limit $\qquad \sqrt{\nu_e} \to \frac{1}{v_s} \qquad t \to t - (E/B)x$

Linearized dynamics at the quantum Hall free surface

$$\partial_t \rho = -\partial_i v_i \qquad \partial_t v_i = -v_s^2 \partial_i \rho + (\nu_o \Delta + B) v_i^*$$

Bulk modes

$$v_x = \left(A|k|e^{|k|y} + Bme^{my} \right) e^{ikx - i\Omega t}$$

$$v_y = -ik \left(Ae^{|k|y} + Be^{my} \right) e^{ikx - i\Omega t}$$

$$\Omega = 0, \quad \pm\sqrt{v_s^2 q^2 + (B - \nu_o q^2)^2}$$

$$q^2 = k^2 - m^2$$

Surface dispersion (leading order)

$$\Omega = 0, \quad -B\mathrm{sgn}(k), \quad B\mathrm{sgn}(k) - 2\nu_o k|k|$$

Gapless boundary mode emerges with confining potential

$$\Omega \approx \frac{E_y}{B} k + \frac{E_y \nu_o}{B^2} k^3$$

$$u_t + \frac{E_y}{B} u_x + \frac{E_y \nu_o}{B^2} u_{xxx} = 0$$

KdV like term

Future goal: To derive the weakly non-linear surface waves (like before) in the presence of magnetic field, confining potential and odd viscosity

Summary

- Parity violating effects through odd viscosity in 2D results in new hydrodynamic phenomena.
- We proved three exact statements that quantify the observable consequences of odd viscosity with flow dependent boundary conditions.
- Odd viscosity manifests at the two fluid interfaces as a boundary phenomena (e.g. Bubble shape, Surface waves) for incompressible fluids. (Non-linear dynamics of surface waves (odd surface waves). (A. Abanov, T. Can, SG arXiv: 1801.10150))

Future directions

- Weakly non-linear surface waves in quantum Hall hydrodynamics.