Collegium Urbis Nov Eborac

ಸೈದ್ಧಾಂತಿಕ ವಿಜ್ಞಾನಗಳ ಅಂತರರಾಷ್ಟ್ರೀಯ ಕೇಂದ್ರ

Mapping the Calogero Model to Anyons

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So on to the talk

Generalized quantum statistics in one and two dimensions

Generalized quantum statistics manifest in both 2d and 1d

- In 2d they play a role in the quantum Hall effect: Fractional quantum Hall excitations obey anyon statistics
- In 1d Calogero particles model fractional statistics AP 1988

Indications of relation between these models

- Similarity of Calogero and anyon LLL wavefunctions
- Real one-dimensional coordinate maps to complex coordinate on the plane

Already some results:

- Alternative realizations of operators in two systems Brink *et al.* 1993
- Matrix model noncommutative realization of FQH states AP 2001

Still...

We would like an **explicit** mapping rather than a formal one

Why is such a mapping useful?

- Instrinsically interesting
- Calculation of quantities in either side may be easier
- E.g., density correlation calculations
- Exploit particle $\leftrightarrow$ hole duality in either system

In the sequel we will derive an N-body kernel that maps Calogero to anyon wavefunctions

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In case you wonder: How can a 2d system map to a 1d system?

LLL $\rightarrow$ phase space reduction effectively to 1d

The LLL anyon model

- N nonrelativistic anyons of unit mass and unit charge
- Anyonic statistical parameter α
- Constant magnetic field $B = 2\omega_c$
- Weak confining harmonic trap of frequency ω_o

In the singular gauge anyon statistics is encoded in the monodromy of the wavefunction and the Hamiltonian is

$$H_{\text{free}} = -2\sum_{i=1}^{N}\left[(\partial_i - \frac{\omega_c}{2}\bar{z}_i)(\bar{\partial}_i + \frac{\omega_c}{2}z_i) - \frac{\omega_c}{2}\right] + \sum_{i=1}^{N}\frac{\omega_t^2}{2}\bar{z}_i z_i$$

$$\omega_{\text{t}}^2 = \omega_{\text{c}}^2 + \omega_o^2$$

Redefine
$$\psi_{\text{free}} = \prod_{k<l}(z_k - z_l)^{\alpha}\, e^{-\omega_t \sum_{i=1}^{N} z_i\bar{z}_i/2}\,\psi$$

with bosonic ψ. The Hamiltonian acting on ψ is

$$H_\alpha = -2\sum_{i=1}^{N}\left[\partial_i\bar\partial_i - \frac{\omega_t+\omega_c}{2}\bar z_i\bar\partial_i - \frac{\omega_t-\omega_c}{2}z_i\partial_i\right] -2\alpha\sum_{i<j}\left[\frac{1}{z_i-z_j}(\bar\partial_i-\bar\partial_j) - \frac{\omega_t-\omega_c}{2}\right] + N\omega_t$$

- LLL states are analytic in z_i
- (unnormalized) 1-body eigenstates are $z_i^{\ell_i}$
- Many-body LLL eigenstates are

$$\psi_{\mathrm{free}} = \prod_{i<j}(z_i-z_j)^\alpha e^{-\omega\sum_{i=1}^{N} z_i\bar z_i/2}\sum_{\pi\in S_N}\prod_{i=1}^{N} z_{\pi(i)}^{\ell_i}$$

with the spectrum

$$E_N = (\omega_t-\omega_c)\left[\sum_{i=1}^{N}\ell_i + \frac{1}{2}N(N-1)\alpha + N\right] + N\omega_t$$

THe Calogero model

- Same spectrum as N-body 1d harmonic Calogero model with $\omega = \omega_t - \omega_o$ and $\ell = \alpha$ (up to a global $N\omega_t 2$ shift)

$$H_c = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j}\frac{\ell(\ell-1)}{(x_i - x_j)^2} + \frac{1}{2}\omega^2\sum_{i=1}^{N} x_i^2$$

- We will seek mapping between anyon and Calogero wavefunctions of same energy (rom now on set $\omega = 1$)

N-body ground state of harmonic Calogero Hamiltonian

$$\psi_0 = \prod_{i>j}(x_i - x_j)^g\, e^{-\sum_{i=1}^{N} x_i^2/2} := \Delta_x{}^g\, e^{-[x^2]/2}$$

where

$$\Delta_x = \prod_{i>j}(x_i - x_j)\ , \qquad [x^2] = \sum_{i=1}^{N} x_i^2$$

- ψ_0 is bosonic and above holds in the “wedge" $x_1 < \cdots < x_N$
- Use g instead of ℓ to avoid confusion with ℓ_i

Constructing Calogero states

Excited states obtained through the action of symmetric products of the ladder operators AP 1992, Brink *et al.* 1992

$$a_i^+ = \Pi_i - x_i \ , \qquad a_i = -\Pi_i - x_i$$

with

$$\Pi_i = \frac{\partial}{\partial x_i} + \sum_{j \neq i} M_{ij} \frac{g}{x_i - x_j} \ , \quad [\Pi_i, \Pi_j] = 0$$

Energy eigenstates ψ_ℓ labeled by the N integers $\ell_i, i = 1, \ldots, N$ (ℓ stands for the set $\ell_1, \ell_2, \ldots, \ell_N$), are

$$\psi_\ell = \sum_{\pi \in S_N} \prod_{i=1}^{N} \left(a^+_{\pi(i)}\right)^{\ell_i} \psi_0$$

- Looking for an N-body kernel $k_g[x, z]$ such that

$$\int k_g[x, z](\psi_\ell[x] e^{[x^2]/2}) \, [dx] = \Delta_z{}^g \sum_{\pi \in S_N} \prod_{i=1}^{N} z_{\pi(i)}^{\ell_i}$$

x and z stand for the collection of variables x_i and z_i

The kernel

Consider the scattering Calogero N-body Hamiltonian is

$$\bar{H}_g = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j}\frac{g(g-1)}{(x_i - x_j)^2}$$

Define $h_g[x,z]$ to be the scattering Calogero eigenstate with asymptotic momenta $z_1 < \cdots < z_N$

$$\bar{H}_g\, h_g[x,z] = \frac{1}{2}[z^2]\, h_g[x,z]$$

- $h_g[x,z]$ behaves as $(x_i - x_j)^g$ when $x_i \to x_j$
- Asymptotically combination of scattered plane waves

$$e^{i\sum_i x_i z_i} \to e^{-i\pi N(N-1)g/4}\sum_{\pi\in S_N} e^{i\pi g\, c(\pi)}\, e^{i\sum_i x_{\pi(i)} z_i}$$

- $c(\pi)$ is the number of particle crossings in $\{x_{\pi(i)}\}$
- Phase chosen symmetric for incident wave $e^{i(x_1 z_N + \cdots + x_N z_1)}$ and scattered wave $e^{i(x_1 z_1 + \cdots + x_N z_N)}$

The appropriate kernel is

$$k_g[x,z] = e^{[z^2]/4} e^{-[x^2]} \, h_g[x,z]$$

- Scattering momenta z_i become anyon coordinates

The proof is rather long (sorry; will cut it short as possible)

Basic steps:

- Conjugate operators $a_i^\dagger$ to the free form Π_i
- Use (anti)hermiticity of Π_i to make them act on $h_g[x,z]$
- Turn their action into eigenvalues of scattering conserved quantities to produce ${z_i}^{\ell_j}$ part
- Show that remaining integral reproduces Δ_z^g

The last step is the most challenging one

- Achieved only for integer g (For $N > 2$)
- Should be true in general by analytic continuation

Write creation ladder operators as

$$a_i^+ = e^{[x^2]/2}\,\Pi_i\, e^{-[x^2]/2}$$

Convolution integral over wedge $x_1 < \cdots < x_N$ becomes

$$\int k_g[x,z]\left(e^{[x^2]/2}\psi_\ell[x]\right)[dx]$$

$$= \int e^{[z^2]/4} e^{-[x^2]/2}\, h_g[x,z] \sum_{\pi\in S_N}\prod_{i=1}^{N}(a_{\pi(i)}^+)^{\ell_i}\Delta_x{}^g\, e^{-[x^2]/2}\,[dx]$$

$$= \int e^{[z^2]/4} h_g[x,z] \sum_{\pi\in S_N}\prod_{i=1}^{N}\Pi_{\pi(i)}{}^{\ell_i}\,\Delta_x{}^g\, e^{-[x^2]}\,[dx]$$

$$= (-1)^{\sum_i \ell_i} e^{[z^2]/4}\int \Delta_x{}^g\, e^{-[x^2]}\sum_{\pi\in S_N}\prod_{i=1}^{N}\Pi_{\pi(i)}{}^{\ell_i} h_g[x,z]\,[dx]$$

In the last step we used the antihermiticity of Π_i

Conserved quantities

- Symmetrized products in Π_i are the conserved integrals of the scattering Calogero model
- Acting on $h_g[x,z]$ gives their eigenvalues
- These are the corresponding symmetrized product of z_i

$$\sum_{\pi \in S_N} \prod_{i=1}^{N} \Pi_{\pi(i)}^{\ell_i}\, h_g[x,z] = \sum_{\pi \in S_N} \prod_{i=1}^{N} (i z_{\pi(i)})^{\ell_i}\, h_g[x,z]$$

(For proofs refer to my "pedagogical" lectures or the paper)

The above lead to

$$\int k_g[x,z]\psi_\ell[x]\,[dx] = (-i)^{\sum_i \ell_i}\, e^{[z^2]/4} \sum_{\pi \in S_N} \prod_{i=1}^{N} z_{\pi(i)}^{\ell_i} \int \Delta_x{}^g\, e^{-[x^2]} h_g[x,z]\,[dx]$$

It remains to show that the integral in the RHS reproduces Δ_z^g

Tools...

To this end, define the exchange-Calogero scattering Hamiltonian

$$\hat{H}_\Pi = -\frac{1}{2}\sum_{i=1}^{N}\Pi_i^2 = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j}\frac{g(g-M_{ij})}{(x_i-x_j)^2}$$

and its harmonic counterpart

$$\hat{H}_a = \sum_{i=1}^{N}\frac{1}{2}(a_i^+ a_i + a_i a_i^+) = \frac{1}{2}\sum_{i=1}^{N}\Big(-\frac{\partial^2}{\partial x_i^2} + x_i^2\Big) + \sum_{i<j}\frac{g(g-M_{ij})}{(x_i-x_j)^2}$$

Define further the intertwining operators Felder & Veselov 1994

$$\hat{\Delta}_\Pi = \prod_{i>j}(\Pi_i - \Pi_j)\ , \qquad \hat{\Delta}_a = \prod_{i>j}(a_i^+ - a_j^+)$$

related as

$$\hat{\Delta}_a = e^{[x^2]/2}\,\hat{\Delta}_\Pi\, e^{-[x^2]/2}$$

and the corresponding bosonic-projected operators $\tilde{\Delta}_g$ and $\bar{\Delta}_g$

$$\hat{\Delta}_a\,\psi_B = \tilde{\Delta}_g\,\psi_B\ , \qquad \hat{\Delta}_\Pi\,\psi_B = \bar{\Delta}_g\,\psi_B$$

Generating scattering states

- The above acting on bosonic states produce fermionic ones
- Exchange-Calogero Hamiltonians acting on fermionic states ψ_F become

$$\begin{aligned} \hat{H}_a \, \psi_F &= \tilde{H}_{-g} \, \psi_F &= \tilde{H}_{g+1} \, \psi_F \\ \hat{H}_\Pi \, \psi_F &= \bar{H}_{-g} \, \psi_F &= \bar{H}_{g+1} \, \psi_F \end{aligned}$$

From the commutativity of Π_i

$$\hat{H}_\Pi \, \hat{\Delta}_\Pi = \hat{\Delta}_\Pi \, \hat{H}_\Pi$$

Applying this relation on a bosonic state and using the projection relations we deduce

$$\bar{H}_{g+1} \, \bar{\Delta}_g \, \psi_B = \bar{\Delta}_g \, \bar{H}_g \, \psi_B$$

and from locality

$$\bar{H}_{g+1} \, \bar{\Delta}_g = \bar{\Delta}_g \, \bar{H}_g$$

Iterating the above for integer g

$$\bar{H}_g \, \bar{\Delta}_{g-1} \ldots \bar{\Delta}_0 = \bar{\Delta}_{g-1} \ldots \Delta_0 \, \bar{H}_0$$

Acting on the symmetrized free plane wave

$$S\, e^{i\sum_i x_i z_i} = \sum_{\pi\in S_N} e^{i\sum_i x_{\pi(i)} z_i}$$

the above gives

$$\bar{H}_g\, \bar{\Delta}_{g-1}\dots\bar{\Delta}_0\, S\, e^{i\sum_i x_i z_i} = \frac{1}{2}[z^2]\, \bar{\Delta}_{g-1}\dots\bar{\Delta}_0\, S\, e^{i\sum_i x_i z_i}$$

This implies (upon determining the overall coefficient)

$$h_g[x,z] = (-1)^{gN(N-1)/2}\, \Delta_z^{-g}\ \bar{\Delta}_{g-1}\dots\bar{\Delta}_0\, S\, e^{i\sum_i x_i z_i}$$

- Operator solution of the scattering Calogero model

Remarkable fact: $h_g[x,z]$ is symmetric ("dual")

$$h_g[x,z] = h_g[z,x]$$

- Not obvious (refer to paper for proof)

This implies

$$h_g[x,z] = (-1)^{gN(N-1)/2}\, \Delta_x^{-g}\ \bar{\Delta}_{g-1}[z]\dots\bar{\Delta}_0[z]\, S\, e^{i\sum_i x_i z_i}$$

The integral of interest becomes, for integer g

$$\int \Delta_x{}^g\, e^{-[x^2]}\, h_g[x,z]\,[dx]$$

$$= \int \Delta_x{}^g\, e^{-[x]^2} (-1)^{gN(N-1)/2}\, \Delta_x^{-g}\, \bar{\Delta}_{g-1}[z] \ldots \bar{\Delta}_0[z]\, S\, e^{i\sum_i x_i z_i} [dx]$$

$$= N!\, (-1)^{gN(N-1)/2} \bar{\Delta}_{g-1}[z] \ldots \bar{\Delta}_0[z] \int e^{-[x]^2 + i\sum_i x_i z_i}\, [dx]$$

$$= (-1)^{gN(N-1)/2}\, \pi^{N/2}\, \bar{\Delta}_{g-1}[z] \ldots \bar{\Delta}_0[z]\, e^{-[z^2]/4}$$

It remains to determine the action of operators $\bar{\Delta}_g[z]$ on $e^{-[z^2]/4}$
WIth similar tricks (see paper) we obtain

$$\bar{\Delta}_{g-1}[z] \ldots \bar{\Delta}_0[z]\, e^{-[z^2]/4} = (-2)^{-gN(N-1)/2} \Delta_z{}^g\, e^{-[z^2]/4}$$

and inserting this in the integral of interest

$$\int \Delta_x{}^g\, e^{-[x^2]}\, h_g[x,z]\,[dx] = 2^{-gN(N-1)/2}\, \pi^{N/2}\, \Delta_z{}^g\, e^{-[z^2]/4}$$

Can be analytically continued to arbitrary values of g

At last…

Putting everything together we obtain our final result

$$\int e^{[z^2]/4} e^{-[x^2]/2} h_g[x,z]\, \psi_\ell[x]\, [dx] = \Delta_z{}^g \sum_{\pi \in S_N} \prod_{i=1}^{N} z_{\pi(i)}^{\ell_i}$$

Calogero $\quad\rightarrow\quad$ Anyons

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Calogero $\rightarrow$ Anyons

Our desired mapping relation!

Conclusions and outlook

- Explicit mapping Calogero $\rightarrow$ Anyons etablished
- State-dependent part fully general
- Vandermonde part only for integer g
- For $N = 2$ established for arbitrary g. $N > 2$ Should be OK too.

Open questions

- Sutherlad to what? (Anyons on cylinder; maybe on sphere??)
- Ellipric to what? (Anyons on torus??)
- Interesting applications?

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Thank You!