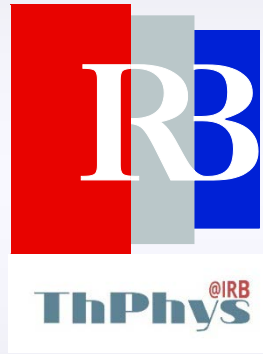


The Frustration of being Odd: Universal Area Law violation in local systems

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Our Claim

- Mix:
 - AFM weakly (non-extensively) frustrated chain
 - Quantum interactions with discrete symmetry
- To obtain:

A new quantum phase of matter
- Which is
 - gapless but non-relativistic
 - with peculiar very long-range correlations
 - area law violating, yet not diverging
 - extended, robust & universal
- All in a very simple, natural setting (Ising chain!)

Outline

1. Basics on Frustration
2. Our results for this weakly frustrated phase:
 - *Local and Quasi-Local* correlation functions
 - The entanglement entropy:
 - ✓ **violates** Area law, but **does not diverge**
 - ✓ **collapses** on a Universal curve
3. The **analytical** case of the Ising Chain
4. Conclusions

Basics on Frustration

- Frustration:

competing interactions favoring **different orders**

⇒ impossible to minimize all energy contributions

- Remark: all **genuine** quantum phases are frustrated (non-commuting terms promote diff. arrangements)

- E.g. Ising Chain:
$$H_{\text{Ising}} = \sum_{l=1}^N (\sigma_l^x \sigma_{l+1}^x - h \sigma_l^z)$$

$[\sigma_l^x \sigma_{l+1}^x, \sigma_l^z] \neq 0$: ground state as a trade-off

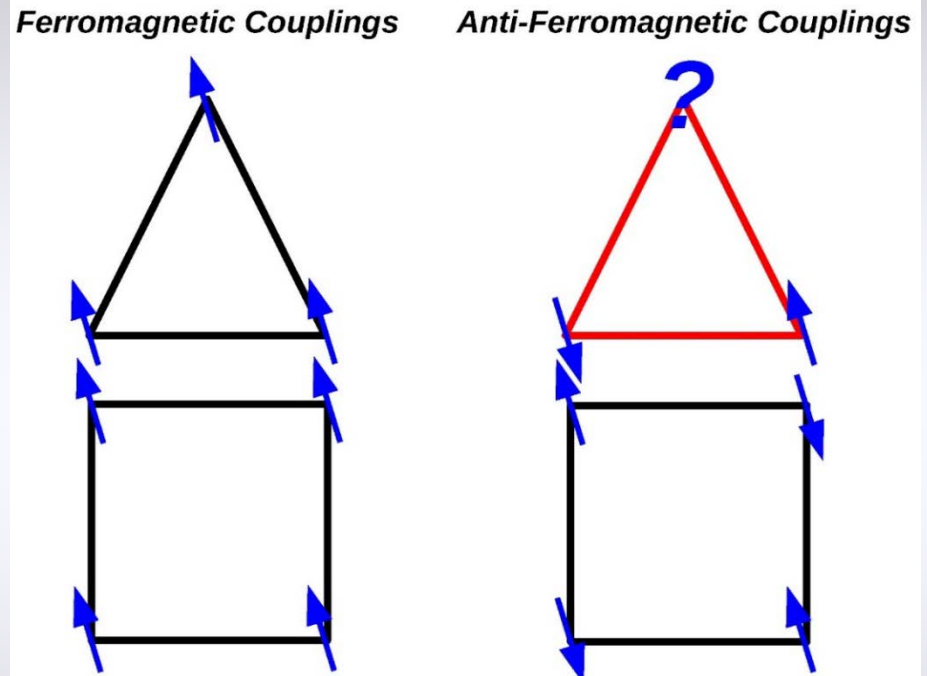
Geometrical Frustration

- Originally, frustration in classical systems:

- Arise from geometry
- Toulouse Criterion:
a classical systems is frustrated if there is a close loop for which
$$-1^{\mathcal{N}_{AFM}} = -1$$

- More loops $\Rightarrow$ more frustration

- Remark: adding one site changes GS degeneracy from 2 to 2N and vice versa (challenges perturbative picture)



Frustrated Systems

- Certain degree of frustration is very common
- In any dimension, due to closed AFM loops
- Typically: **extensive frustration**
(# loops scale with system size)
 - **Ordered** (ANNNI model, spin-ice...)
 - **Disordered** (Sherrington-Kirkpatrick model, spin glasses...)
- Peculiar physics: residual entropy, local zero-modes, algebraic decay, artificial EM, monopoles, Dirac strings...

Frustration & Quantum Interactions

- In quantum systems, subtle **interplay** between **geometrical** frustration and **quantum** interactions
- Hard Problem
- We consider the simplest setting:
 - 1 loop (1D chain): **non-extensive** frustration
 - Most **generic** n.n. AFM quantum chain ($N = 2M + 1$)

$$H = \frac{1}{2} \sum_{l=1}^N \left[\left(\frac{1+\gamma}{2} \right) \sigma_l^x \sigma_{l+1}^x + \left(\frac{1-\gamma}{2} \right) \sigma_l^y \sigma_{l+1}^y \right] + \frac{\Delta}{2} \sum_{l=1}^N \sigma_l^z \sigma_{l+1}^z - \sum_{l=1}^N h \sigma_l^z$$

with periodic boundary conditions: $\sigma_{l+N}^\alpha = \sigma_l^\alpha$

1D Spin Chain

$$H = \frac{J}{2} \sum_{l=1}^N \left[\left(\frac{1+\gamma}{2} \right) \sigma_l^x \sigma_{l+1}^x + \left(\frac{1-\gamma}{2} \right) \sigma_l^y \sigma_{l+1}^y \right] + \frac{\Delta}{2} \sum_{l=1}^N \sigma_l^z \sigma_{l+1}^z - \sum_{l=1}^N h \sigma_l^z$$

- $J = -1$: Easy-Plane Ferromagnet
- $J = 1$: Easy-Plane Anti-Ferromagnet (AFM)
- $\Delta = 0$: XY Chain (Free Fermions)
- $h = 0$: Integrable XYZ chain
- $\gamma = 0$: U(1) global symmetry (Integrable XXZ chain)
- $\gamma \neq 0$: $\mathbb{Z}_2$ global residual symmetry

$$\text{Parity Operator: } P \equiv \prod_{l=1}^N \sigma_l^z, [H, P] = 0$$

Weakly frustrated Quantum Chain

$$H = \frac{J}{2} \sum_{l=1}^N \left[\left(\frac{1+\gamma}{2} \right) \sigma_l^x \sigma_{l+1}^x + \left(\frac{1-\gamma}{2} \right) \sigma_l^y \sigma_{l+1}^y \right] + \frac{\Delta}{2} \sum_{l=1}^N \sigma_l^z \sigma_{l+1}^z - \sum_{l=1}^N h \sigma_l^z$$

with PBC: $\sigma_{l+N}^\alpha = \sigma_l^\alpha$

- $J = -1$: Ferromagnet $\rightarrow$ no geometrical frustration
- $J = 1$: AFM $\rightarrow \begin{cases} N = 2M & : \text{no geometrical frustration} \\ N = 2M + 1 : \text{weak frustration} \end{cases}$
- Not much in literature on frustrated case
 - $\gamma = 0 \rightarrow U(1)$: trivial effect of frustration ($S^Z = \pm \frac{1}{2}$)
 - $\gamma \neq 0 \rightarrow Z_2$: only now, very non-trivial

Weakly frustrated Quantum Chain

$$H = \frac{1}{2} \sum_{l=1}^N \left[\left(\frac{1+\gamma}{2} \right) \sigma_l^x \sigma_{l+1}^x + \left(\frac{1-\gamma}{2} \right) \sigma_l^y \sigma_{l+1}^y \right] + \frac{\Delta}{2} \sum_{l=1}^N \sigma_l^z \sigma_{l+1}^z - \sum_{l=1}^N h \sigma_l^z$$

with PBC: $\sigma_{l+N}^\alpha = \sigma_l^\alpha$. For $|h| < 1$:

- $N = 2M$: No frustration $\Rightarrow$ SSB of Z_2 symmetry
 - Gapped
 - Doubly degenerate GS $\rightarrow$ Spontaneous magnetization
 - Exponential decay of correlations
- $N = 2M + 1$: Weak frustration + Z_2 quantum symmetry $\Rightarrow$
 - Gapless, but not relativistic (Galilean)
 - Non-degenerate GS $\rightarrow$ No order parameter
 - Mixture of exponential and algebraic correlations

Local and Quasi-Local Correlators

- Ising Chain: $H_{\text{Ising}} = \frac{1}{2} \sum_{l=1}^{2M+1} (\sigma_l^x \sigma_{l+1}^x - h \sigma_l^z)$
- Mappable to free fermions (more later):
unique GS, gapless (with quadratic dispersion) for $|h| < 1$
- “Local” correlators:
$$C^{zz}(R) \equiv \langle \sigma_l^z \sigma_{l+R}^z \rangle = m_z^2 - \frac{c_1^z(h)}{R^2} \left(\frac{h^2}{J^2} \right)^R + \frac{4m_z}{N} \left[1 + c_2^z(h)(-1)^R \left| \frac{h}{J} \right|^R \right]$$
- “Quasi-Local” correlators (Dong et al. JSTAT '16)
$$C^{xx}(R) \equiv \langle \sigma_l^x \sigma_{l+R}^x \rangle = (-1)^R \left(1 - \frac{h^2}{J^2} \right)^{1/4} \left[1 + \frac{c^x(h)}{R^2} \left(\frac{h^2}{J^2} \right)^R \right] \left(1 - \frac{2R}{N} \right)$$
- Depending on the number of fermions appearing after Jordan-Wigner transformation (more later)

Improved Thermodynamic Limit

- “Local” correlators:

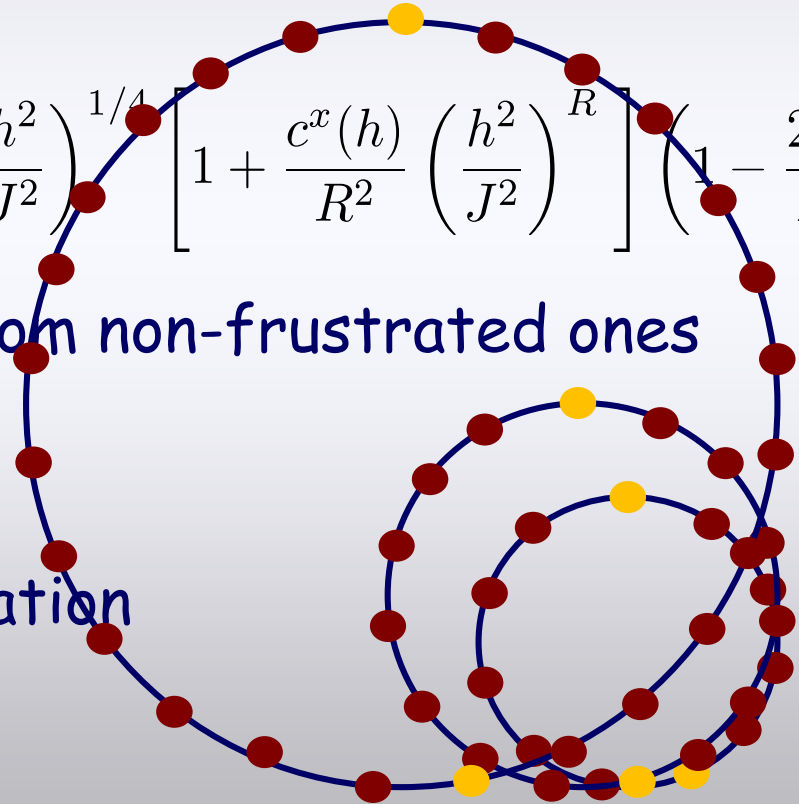
$$C^{zz}(R) \equiv \langle \sigma_l^z \sigma_{l+R}^z \rangle = m_z^2 - \frac{c_1^z(h)}{R^2} \left(\frac{h^2}{J^2} \right)^R + \frac{4m_z}{N} \left[1 + c_2^z(h)(-1)^R \left| \frac{h}{J} \right|^R \right]$$

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- Locally: indistinguishable from non-frustrated ones

- Order parameter/
Spontaneous Magnetization



Improved Thermodynamic Limit

- “Local” correlators:

$$C^{zz}(R) \equiv \langle \sigma_l^z \sigma_{l+R}^z \rangle = m_z^2 - \frac{c_1^z(h)}{R^2} \left(\frac{h^2}{J^2} \right)^R + \frac{4m_z}{N} \left[1 + c_2^z(h)(-1)^R \left| \frac{h}{J} \right|^R \right]$$

- “Quasi-Local” correlators

$$C^{xx}(R) \equiv \langle \sigma_l^x \sigma_{l+R}^x \rangle = (-1)^R \left(1 - \frac{h^2}{J^2} \right)^{1/4} \left[1 + \frac{c^x(h)}{R^2} \left(\frac{h^2}{J^2} \right)^R \right] \left(1 - \frac{2R}{N} \right)$$

- Locally: indistinguishable from non-frustrated ones

- Order parameter:

$$\langle \sigma^x \rangle = \lim_{N \rightarrow \infty} \sqrt{C^{xx} \left(\frac{N-1}{2} \right)} = 0$$

- Improved Thermodynamic Limit: $N \rightarrow \infty$ as

$$r \equiv \frac{R}{N} = \text{const}$$

Entanglement Entropy

- Measures genuine quantum correlation
- Consider a pure (ground-state) and bi-partition ($A|B$)
- If system wave-function:

$$|\Psi^{A,B}\rangle = |\Psi^A\rangle \otimes |\Psi^B\rangle \quad \rightarrow \text{No Entanglement}$$

$$|\Psi^{A,B}\rangle = \sum_{j=1}^{\mathcal{D}} \sqrt{\lambda_j} |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle \quad \rightarrow \text{Entangled}$$

(with $\mathcal{D} > 1$, $|\Psi_j^A\rangle$ & $|\Psi_j^B\rangle$ linearly independent):

- Entangled: Measurements on B affect A

Von Neumann Entropy

- To quantify entanglement with a number:

$$|\Psi^{A,B}\rangle = \sum_{j=1}^{\mathcal{D}} \sqrt{\lambda_j} |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle$$

$$\begin{aligned} \rho_A &= \text{tr}_B |\Psi^{A,B}\rangle \langle \Psi^{A,B}| \quad \text{➤ Reduced Density Matrix} \\ &= \sum \lambda_j |\Psi_j^A\rangle \langle \Psi_j^A| \end{aligned}$$

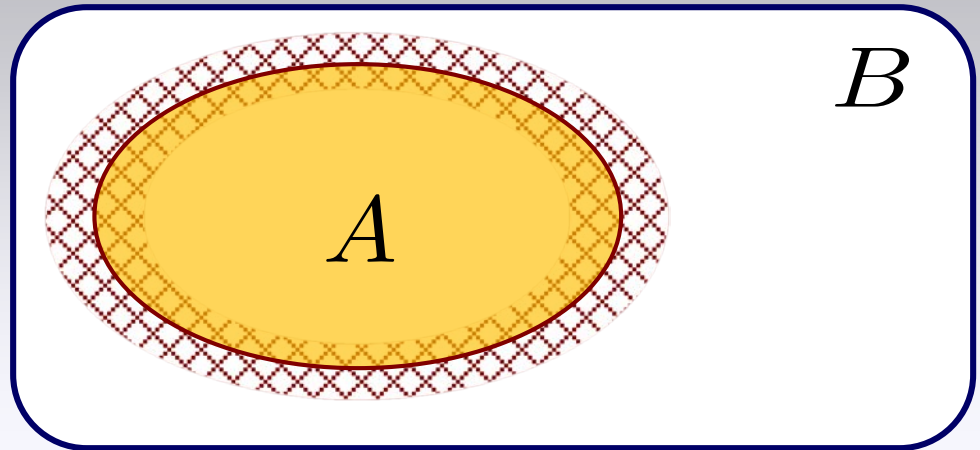
- Von Neumann (Quantum analog of Shannon Entropy):

$$S = -\text{tr}_A (\rho_A \log \rho_A) = -\sum \lambda_j \log \lambda_j$$

- Non-local, but accounts for all different correlations!
- Universality

Area Law

- For GS, correlations between two systems localized around the boundary dividing them

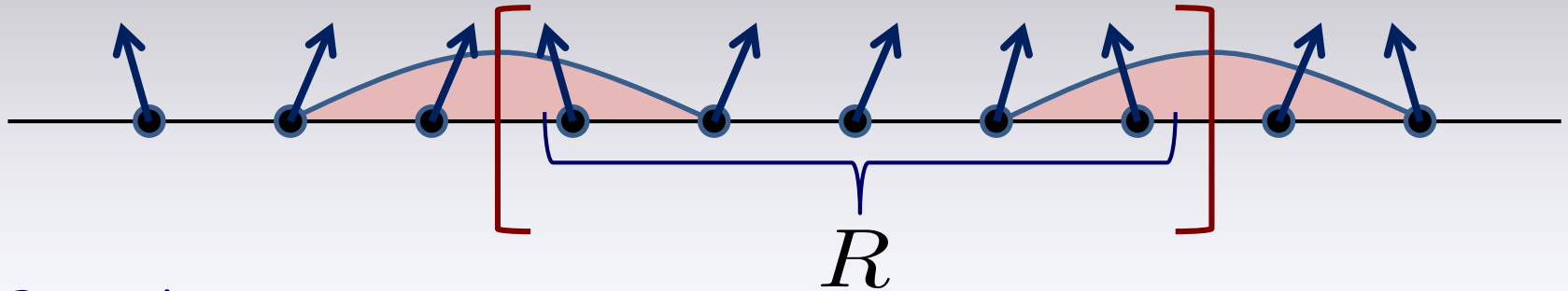


- Gapped systems:
correlations decay within few correlation length
 $\Rightarrow$ EE proportional to **Area**
- Critical systems:
correlation decay algebraically
 $\Rightarrow$ **corrections** to area law
(logarithmic with **Fermi surfaces**)

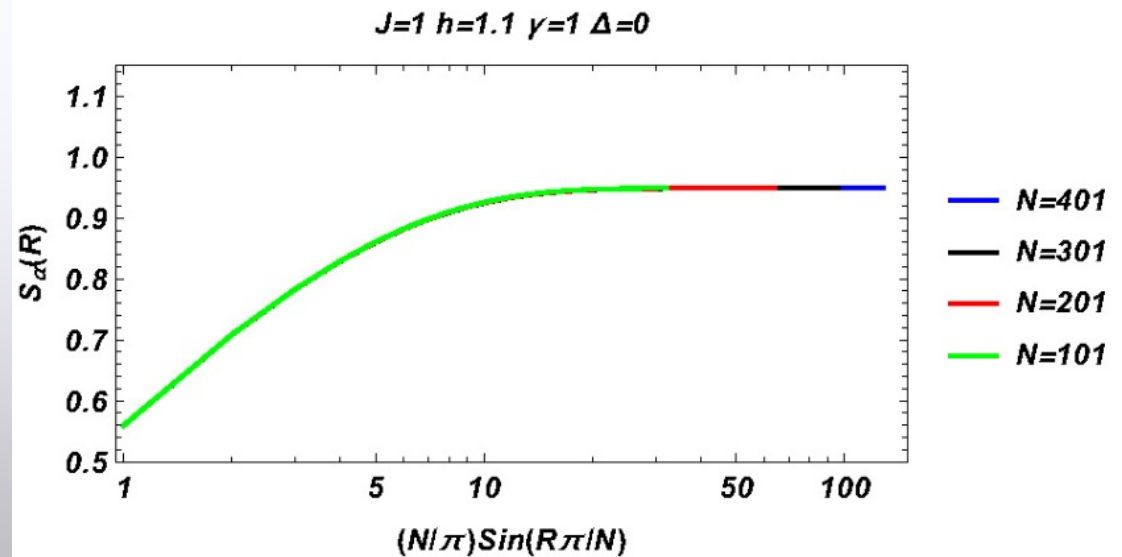
$$\begin{aligned} \triangleright \rho_A &= \text{Tr}_B |GS\rangle \langle GS| \\ &\text{Reduced Density Matrix} \\ \triangleright S_A &= -\text{Tr}_A [\rho_A \ln \rho_A] \\ &\text{Von Neumann Entropy} \end{aligned}$$

Area law in 1D

- Divide system into two connected sets of R and $N-R$ sites

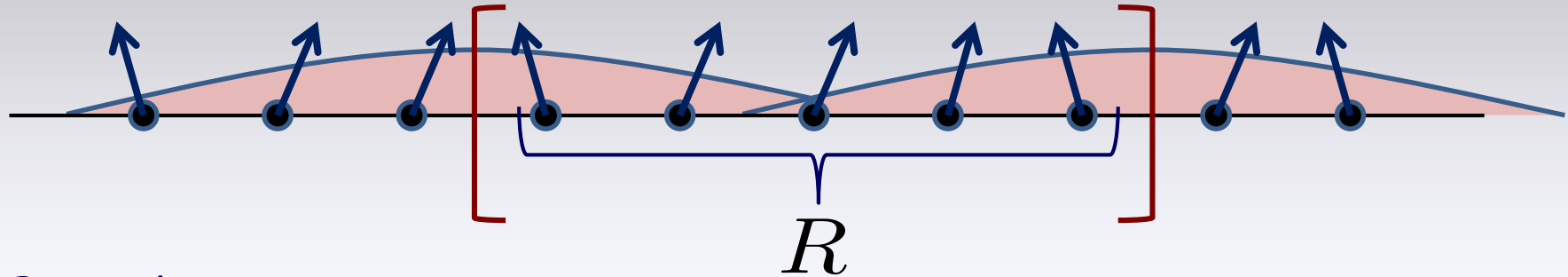


- Boundary is 2 points
- Gapped: area law
→ Saturation



Area law in 1D

- Divide system into two connected sets of R and $N-R$ sites

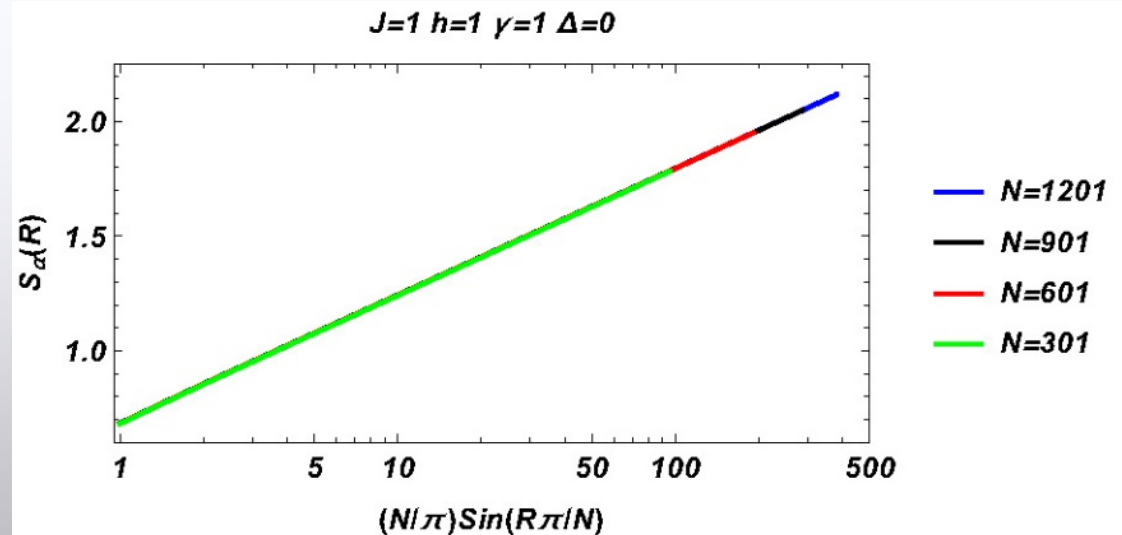


- Boundary is 2 points
- Critical (CFT)

→ Log-divergence

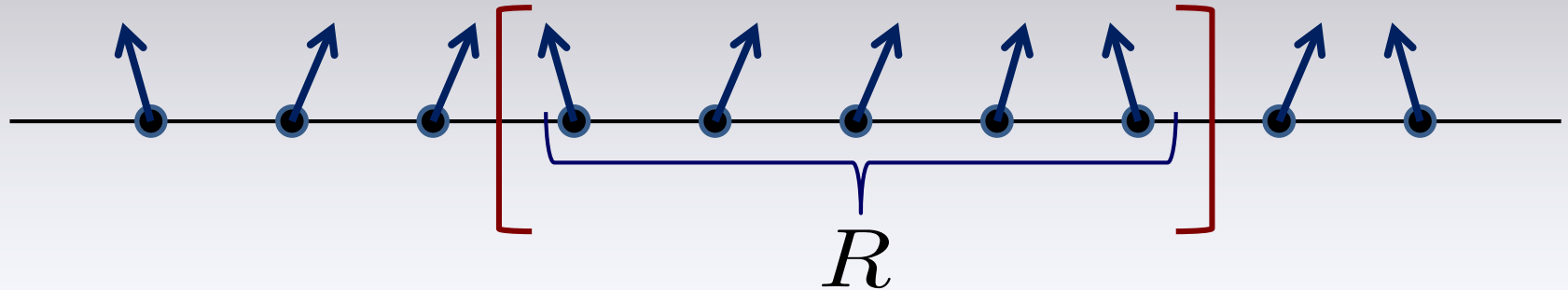
$$S(R) = \frac{c}{6} \ln \frac{N}{\pi} \sin \frac{\pi R}{N}$$

c : central charge



Area law Violation in 1D

- Divide system into two connected sets of R and $N-R$ sites



- Area law violation for

➤ Breaking of translational invariance

(Vitagliano et al. NJP '10)

(Ramirez et al. JSTAT '14)

➤ Long-range/non-local interactions
(complex Fermi surface)

(Vodola et al. PRL '14)

(Gori et al. PRB '15)

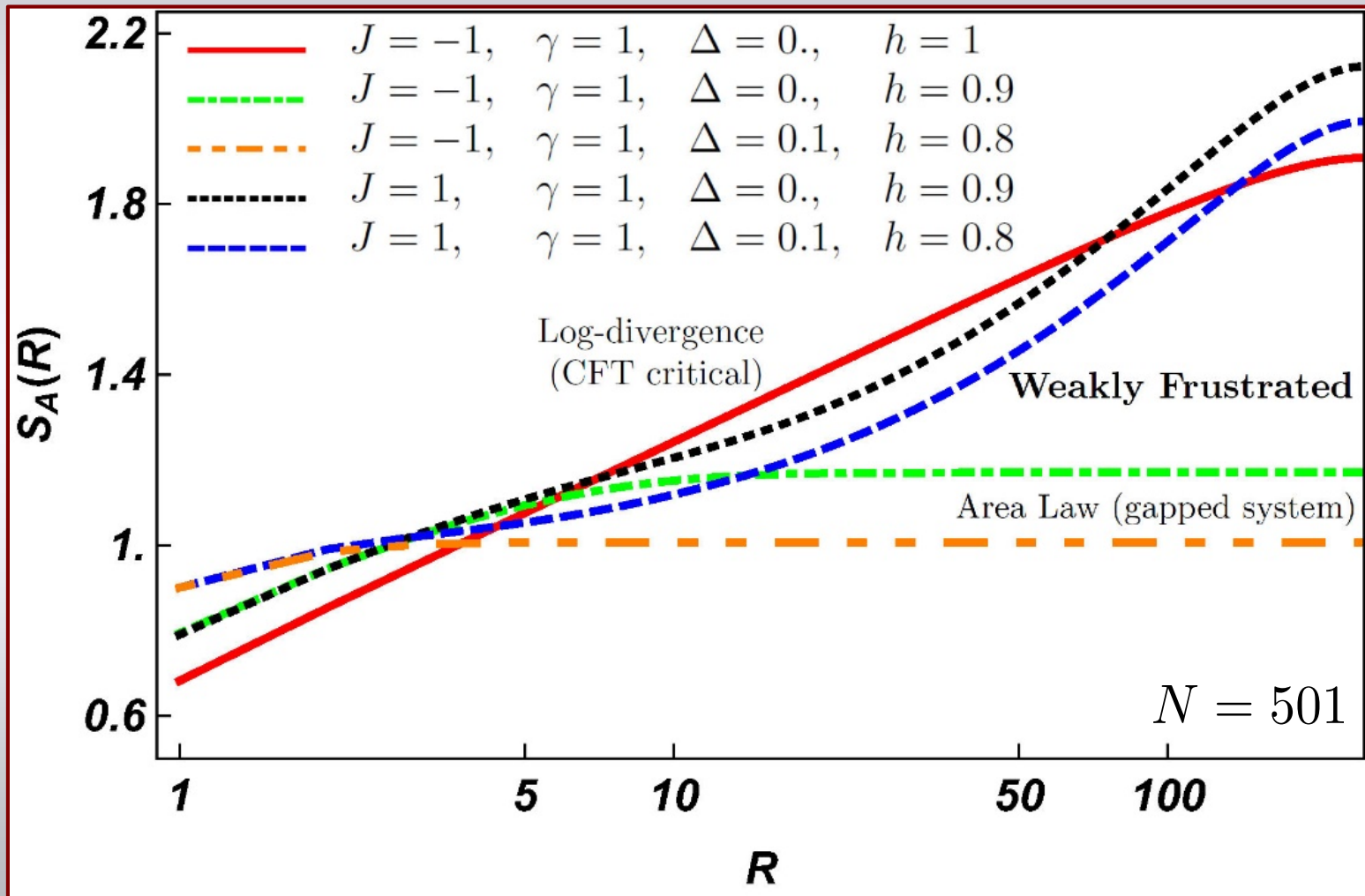
➤ Frustration Free systems with **massive degeneracy**
(Motzkin & Fredkin chains)

(Bravyi et al. PRL '12)

(Dell'Anna et al. PRB '16)

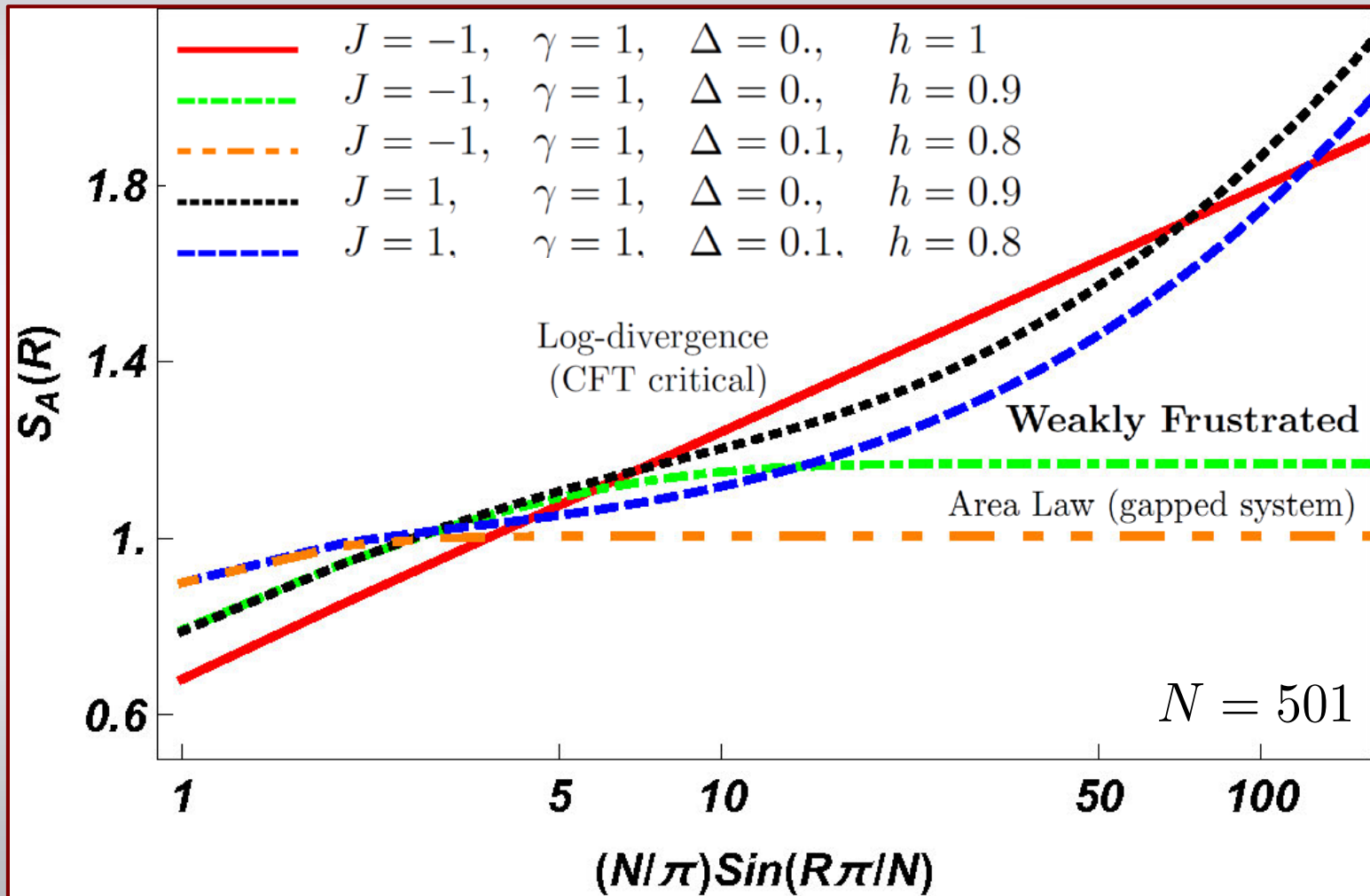
EE in weakly frustrated phase

- Novel behavior: area law violation!



EE in weakly frustrated phase

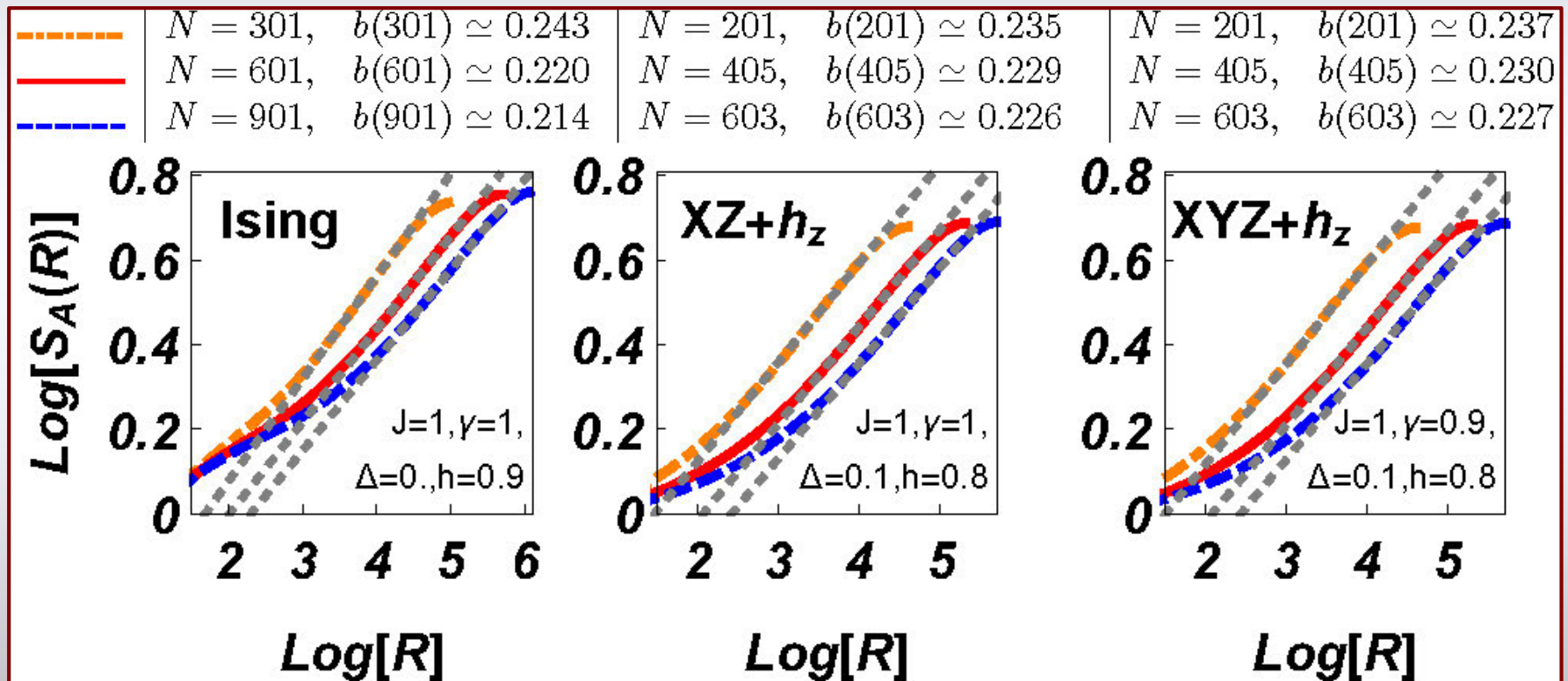
- Novel behavior: area law violation!



Algebraic Area Law Violation

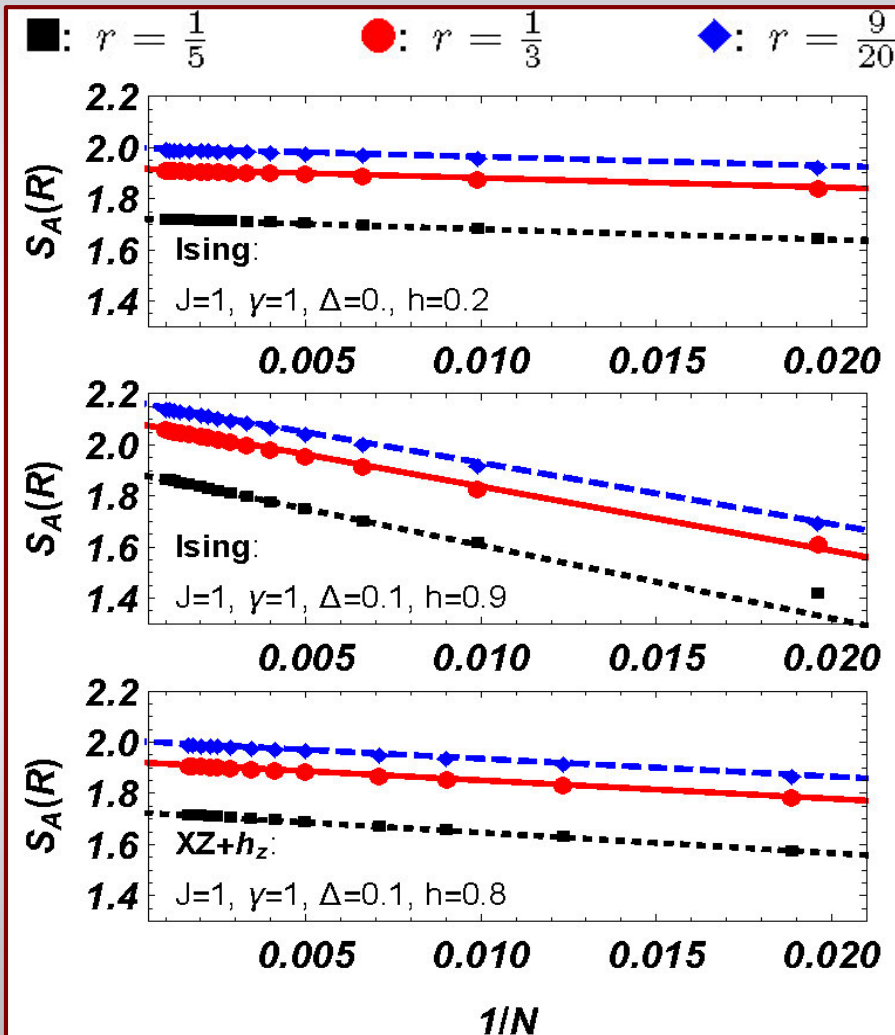
- Log-Log plots show power-law behavior in the bulk:

(grey dashed lines: best fit with $S_A(R) \simeq a(N)R^{b(N)}$)



And yet it saturates

- Despite area law violation, the EE does not diverge



(lines are best fit with

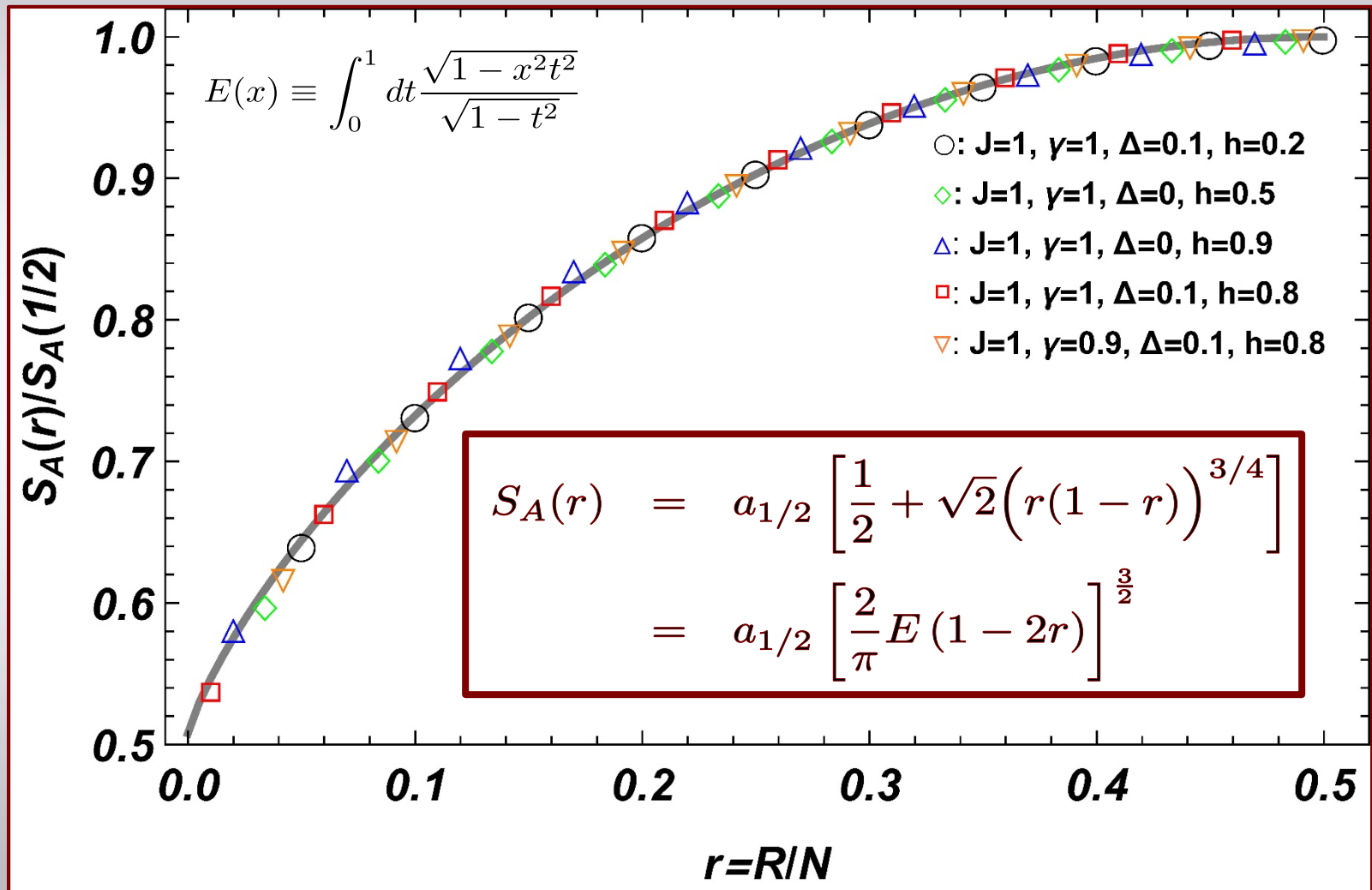
$$a_r + \frac{b_r}{N} :$$

for $N \rightarrow \infty$ EE saturates
at finite a_r)

- Improved Therm. Limit provides limiting EE value to compare behavior in parameter space

Universal Behavior

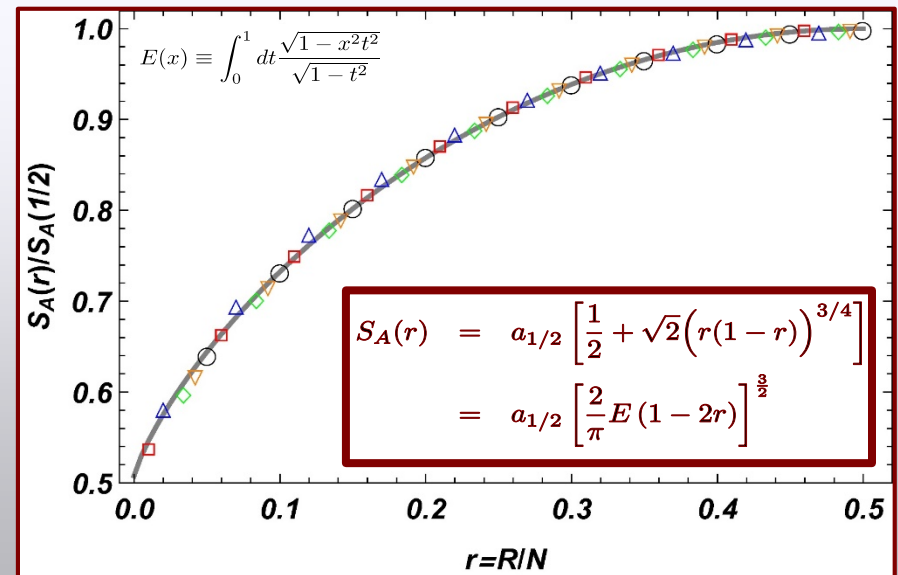
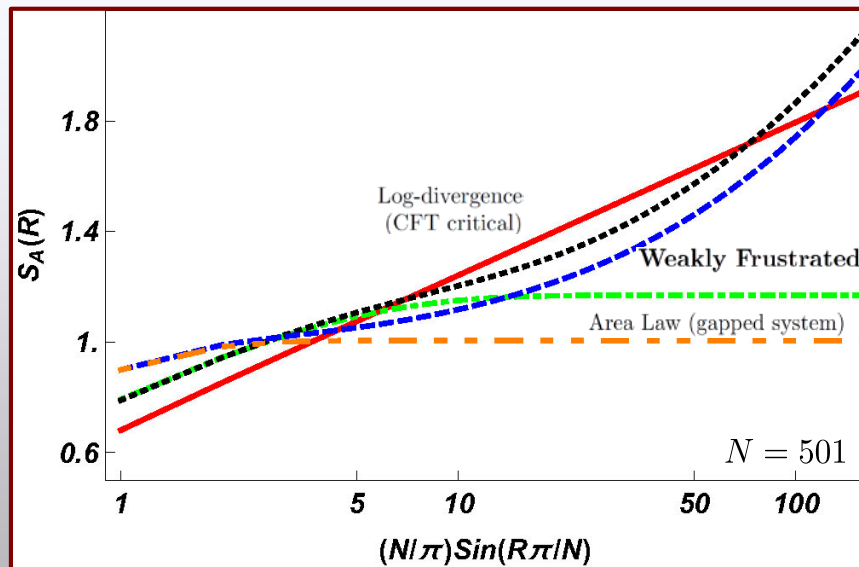
- Every saturation point collapses on the same curve!



Interpretation of the results

- Frustrated system has a "finite amount" of entanglement (like in a gapped case, it does not diverge)
- Correlations span the whole systems (like a critical phase, but not power-law correlation): technological applications?

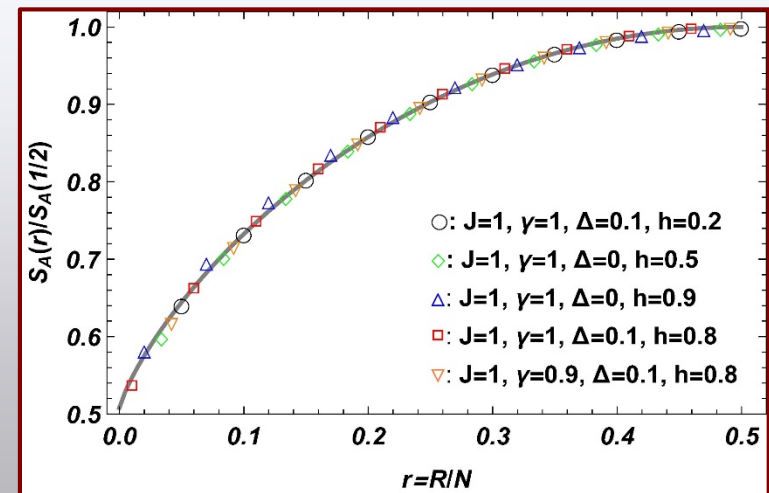
$$C^{xx}(R) \equiv \langle \sigma_l^x \sigma_{l+R}^x \rangle = (-1)^R \left(1 - \frac{h^2}{J^2}\right)^{1/4} \left[1 + \frac{c^x(h)}{R^2} \left(\frac{h^2}{J^2}\right)^R\right] \left(1 - \frac{2R}{N}\right)$$



Conclusions

- Weak frustration + discrete quantum symmetry
⇒ New critical quantum phase
- Both **exponential** and **algebraic** (not power-law!) correlations
- Entanglement Entropy to study phase properties
- Algebraic area law violation, but no divergence
- **Robust** against perturbation: very natural systems!
- **Universal collapse** with improved thermodynamic limit/scaling

arXiv:1807.07055



Frustrated Ising Chain

$$H_{\text{Ising}} = \frac{1}{2} \sum_{l=1}^{2M+1} (\sigma_l^x \sigma_{l+1}^x - h \sigma_l^z)$$

- Jordan-Wigner transformation turns spins into spinless fermions:

$$\sigma_l^+ = e^{i\pi \sum_{j<l} \psi_j^\dagger \psi_j} \psi_l, \quad \sigma_l^z = 1 - 2\psi_l^\dagger \psi_l$$

- Separate Hilbert space according to parity:

$$H = \frac{1+P}{2} H^+ + \frac{1-P}{2} H^- \quad P \equiv \prod_{l=1}^N \sigma_l^z, [H, P] = 0: \text{Parity}$$

- Rotation in Fourier space (Bogoliubov rotation) to get:

$$H^\pm = \sum_{q \in \Gamma_\pm} \varepsilon\left(\frac{2\pi}{N} q\right) \left\{ \chi_q^\dagger \chi_q - \frac{1}{2} \right\}, \quad \Gamma_P = \left\{ n + \frac{1+P}{4} \right\}_{n=0}^{N-1}$$

$$\varepsilon(\alpha) \equiv \sqrt{(h + \cos \alpha)^2 + \sin^2 \alpha}, \quad \varepsilon(0) = h + 1, \quad \varepsilon(\pi) = h - 1$$

Even Parity

- Absolute GS in even parity sector (P=1): $\chi_q|GS\rangle = 0, \forall q \in \mathbb{N} + \frac{1}{2}$
- GS never degenerate!
- For $h < 1$, occupation of π -mode lowers the energy

$$|GS\rangle \rightarrow E_0 = -\frac{1}{2} \sum_{q=0}^{2M} \varepsilon \left[\frac{2\pi}{N} \left(q + \frac{1}{2} \right) \right] + 1 - h$$

excited states with P=1 lie arbitrarily close in energy to GS, forming a band with quadratic dispersion:

$$\chi_{M+1/2}^\dagger \chi_{p+1/2}^\dagger |GS\rangle \rightarrow E_p = -\frac{1}{2} \sum_{q=0}^{2M} \varepsilon \left[\frac{2\pi}{N} \left(q + \frac{1}{2} \right) \right] + \varepsilon \left[\frac{2\pi}{N} \left(p + \frac{1}{2} \right) \right]$$

$$E(k) \simeq E_0 + \frac{1}{2} \left(\frac{h}{1-h} \right) (k - \pi)^2 + \dots$$

Odd Parity

- Vacuum does not belong to odd parity sector ($P=-1$):

$$\chi_q|0'\rangle = 0, \forall q \in \mathbb{N}$$

- Low energy states have one excitation: $\chi_p^\dagger|0'\rangle$
- Lowest energy state(s) for $p=M/M+1$:

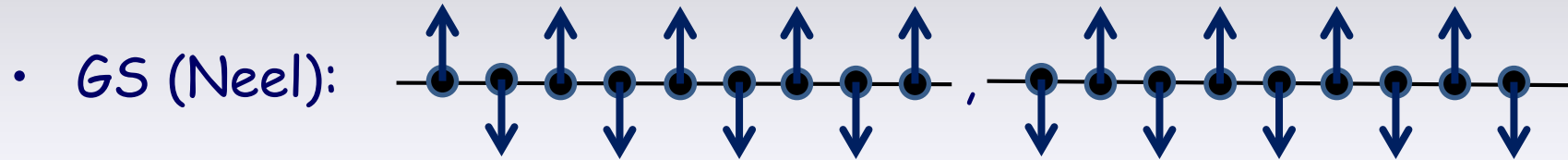
$$\chi_{M,M+1}^\dagger|0'\rangle = |GS'\rangle \rightarrow E'_0 = -\frac{1}{2} \sum_{q=0}^{2M} \varepsilon \left[\frac{2\pi}{N} \left(q + \frac{1}{2} \right) \right] + \varepsilon \left(\pi \pm \frac{\pi}{N} \right)$$

which is bigger than E_0 , closing in polynomially!

- Low energy states also form a **band** above $|GS'\rangle$ with quadratic dispersion, **intertwining** with that of the even parity sector
- In total: Even + Odd produce a gapless **band of $2N$ states**

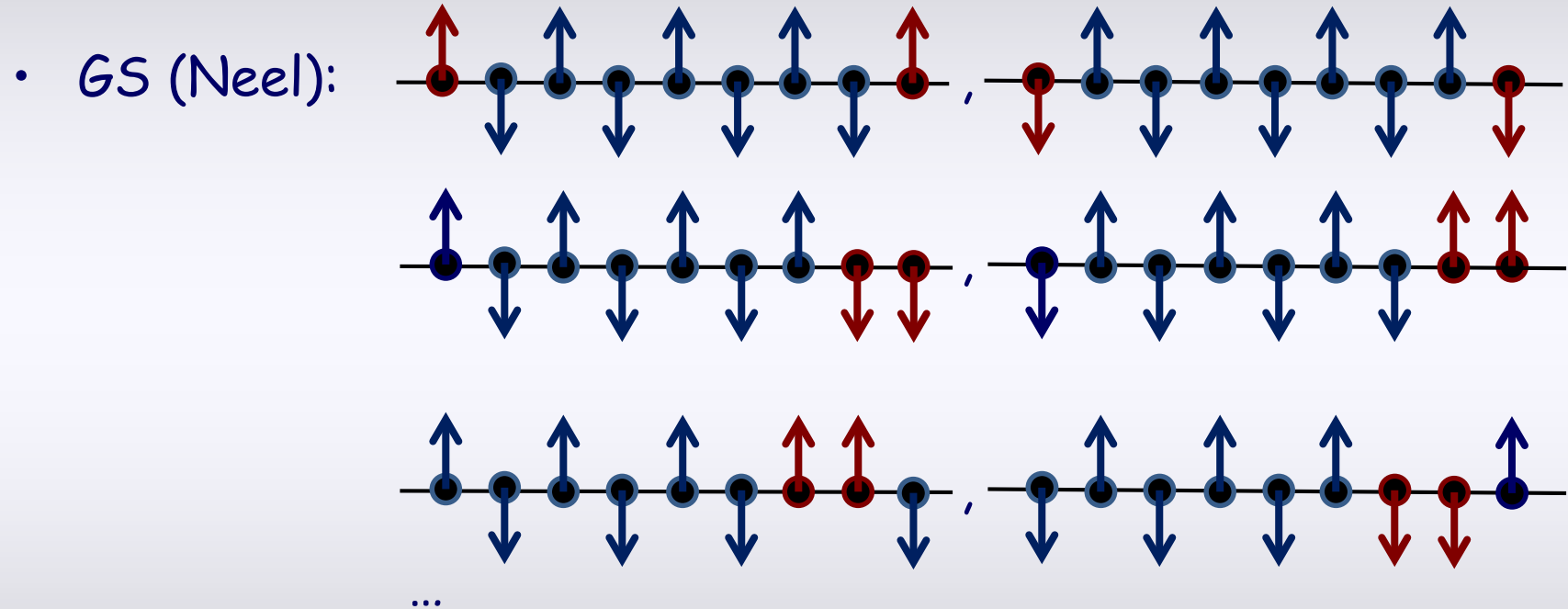
Physical picture for frustrated spectrum

- $H_{\text{Ising}} = \frac{1}{2} \sum_{l=1}^{2M+1} (\sigma_l^x \sigma_{l+1}^x - h \sigma_l^z)$: start from $h=0$



Physical picture for frustrated spectrum

- $H_{\text{Ising}} = \frac{1}{2} \sum_{l=1}^{2M+1} (\sigma_l^x \sigma_{l+1}^x - h \sigma_l^z)$: start from $h=0$



2N States!

Physical picture for frustrated spectrum

$$H_{\text{Ising}} = \frac{1}{2} \sum_{l=1}^{2M+1} (\sigma_l^x \sigma_{l+1}^x - h \sigma_l^z)$$

- At $h=0$: **2N-degenerate** GS (2 x Neel with 1 domain wall)
(compare to 2-degenerate in non-frustrated)
- Turn on $h>0$: it **does not open a gap** proportional to h !
(because of Z_2 symmetry)
- Low-energy eigenstates are in continuity with plane wave
superposition of domain walls

Correlation functions

$$H_{\text{Ising}} = \frac{1}{2} \sum_{l=1}^{2M+1} (\sigma_l^x \sigma_{l+1}^x - h \sigma_l^z)$$

$$\begin{aligned} \sigma_l^+ &= e^{i\pi \sum_{j<l} \psi_j^\dagger \psi_j} \psi_l \\ \sigma_l^z &= 1 - 2\psi_l^\dagger \psi_l \end{aligned}$$

- Correlation functions can be calculated starting from FF picture
- Introduce **Majorana Fermions**: $A_l \equiv \psi_l^\dagger + \psi_l$, $B_l \equiv i(\psi_l - \psi_l^\dagger)$

$$\langle A_l A_m \rangle = \langle B_l B_m \rangle = \delta_{l,m} ,$$

$$\langle A_{l+R} B_l \rangle = iG(R, J, h) , \quad \nu(h, R) = \begin{cases} (-1)^R & h > 0 \\ -1 & h < 0 \end{cases}$$

$$G(R, 1, h) = -G(R, -1, -h) + \frac{2}{N} \nu(h, R)$$

- Compared to the standard case, the frustrated GS correlators have **1 additional contribution** as for **1 (π -)mode excited state**

Local and Quasi-Local Correlators

$$\langle A_{l+R} B_l \rangle = iG(R, J, h), \quad G(R, 1, h) = -G(R, -1, -h) + \frac{2}{N} \nu(h, R)$$

- Local Correlation functions have a finite number of Majoranas

$$\begin{aligned} \langle \sigma_l^z \sigma_m^z \rangle &= \langle A_l B_l A_m B_m \rangle \\ &= m_z^2 - \frac{c_1^z(h)}{R^2} \left(\frac{h^2}{J^2} \right)^R + \frac{4m_z}{N} \left[1 + c_2^z(h)(-1)^R \left| \frac{h}{J} \right|^R \right] \end{aligned}$$

- Quasi-local ones have # of Majorana growing with distances

$$\begin{aligned} \langle \sigma_l^x \sigma_m^x \rangle &= \langle B_l A_{l+1} B_{l+1} \dots A_{m-1} B_{m-1} A_m \rangle \\ &= (-1)^R \left(1 - \frac{h^2}{J^2} \right)^{1/4} \left[1 + \frac{c^x(h)}{R^2} \left(\frac{h^2}{J^2} \right)^R \right] \left(1 - \frac{2R}{N} \right) \end{aligned}$$

- The $\frac{1}{N}$ contributions add up to be finite in improved therm. limit
- Same for Entanglement Entropy

Conclusions & Outlook

- Weak frustration + discrete quantum symmetry
⇒ New critical quantum phase arXiv:1807.07055
- Both **exponential** and **algebraic** (not power-law!) correlations
- Entanglement Entropy to study phase properties
- Algebraic area law violation, but no divergence
- **Robust** against perturbation: very natural systems!
- **Universal collapse** with improved thermodynamic limit/scaling
- Other symmetries? Stronger frustration? Higher D?
- Spontaneous breaking of translational symmetry?
- Origin/description of universality?

Thank you!

Quantifying Frustration

- First quantify “quantum” frustration:
 - Write Hamiltonian as sum of local terms
 - Find GS of H and of all the H_j separately and construct projectors

$$H = \sum_j H_j \longrightarrow \begin{cases} H \rightarrow \Pi \equiv |GS\rangle\langle GS| \\ H_j \rightarrow \Pi_j \equiv \sum_{\alpha} |GS_j^{\alpha}\rangle\langle GS_j^{\alpha}| \end{cases}$$

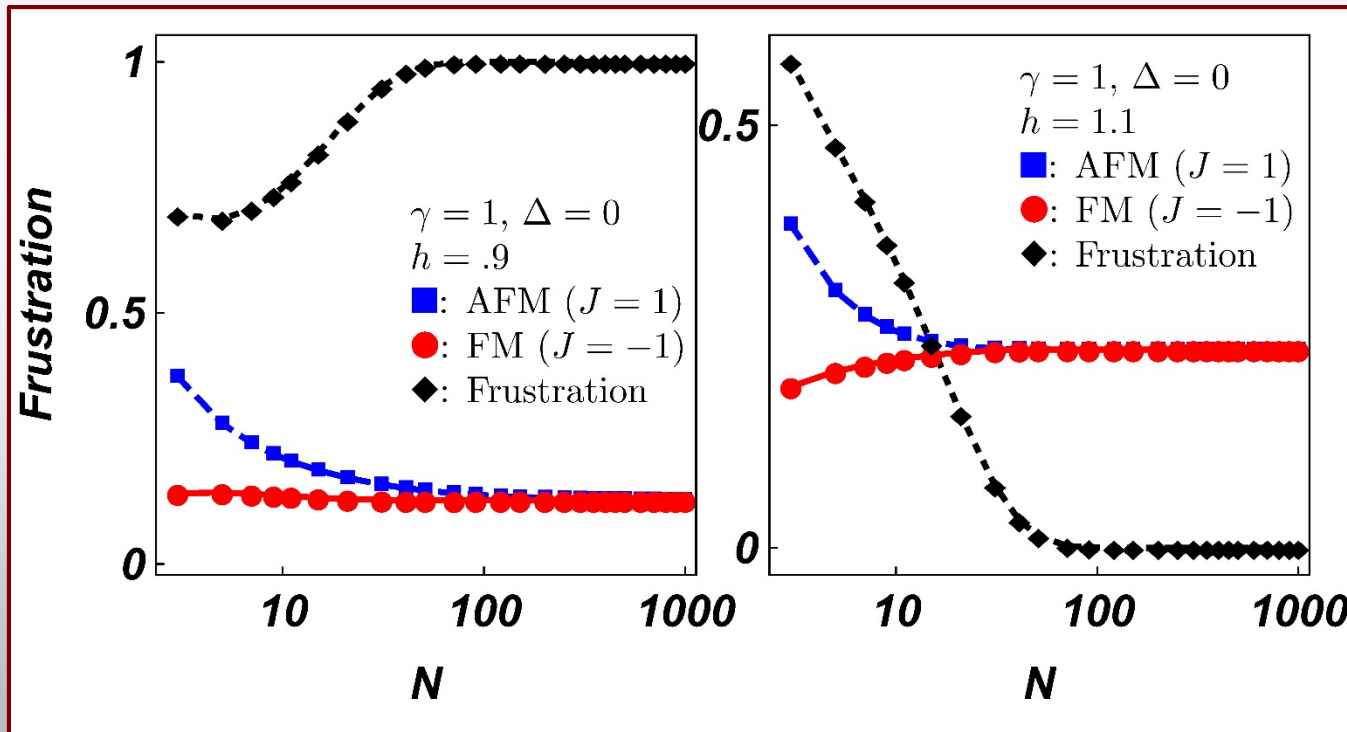
- Measure Hilbert-Schmidt distance between them

$$F_j \equiv \text{Tr}(\Pi_j \Pi)$$

- If translational invariance: $F \equiv F_j$ (Giampaolo et al. PRL '11)

Quantifying Frustration

- Consider frustration of Ferromagnetic ($J=-1$) $F(J = -1)$
and AFM system ($J=1$) $F(J = 1)$
- Geometrical frustration: $g_F = \sum_{j=1}^N [F(J = 1) - F(J = -1)]$



Frustration **does not increase** with the system's length and **vanishes** in non-frustrated phase

Approaching $h \rightarrow 1$

- CFT behavior up to (non-frustrated) correlation length scale & deviation beyond it

