Integrabililty with confined potentials: Duality, Solitons, Field Theory and Growth of Perturbations

Manas Kulkarni

International Centre for Theoretical Sciences

ICTS- TIFR, Bangalore

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Very Recent / Ongoing Work

Duality in Weirstrass-Calogero models
(ongoing with Polychronakos)
M. K , Polychronakos, J. Phys. A: Math (2017)

Multi-component PDEs and relations to cold gases and nonlinear optics
(ongoing with Swarup & Vasan)

Quenches & Thermalizaion in Fermions
(with Mandal & Morita, arXiv:1806.09343)

Aubry-Andre-Harper Model
(with Purkayastha, Sanyal, Dhar
PRB 2018)

Integrable Models

Correlations in lattice integrable models,
for e.g., Fadeev-Takhthajan spin chain.
(ongoing with Das, Dhar, Mendl, Damle, Spohn, Huse)

Hyperbolic Calogero in box-like potential
(ongoing with Gon)

Part-A

Spread of perturbations in integrable models (classical OTOC)
(ongoing with Sundar, Agarwal, Huse) , very preliminary

Part-B

Short-Ranged Model in a box-like potential

PART - A

A. Gon & **M. K** (2018, in preparation)

- Short Ranged Models ubiquitous in nature
- Confining particles unavoidable (both practically and for calculations)
- Can we retain integrability even after confining
- Harmonic traps, quartic traps, box-like potentials (uniform) have now become realistic

$$H = \sum_{i=1}^{N} \left[\frac{p_i^2}{2} + V(x) + \sum_{i,\, j\neq i}^{N} \frac{1}{2L^2} \left(\frac{g^2}{\sinh^2\left(\frac{x_i - x_j}{L}\right)} \right) \right]$$

Hyperbolic – Calogero Model (exponentially decaying interaction)

$$V(x_i) = a_1 \cosh\left(\frac{2x_i}{L}\right) + b_1 \sinh\left(\frac{2x_i}{L}\right) + a_2 \cosh\left(\frac{4x_i}{L}\right) + b_2 \sinh\left(\frac{4x_i}{L}\right)$$

Box-like trap !

The above system is integrable -- Reduces to showing integrability of positive-definte matrices in external potentials (Polychronakos, 1991)

Short-Ranged Model in a box-like potential

$$H = \sum_{i=1}^{N}\left[\frac{p_i^2}{2} + V(x) + \sum_{i,j\neq i}^{N}\frac{1}{2L^2}\left(\frac{g^2}{\sinh^2\left(\frac{x_i - x_j}{L}\right)}\right)\right]$$

Hyperbolic – Calogero Model (exponentially decaying interaction)

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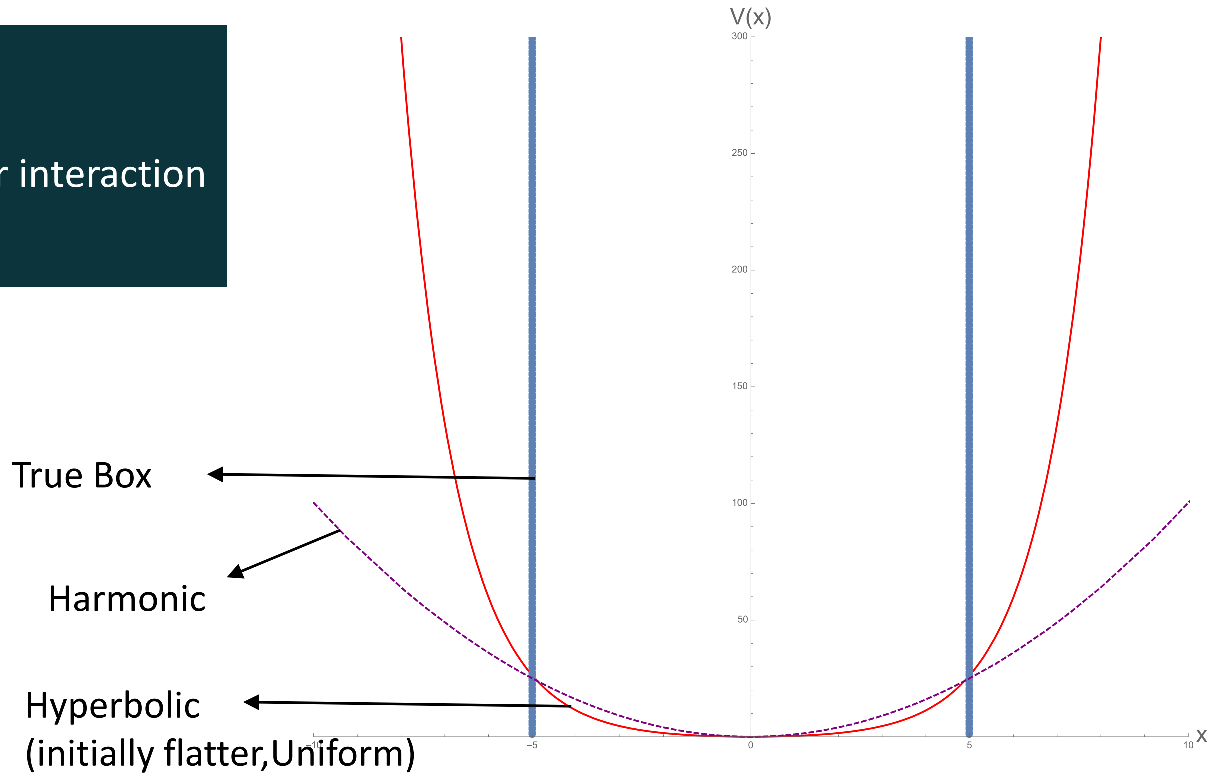
Box-like trap !

- g: coupling constant .
- Box like external potential
- Short-ranged interaction but NOT just nearest neighbour interaction
- Integrable -N integrals of motion exists.



- Duality
- Solitons
- Quenches
- Field Theory

Box Potential Experiments –Hadzibabic Group (Cambridge)

Meaning of a soliton solution for finite number of particles

A. G. Abanov, A. Gromov, **M. K**, J. Phys. A: Math. Theor. 44 (2011)

- Very special space of initial conditions for the list of x's and p's for N particles
- These special initial conditions can be characterized by a few number of "dual particles", say M values of z
- Using the special initial conditions perform Newtonian dynamics of the Calogero particles

Meaning of a soliton in field theory limit

$$\rho(x,t) = \rho(x;\{z_j\})$$
$$v(x,t) = v(x;\{z_j\})$$

For e.g., in flat background (straight line, no potential)

$$\rho(x,t) = \rho(x - z(t)) \qquad z(t) = vt$$

A. G. Abanov, A. Gromov, **M. K**, J. Phys. A: Math. Theor. 44 (2011)
M. K , Polychronakos, J. Phys. A: Math (2017)

Solitons ⟷ M<N Duality

M= N Duality is interesting in general but not for soliton solutions

Duality

$$\dot{x}_i - i\frac{A}{L}\sinh\left(\frac{2x_i}{L}\right) = -i\frac{g}{L}\sum_{j\neq i}^{N}\coth\left(\frac{x_i - x_j}{L}\right) + i\frac{g}{L}\sum_{n=1}^{M}\coth\left(\frac{x_i - z_n}{L}\right)$$

- Z_n 's: dual particles moving in complex planes.
- Very close connection between real particles the dual particles.

$$\dot{z}_n - i\frac{A}{L}\sinh\left(\frac{2z_n}{L}\right) = i\frac{g}{L}\sum_{m\neq n}^{M}\coth\left(\frac{z_n - z_m}{L}\right) + i\frac{g}{L}\sum_{i=1}^{N}\coth\left(\frac{z_n - x_i}{L}\right)$$

- At the first order level x's and z's are coupled
- We take one more derivative the get second order equations
- We use below "Addition Theorems" to simplify the equations [for e.g., see below]

Let there be a functional equation such that $\tilde{f}_{cb}\tilde{f}_{db} + \tilde{f}_{dc}\tilde{f}_{bc} + \tilde{f}_{bd}\tilde{f}_{cd} = C_{bcd}$

If $C_{bcd} = g^2$ then, $\tilde{f}(x_{ab}) = g\coth(x_{ab})$

M. K , Polychronakos, J. Phys. A: Math (2017)

Addition theorems generate a big class of models with Duality

Duality

$$\ddot{x}_i = -\frac{2A^2}{L^3}\sinh\left(\frac{2x_i}{L}\right)\cosh\left(\frac{2x_i}{L}\right) + \frac{2Ag}{L^3}(N-M-1)\sinh\left(\frac{2x_i}{L}\right)$$

$$+\frac{2g^2}{L^3}\sum_{j\neq i}^{N}\left(\frac{\cosh\left(\frac{x_i-x_j}{L}\right)}{\sinh^3\left(\frac{x_i-x_j}{L}\right)}\right) \qquad j=1.....N$$

$$\ddot{z}_n = -\frac{2A^2}{L^3}\sinh\left(\frac{2z_n}{L}\right)\cosh\left(\frac{2z_n}{L}\right) + \frac{2Ag}{L^3}(N-M+1)\sinh\left(\frac{2z_n}{L}\right)$$

$$+\frac{2g^2}{L^3}\sum_{m\neq n}^{M}\left(\frac{\cosh\left(\frac{z_n-z_m}{L}\right)}{\sinh^3\left(\frac{z_n-z_m}{L}\right)}\right) \qquad n=1.....M$$

$$\mathcal{H} = \sum_{i=1}^{N}\left(\frac{p_i^2}{2} + \frac{A^2}{2L}\sinh^2\left(\frac{2x_i}{L}\right) - \frac{Ag}{L^2}(N-M-1)\cosh\left(\frac{2x_i}{L}\right)\right)$$

$$+\sum_{i,j\neq i}^{N}\frac{1}{2L^2}\left(\frac{g^2}{\sinh^2\left(\frac{x_i-x_j}{L}\right)}\right)$$

Soliton Solutions

$$\frac{A}{L}\sinh\left(\frac{2x_i}{L}\right) = \frac{g}{L}\sum_{j\neq i,\, j=1}^{N}\coth\left(\frac{x_i - x_j}{L}\right) - \frac{g}{L}Re\sum_{m\neq n}^{M}\left[\coth\left(\frac{x_i - z_n}{L}\right)\right]$$

$$p_i = \frac{g}{L}Im\left[\sum_{n=1}^{M}\coth\left(\frac{x_i - z_n}{L}\right)\right]$$

- Equating the imaginary part we get the first equation
- Equating the real we get momentum
- These will be our "Special Initial Conditions"

If we specify a few z's then we can find all the initial positions (first eq) and therefore initial momentum (second eq)

$$\dot{x}_i = -\gamma\left(\frac{A}{L}Sinh\left(\frac{2x_i}{L}\right) + \frac{g}{L}\sum_{j\neq i,\, j=1}^{N} Coth\left(x_i - x_j\right) - \frac{g}{L}Re\left[Coth\left(x_i - z_n\right)\right]\right)$$

like an attractor

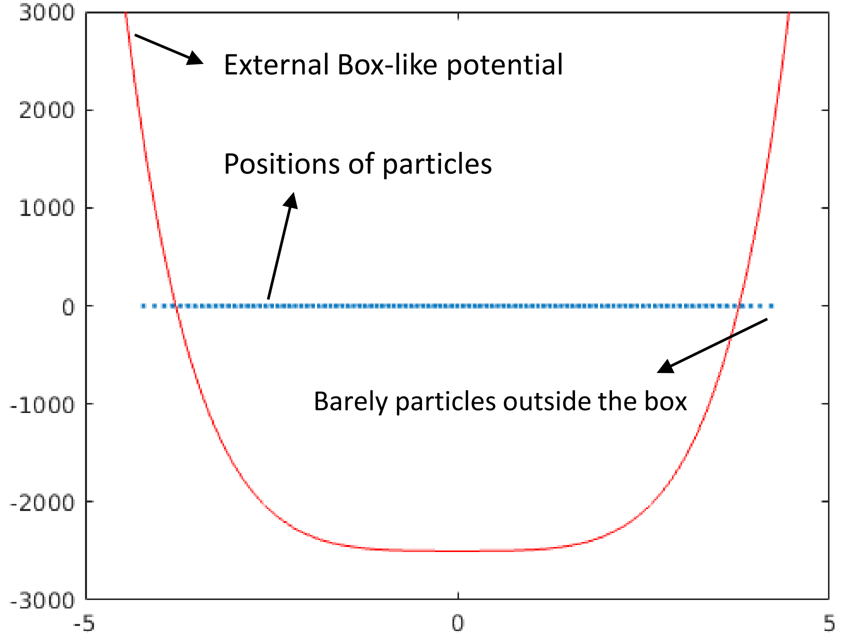
We get all the "special initial conditions" and then do Newtonian dynamics

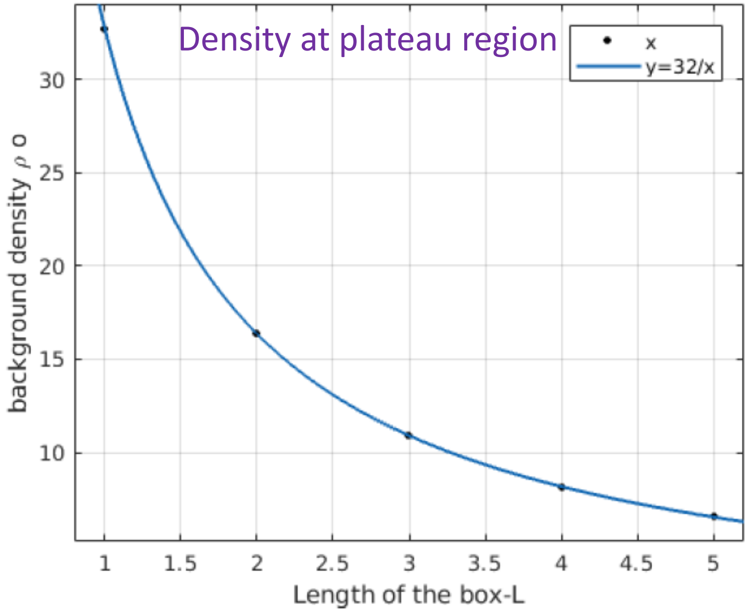
Background Solutions

$$\frac{A}{L}\sinh\left(\frac{2x_i}{L}\right) = \frac{g}{L}\sum_{j\neq i,\, j=1}^{N}\coth\left(\frac{x_i - x_j}{L}\right)$$

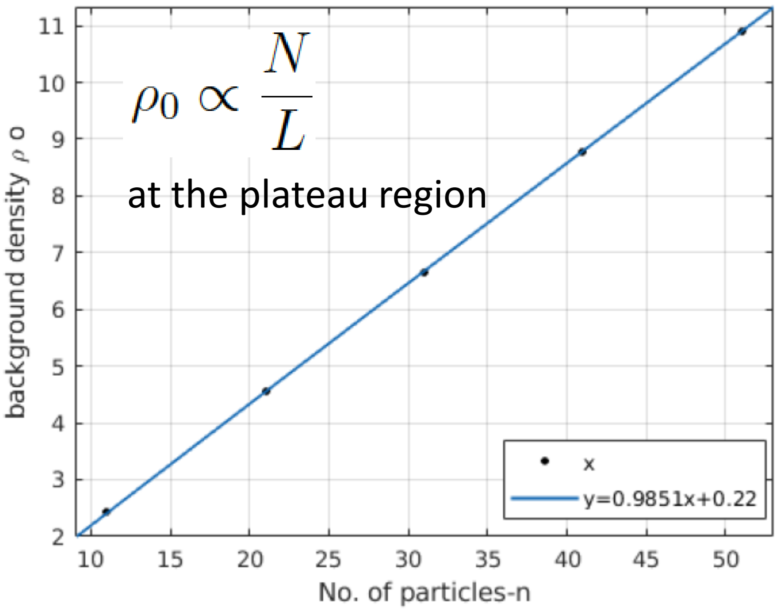
$p_i = 0$ Take all z's to infinity

Are the solutions of above zeros of some orthogonal polynomials ?
(probably, but not done yet)
[For Rational case with harmonic trap → Zeros of Hermite Polynomials]

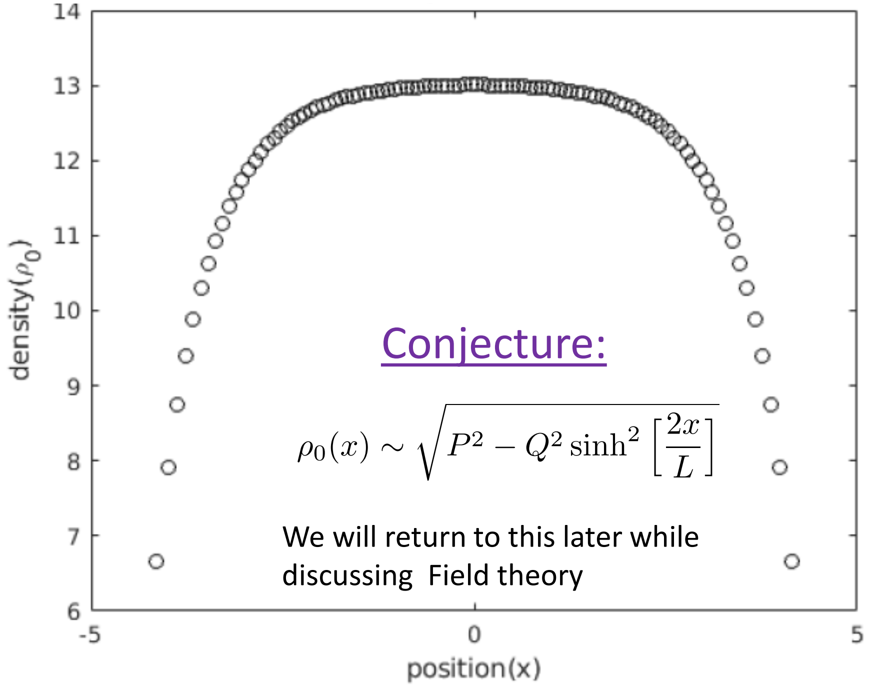




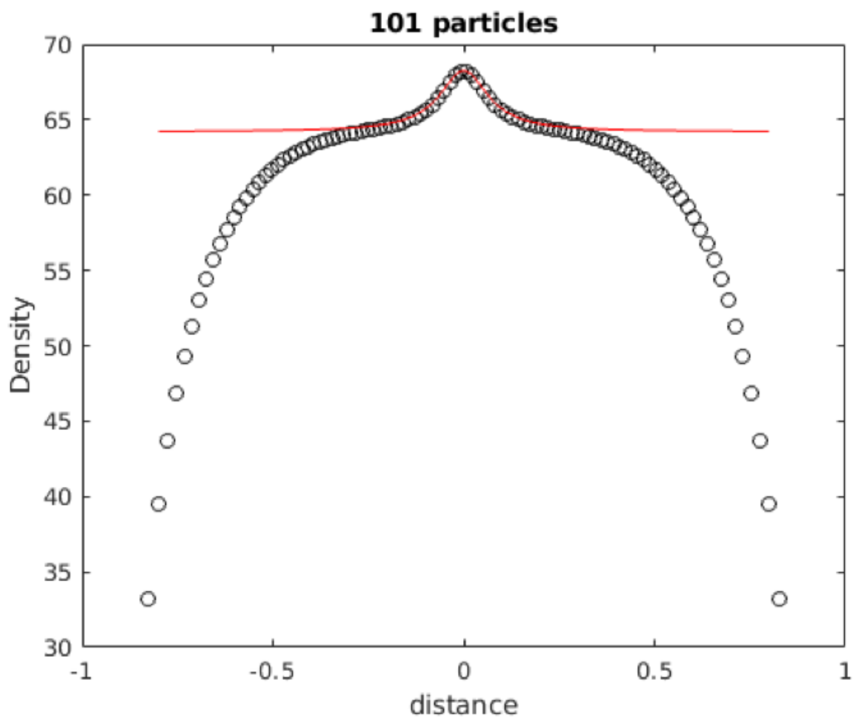
$\rho_0 \propto \frac{1}{L}$

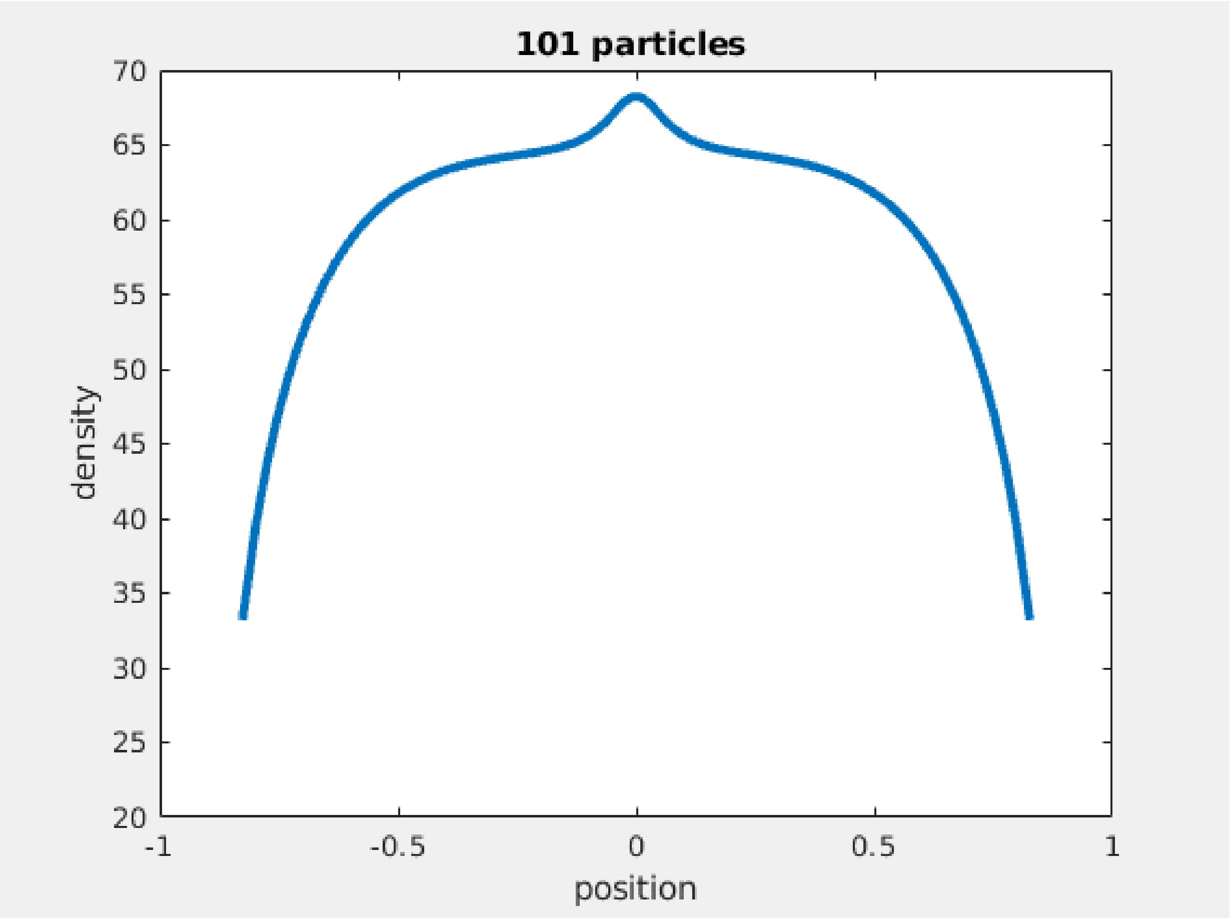


$\rho_0 \propto N$



One Soliton Solution





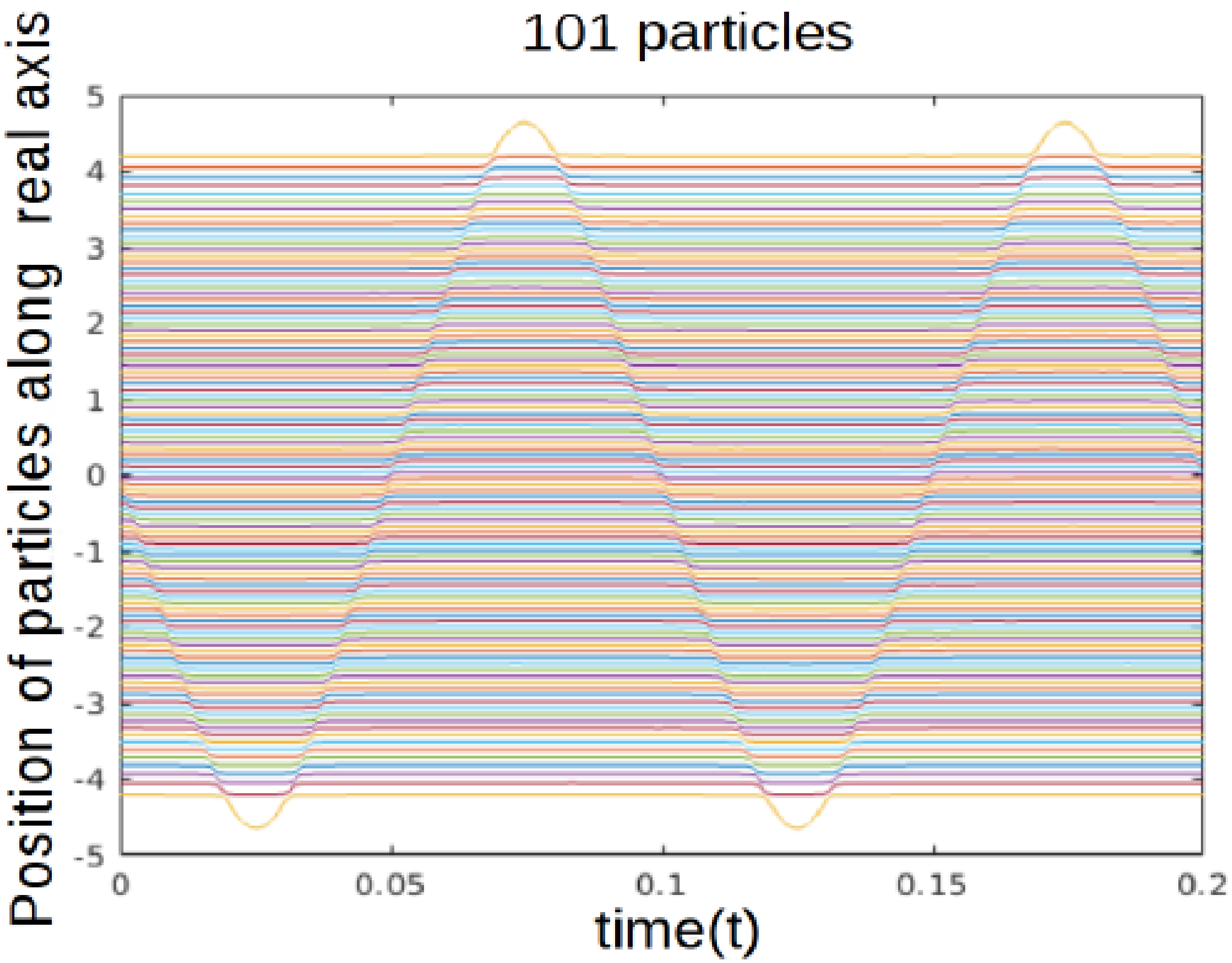
$$\frac{A}{L}\sinh\left(\frac{2x_i}{L}\right) = -\frac{g}{L}\sum_{i,\, i\neq i}^{N}\coth\left(x_i - x_j\right) + \frac{g}{L}Re\left[\coth\left(\frac{x_i - z}{L}\right)\right]$$

$$p_i = \frac{g}{L}Im\left[\coth\left(\frac{x_i - z}{L}\right)\right]$$

- One single z specifies 101 particle positions and momentum
- We plot density only for visualization as inverse of inter-particle distance
- The single dual variable evolves in its hyperbolic potential makes a rectangular-like trajectory

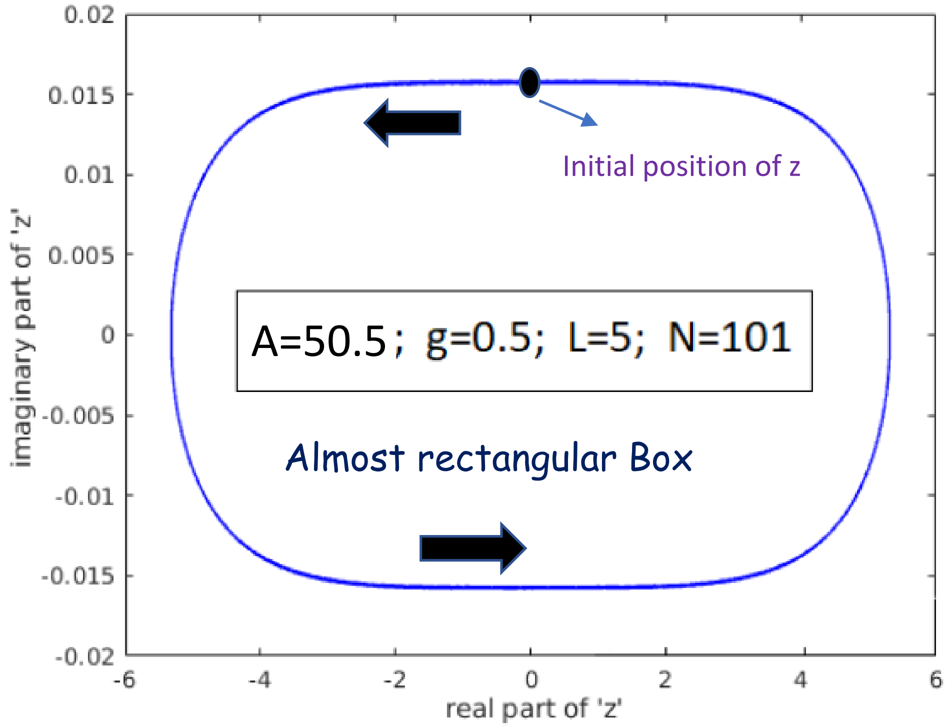
- Initial value problem was numerically solved using RK-4
- Approach of elevating this problem to matrices (Olshanetsky & Perelemov) does not work because of complicated external potential

One Soliton Solution – World Line



Motion of the single dual -z

$$\ddot{z} = -\frac{2A^2}{L^3}\sinh\left(\frac{2z}{L}\right)\cosh\left(\frac{2z}{L}\right) + \frac{2Ag}{L^3}N\sinh\left(\frac{2z}{L}\right)$$



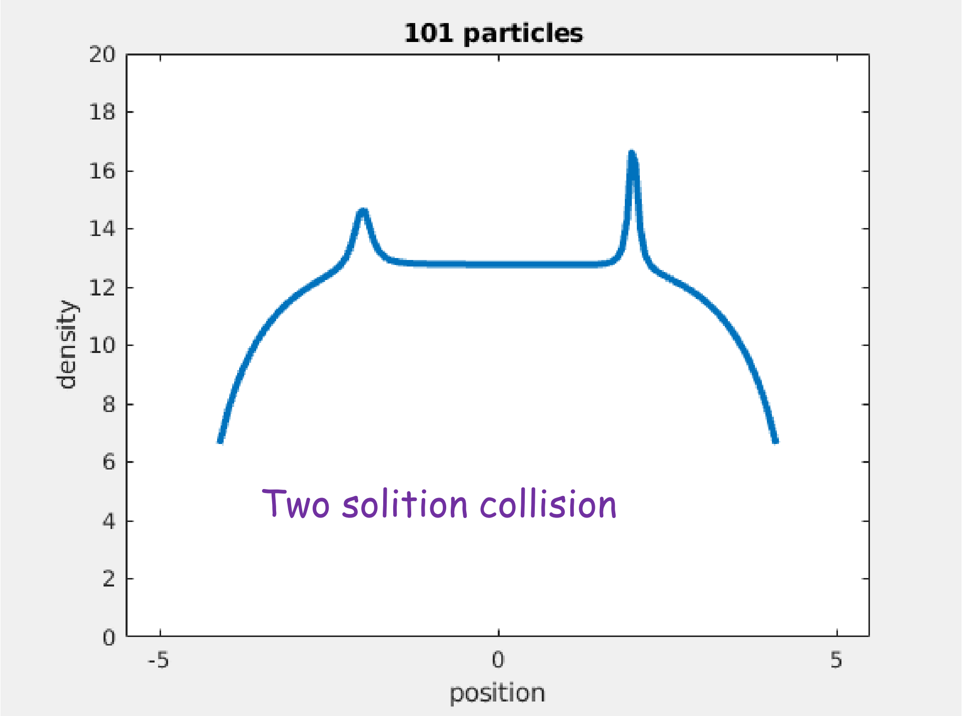
In small y-limit, we have analytical solutions:

$$x = c_1 i\left[\tan^{-1}\left(c_2 \text{JacobiSN}(c_3 it, m)\right)\right]$$

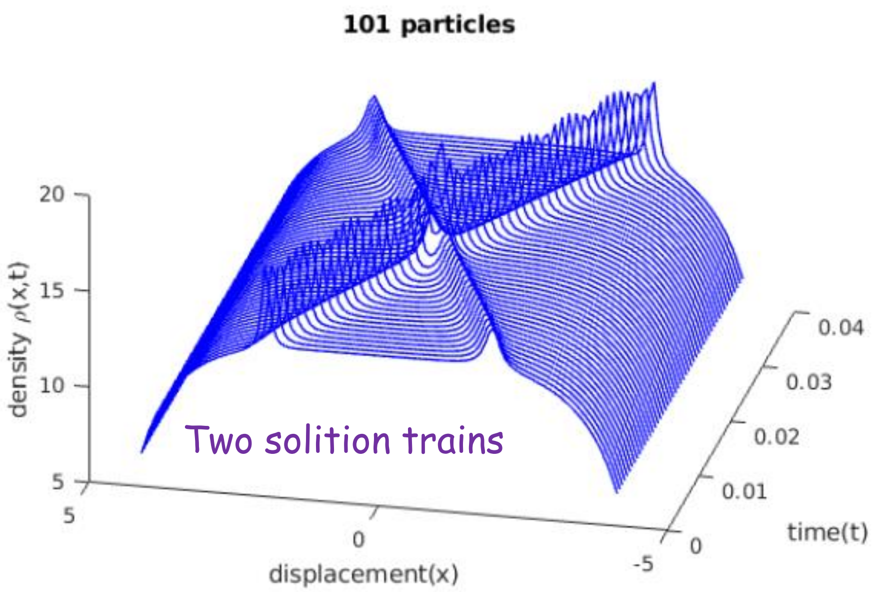
$$T = \frac{Re\left[4\text{EllipticK}(1-\text{m})\right]}{c_3}$$

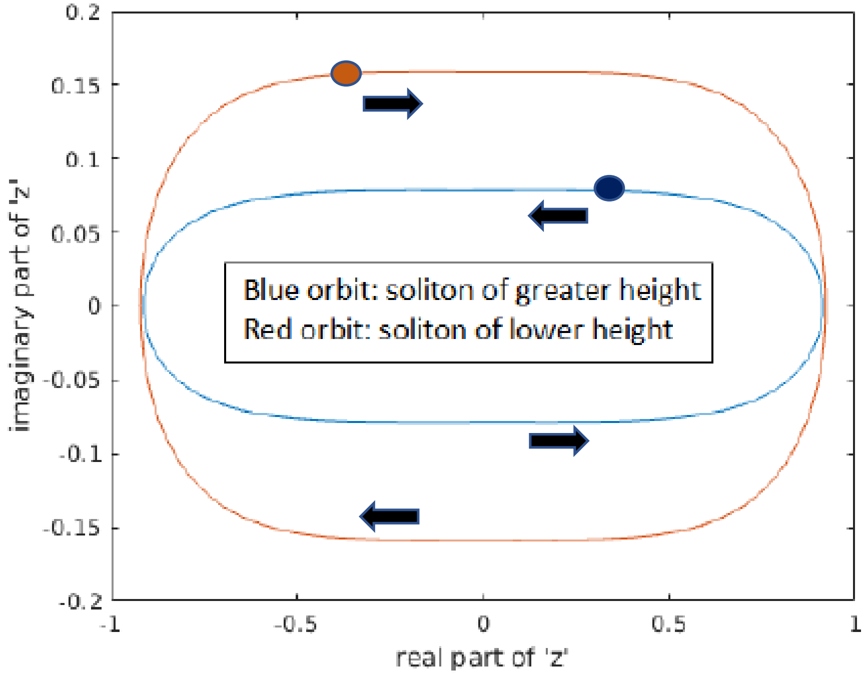
Complete elliptic integral of 1st kind

Multi-Soliton Solutions

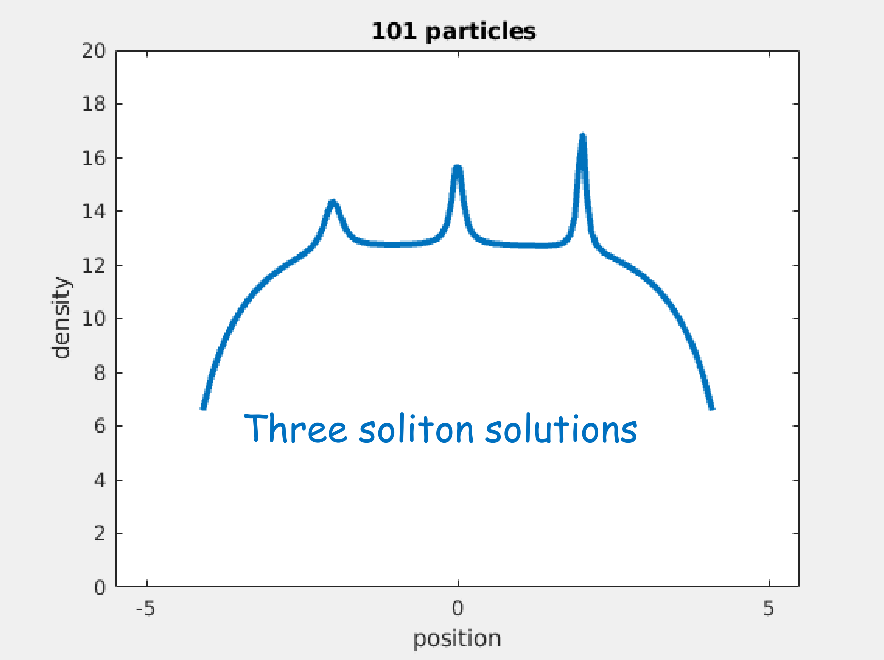


- For two soliton solution we need two dual z's
- Real[z] fixes the position and Im[z] fixes the magnitude and direction of momentum

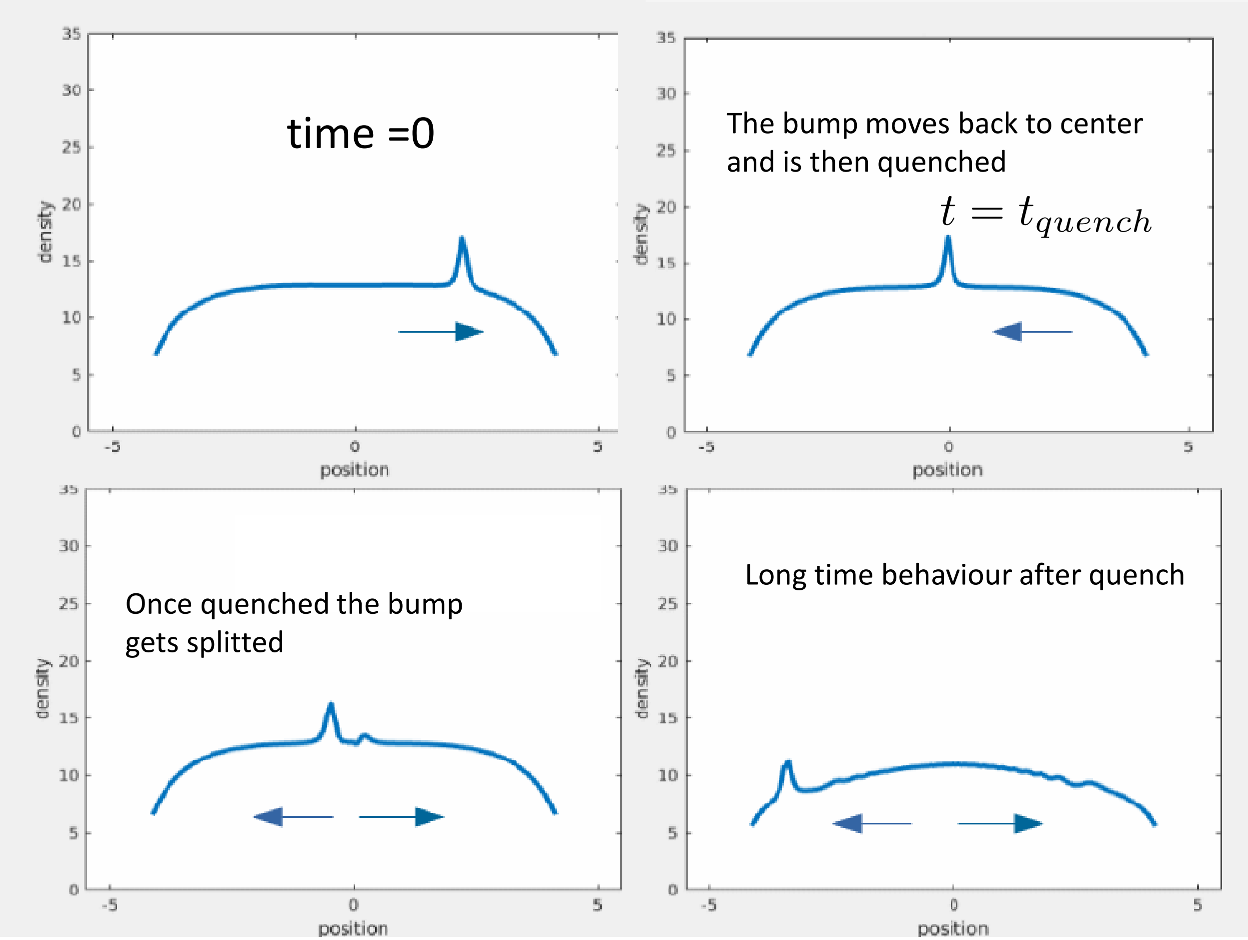


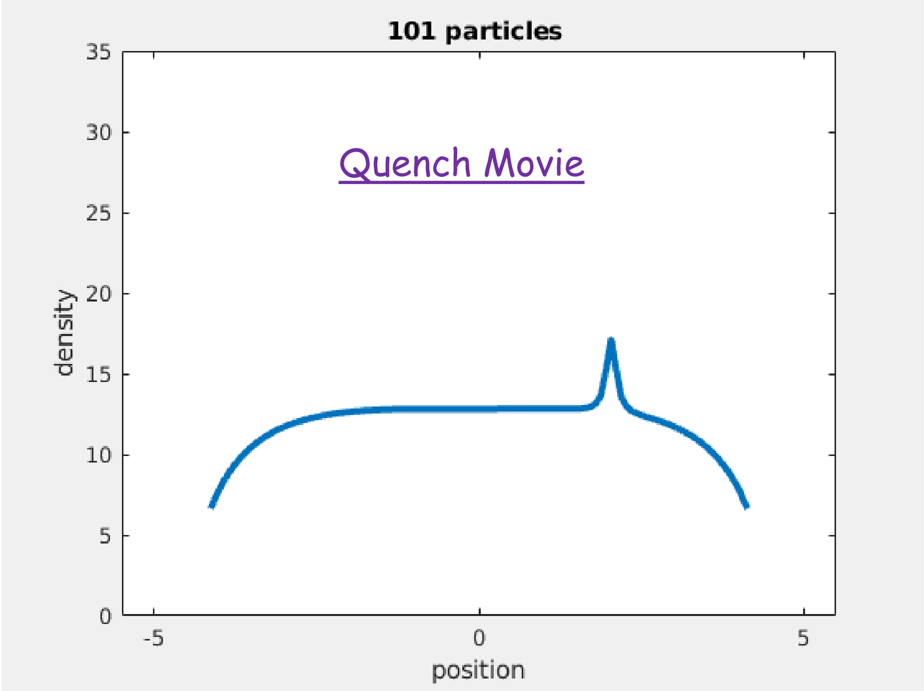


- Similarly, we can find multi-soliton solutions, any M<N.
- If M=N we end up spanning entire space of initial conditions
- Duality closely connected to Integrability

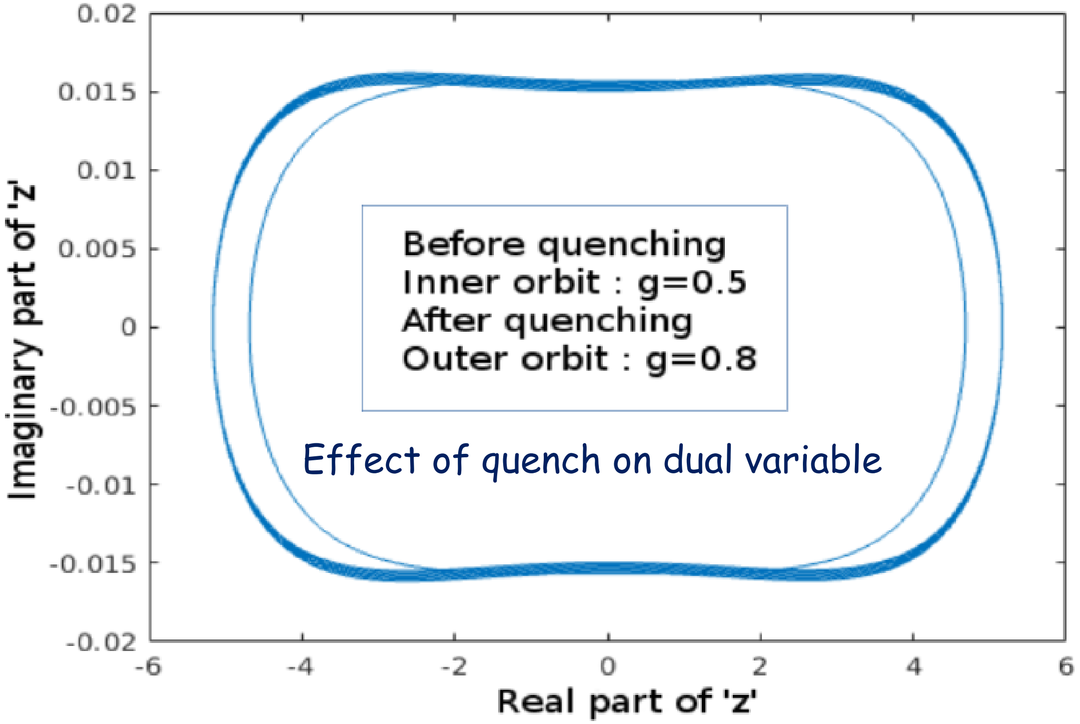


Quenches





- Higher quench, z jumps to outer orbit, and lower quench z jumps to a inner orbit
- Although the z gets deformed, a single z no more sufficient after quench
- Quenching effective generates additional N=1 dual variables



See also, F. Franchini, **M. K**, A. Trombettoni, New J. Phys (2016)

Field Theory

$$\mathcal{H} = \int dx \left[\frac{1}{2}\rho v(x)^2 + \frac{1}{2}\left(\pi\rho^H - \partial_x log\sqrt{\rho(x)}\right)^2 + V_{ext}(x)\rho(x)\right]$$

$$\rho(x)^H = \frac{1}{\pi L} P\left\{\int_{-\infty}^{\infty}\left[\rho(\tau) Coth\left(\frac{\tau - x}{L}\right) d\tau\right]\right\}$$

Thanks –
Lectures of Sasha, Alexios !

Without external potential, soliton solution in hydrodynamic limit must satisfy

$$g\left(\pi\rho^H - \partial_x log\sqrt{\rho(x)}\right) = -\frac{g}{2L}\left(\coth\left(\frac{x-z}{L}\right) + \coth\left(\frac{x+z}{L}\right)\right)$$

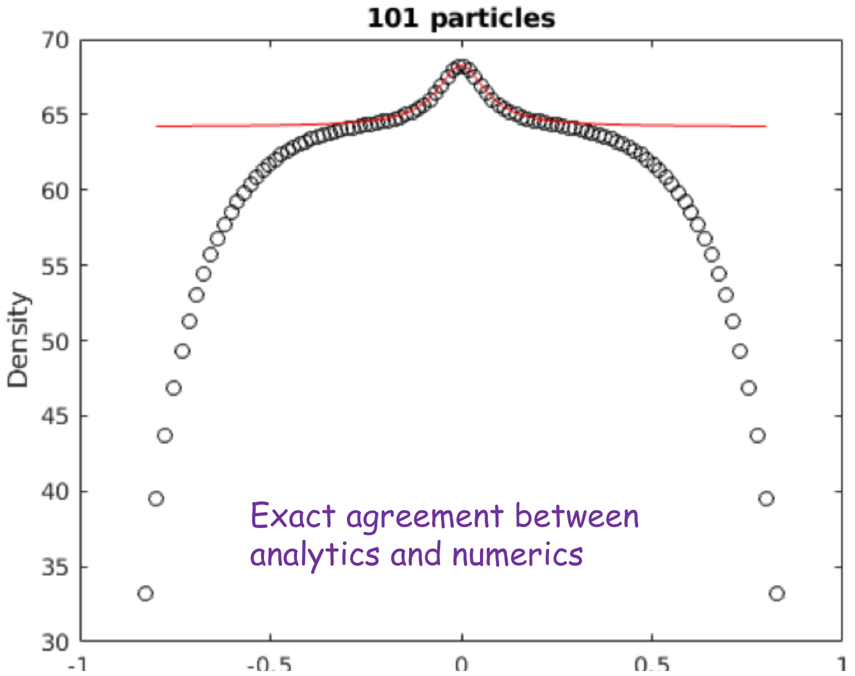
The solution of above equation is shown to be

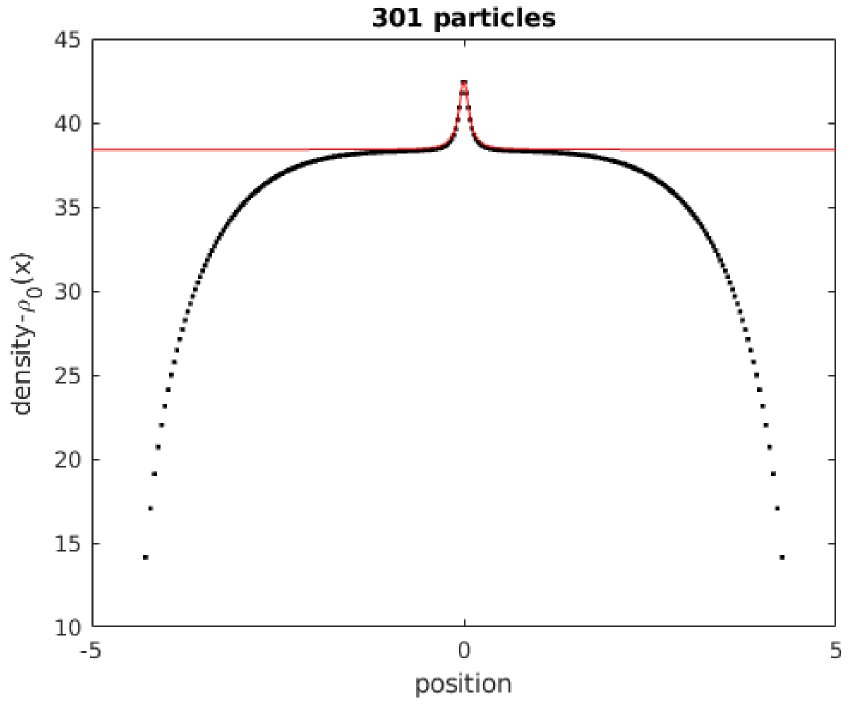
$$\rho(x) = \frac{1}{i\pi L}\left[\coth\left(\frac{x - z_1}{L}\right) - \coth\left(\frac{x - \bar{z}_1}{L}\right)\right]$$

- We can find analytical solitons without potential

- May able to find analytical solution of the background (no soliton)

with external potential but without Log term above (conjecture). In other words, a continuum limit of zeros of the orthogonal polynomials (if we find them). For e.g., in rational case in Harmonic trap, the contiunuum limit of zeros of Hermite polynomials is a semi-circle (if Log is neglected)





Part A

A. Gon, **M. K** (2018, in preparation)

Conclusions

- Exploring integrable nature of models in confined potentials
- M<N dualities in a short-ranged model confined in strong potentials
- Notion of solitons for finite number of particles
- Properties of dual variables
- Quenches, orbits and universality
- Systematic derivation of field theory and subsequent analytical solutions

Outlook

- Are solutions of the equilibrium zeros of known orthogonal polynomials ?
- Analytical forms of background solutions (in field theory limit)
- What about solitons when dual equations are unstable ?
- Quantum mechanical versions ?
- M<N seems to be missing in periodic version of Hyperbolic model (elliptic-Weirstrass Calogero). Can we find soliton solutions there ? [ongoing work with Polychronakos]
- Connections with ILW equations ?

PART - B [Very preliminary results]

Ongoing with Sundhar, Agarwal, Huse

- Effect of localized perturbations in classical integrable many body systems [spatially extended systems]
- Discuss two cases: Rational Calogero (somewhat long-ranged) and hyperbolic Calogero (somewhat short ranged)
- Is there a notion of butterfly speed ?
- What happens when we strongly break integrability of the Calogero family ?
- Lyaponov (or lack of it) contains information of sensitivty to initial conditions
- Butterfly speed contains information of spacial spreading of pertubations

Classical Spin Chain, Das *et al* [PRL 2018]

- Calculate spacio-temoral evolution of difference in trajectories
- Generally an interesting question that seems to have not been addressed for "particle models"

Models and Protocol

Relatively long-ranged model integrable even in Harmonic trap [Rational Case]

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 x_i^2 + \frac{1}{2} \sum_{i,j,i\neq j}^{n} \frac{g^2}{(x_i - x_j)^2}$$

Once we know the initial positions and momentum
the dynamics can be computed by elevating the systems
to higher dimension [Matrices, Powerful Method]

$$X_{ij}(t) = \delta_{ij} x_i(t)$$

$$L_{ij}(t) = \delta_{ij} p_i(t) + (1 - \delta_{ij}) \frac{ig}{x_i(t) - x_j(t)} \quad .$$

$$Q(t) = X(0)cos(\omega t) + \omega^{-1} L(0) sin(\omega t)$$

Matrix Q is moving inside a harmonic trap, i.e,

$$\ddot{Q} = -\omega^2 Q$$

Eigenvalues of Q(t) are Calogero particles

(Dimensional Reduction)

Relatively short-ranged model integrable even in box-like potential [Hyperbolic case]

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m} + Am \cosh(2\omega x_i) + \frac{1}{2} \sum_{i,j,i\neq j}^{n} \frac{g^2}{sinh^2(x_i - x_j)}$$

Once we know the initial positions and momentum
the dynamics can be computed by RK4

$$\frac{dx_i}{dt} = p_i$$

$$\frac{dp_i}{dt} = -2A\omega \sinh(2\omega x_i) + \sum_{i,j,j\neq i}^{n} \frac{2g^2}{\sinh(x_i - x_j)^3} \cosh(x_i - x_j)$$

Next we discuss how to prepare ensemble of initial conditions for a given temperature

Models and Protocol

Monte-Carlo Metropolis

Start with some random set of initial positions

↓

Calculate the potential energy say V

↓

Perturb the configuration to compute new potential energy V'

↓

If new potential energy is lower, then accept it
If not, then accept/reject with a probability $P = e^{\beta(V-V')}$

↓

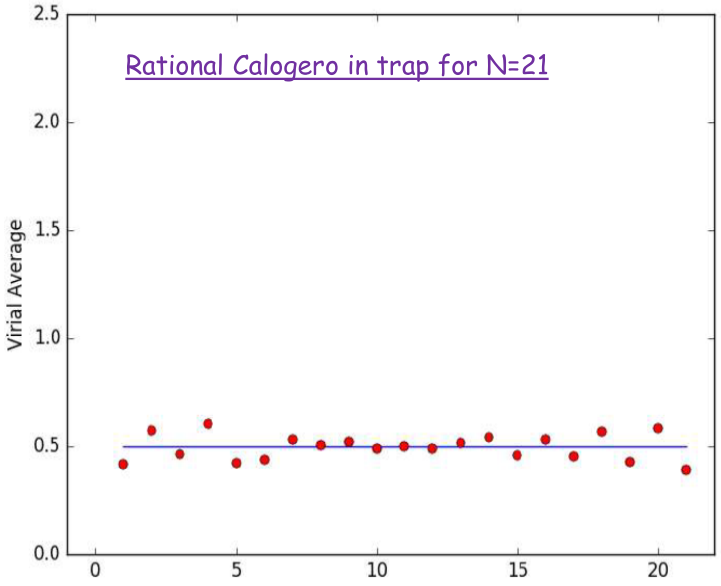
Repeat this test many times

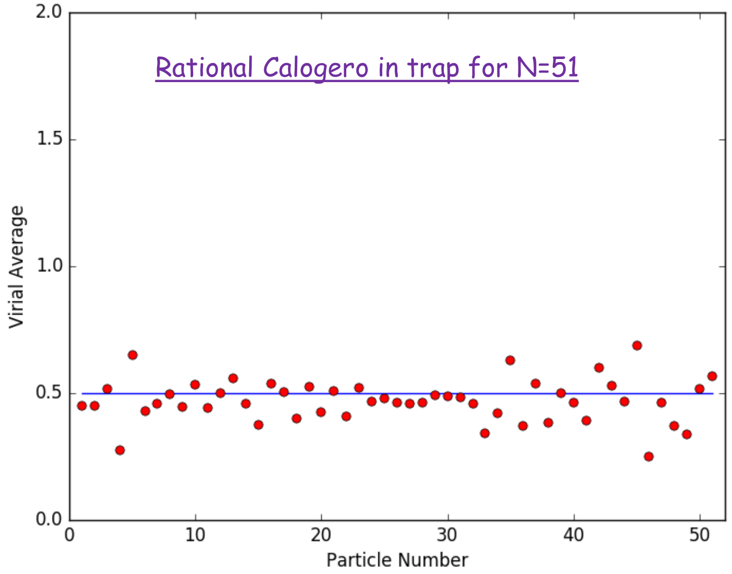
↓

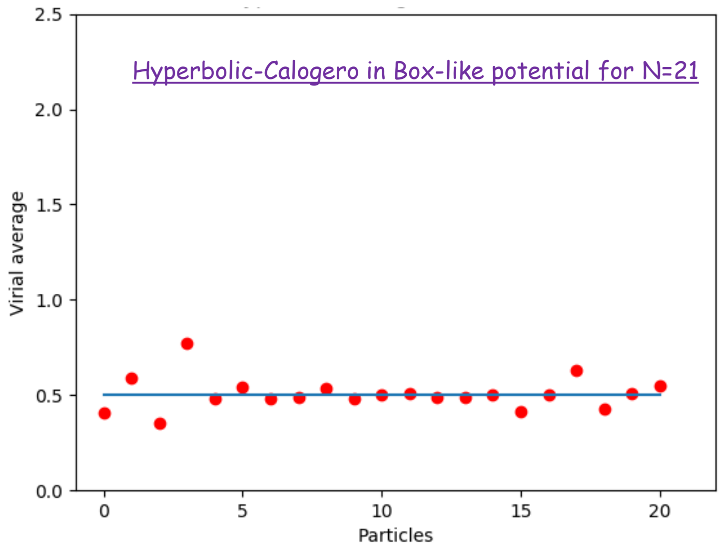
Check generalized virial theorem

$$< q_i \frac{\partial H}{\partial q_j} > = kT\delta_{ij}$$

Complicated long-ranged expression for us







The final sample of data from above will form our initial conditions

Note: Momenta are obtained from an appropriate Gaussian distribution $e^{-\beta p^2}$

Models and Protocol

The final sample of data from above will form our initial conditions

Sample 1: $\{x_1, x_2 .....x_{\frac{N-1}{2}}, ...x_{N-1}, x_N\},$

$\{p_1, p_2 .....p_{\frac{N-1}{2}}, ...p_{N-1}, p_N\}$

Sample 1^{ϵ}: $\{x_1, x_2 .....x^{\epsilon}_{\frac{N-1}{2}}, ...x_{N-1}, x_N\},$

(Perturbed) $\{p_1, p_2 .....p_{\frac{N-1}{2}}, ...p_{N-1}, p_N\}$

where the middle particle $x^{\epsilon}_{\frac{N-1}{2}}(t=0) = x_{\frac{N-1}{2}}(t=0) + \epsilon$

Quantity which has the spacio-temporal information of the difference of trajectories

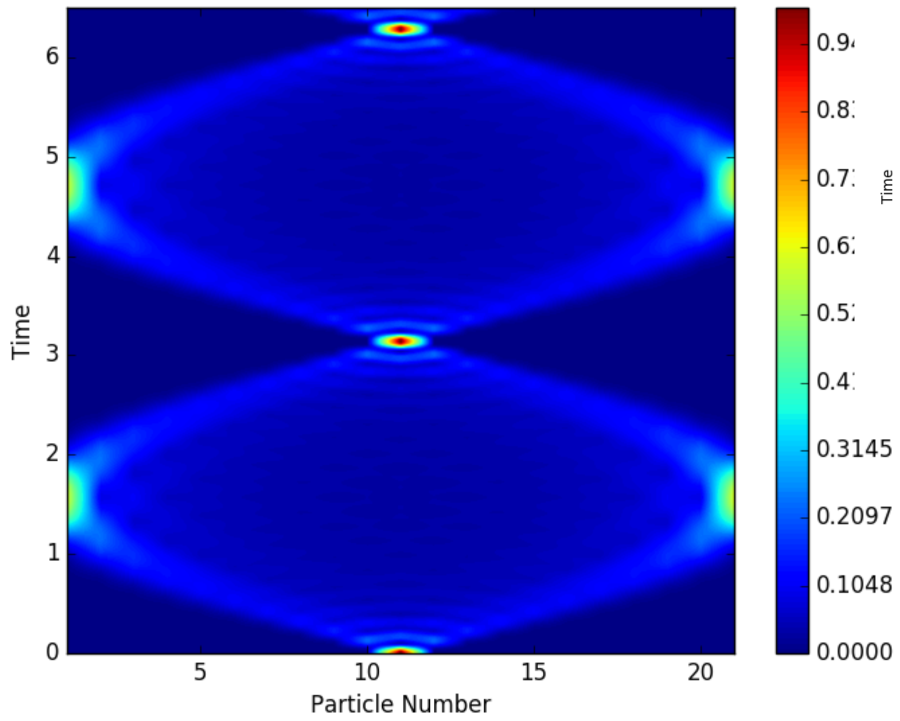
$$D(i,t) = \left\langle \left| \frac{x_i^{\epsilon}(t) - x_i(t)}{\epsilon} \right|^2 \right\rangle$$

Average over samples chosen from the equilibrium distribution $e^{-\beta H}$

Rational Case [Very prelimnary]

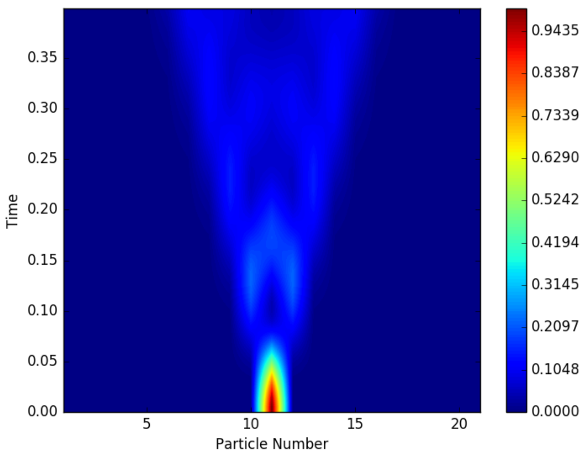
After checking virial (21 particles)

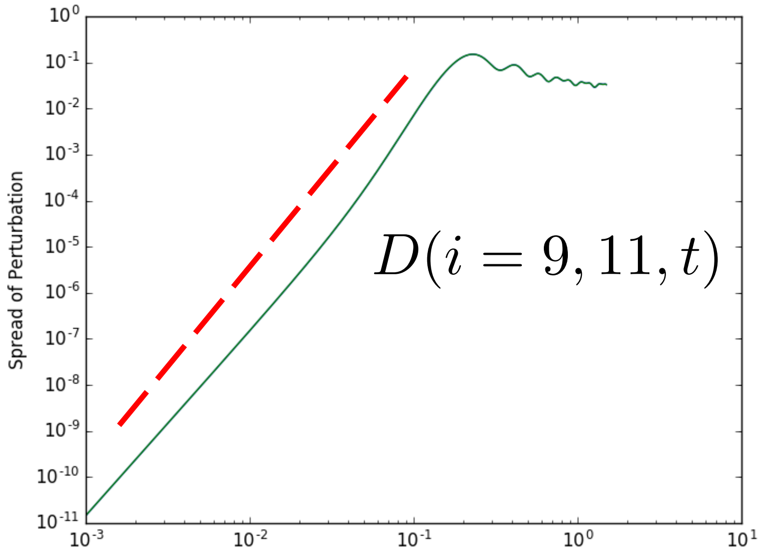
$D(i,t)$

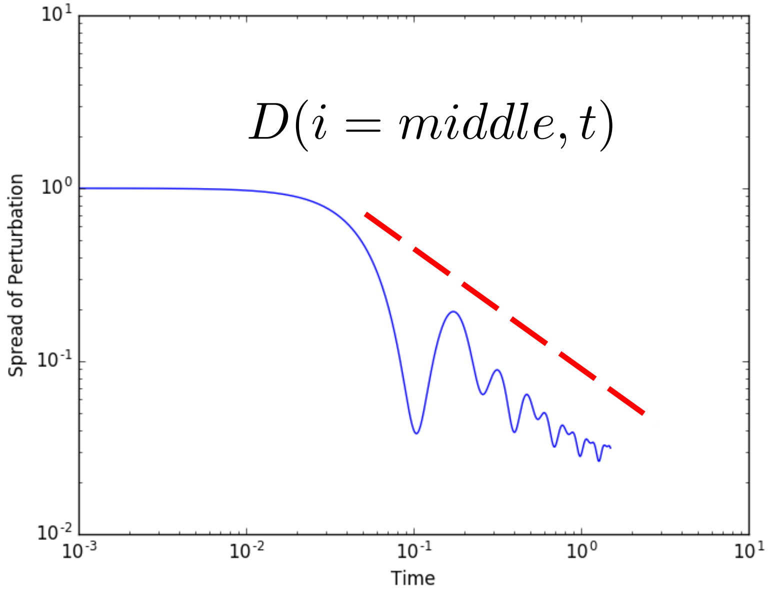


Periodic color plot because model is inherently periodic

$D(i,t)$





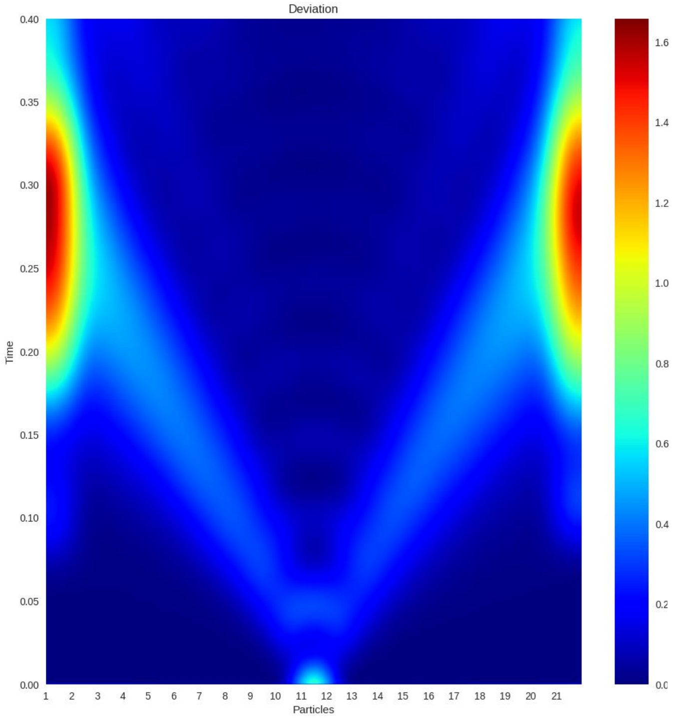


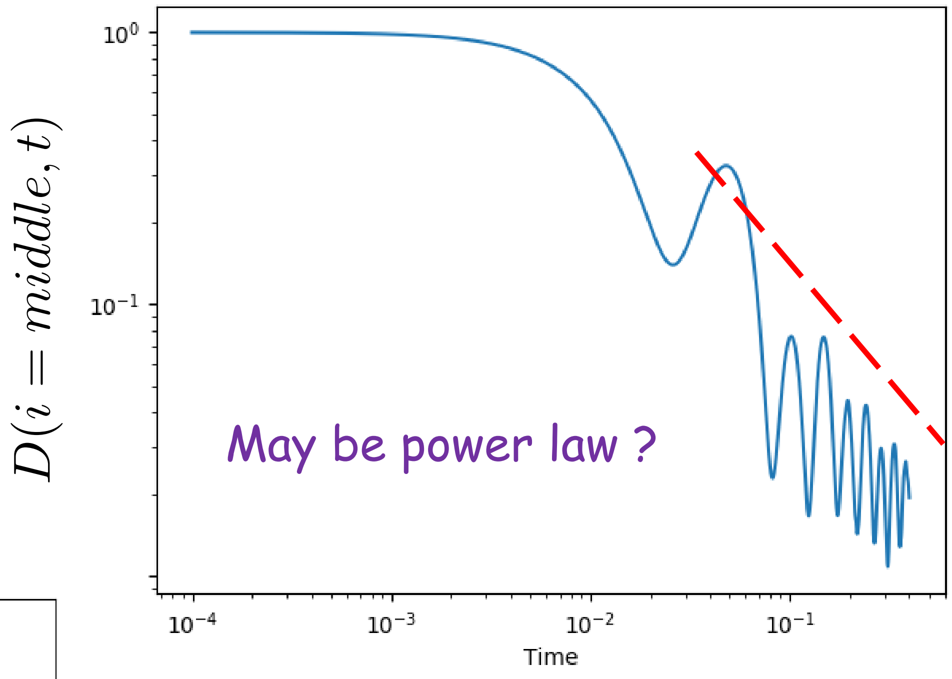
- Some interesting pattern in color plot
- Possibiluty/Hint of butterfly speed
- Hint of power law (zero Lyapunov)

Hyperbolic Case [Very prelimnary]

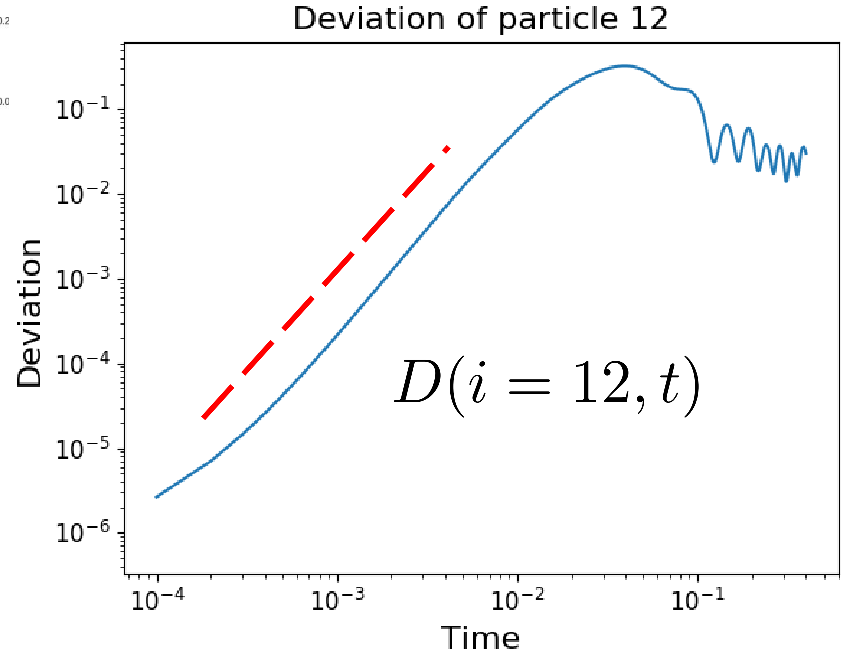
After checking virial (21 particles)

$D(i,t)$





time



- Some interesting pattern in color plot
- Possibiluty/Hint of butterfly speed
- Hint of power law (zero Lyapunov)

Part B

Conclusions and Outlook

(ongoing with Sundar, Agarwal and Huse)

[Very prelimnary]

- Successful Monto-Carlo Metropolis for Calogero family
- Understanding spread of perturbations (color plot)
- Possibility / Hint of butterfly speed ?
- Need to go to larger systems sizes, both Monte-Caro and computing D(i,t)
- Hint of "power-law"
- Effect of strongly breaking integrability ?
  [In general, a particle model that is chaotic]