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Calogero Particles and Fluids
A Review Lecture 3

August 6, 2018

Review

We had obtained through our exchange operator formulation

$$H_c = \sum_i \frac{1}{2} p_i^2 + \sum_i \frac{1}{2}\omega^2 x_i + \sum_{i<j} \frac{\ell(\ell \mp \sigma_{ij})}{x_{ij}^2}$$

with
$$\sigma_{ij} = \vec{S}_i \cdot \vec{S}_j + \frac{1}{q}$$

- Ferro/Antiferromagnetic spin interaction models (as in Matrix)
- Arbitrary coefficient strength (ℓ)
- Spins necessarily in the fundamental of $SU(q)$
- Ferro $\rightarrow$ Antiferro: B $\rightarrow$ F or $\ell \rightarrow -\ell$
- $\ell = 1$: Matrix and Exchange models agree
  (B $\leftrightarrow$ fundamental, F $\leftrightarrow$ antifundamental)

Spectrum of spin-particle model *in brief*

- Solution of these models can be obtained explicitly
- For spin-Sutherland model: ABA (else Yangian symmetry)
- For spin-Calogero model: solution obtained algebraically

Define the operators

$$A_n^\dagger = \sum_i (a_i^\dagger)^n \ , \quad (A_n^a)^\dagger = \sum_i (a_i^\dagger)^n S_i^a$$

and their hermitian conjugates

- Complete set for all permutation-symmetric creation and annihilation operators for all internal states
- Commutators among themselves and with H do not involve ℓ
- Create the same spectrum of excitations as N noninteracting bosons or fermions with q species
- Ground state depends on type of interaction (ferro-antiferro)

Ferro: $\ell > 0$, bosons. Ground state:

$$\psi_B = \prod_{i<j} |x_{ij}|^{\ell} e^{-\frac{1}{2}\omega \sum_i x_i^2} \chi_s(\{\sigma_i\})$$

- $\chi_s(\{\sigma_i\})$ symmetric in the spins σ_i
- Set of all χ_s: N-fold symmetric irrep of total spin $S = \sum_i S_i$
- Ground state is $(N+q-1)!/N!(q-1)!$-degenerate

Antiferro: $\ell > 0$, fermions. Ground state:

$$\psi_F = \sum_M (-1)^M \left(\prod_i \delta_{\sigma_i,\alpha_i} \right) \prod_{i<j} |x_{ij}|^{\ell} x_{ij}^{\delta_{\alpha_i,\alpha_j}} e^{-\frac{1}{2}\omega \sum_i x_i^2}$$

- M are ($N!$ in number) total particle permutations
- α_i a set of values for σ_i that determine the state
- Minimal total power of $x_i \Rightarrow$ maximally distinct set of α_i
- States form n-fold antisymmetric irrep total spin S
- Ground state is $q!/n!(q-n)!$-degenerate, $n = N (mod\, q)$

What about Spin-Sutherland model? Better call ABA!

Asymptotic Bethe Ansatz approach

ABA: physically lucid method, works well for periodic models

Consider distinguishable exchange-Calogero particles without external potential coming in with asymptotic momenta k_i 5

- Key fact: particles 'go through' each other, no backscattering
- Impenetrable $1/x^2$ potential became completely penetrable!

[Proof: simultaneous eigenstate of π_i becomes asymptotically an eigenstate of p_i at both $t \to \pm\infty$: no shuffling of p_i]

(Puzzle: what happens with the correspondence principle? The interaction coefficient is $\ell(\ell - \hbar M_{ij})$. How can $\hbar$ produce such a dramatic effect, particles going through each other?)

YB equation trivially satisfied; Scattering phase shift is sum of two-body phases

$$\theta_{sc} = \frac{N(N-1)}{2}\pi\ell$$

Sidebar: Yang-Baxter equation and integrability

The fact that particles go through each other means that their scattering trivially satisfies the Yang-Baxter relation:

$$S_{12}\, S_{13}\, S_{23} = S_{23}\, S_{13}\, S_{12}$$

- YB equation is considered a hallmark of integrability
- It is actually necessary for integrablity but not sufficient

YB condition can be viewed as absence of Aharonov-Bohm effects around triple coincidence points

- For zero-range interactions it is enough:
- Lieb-Liniger: triple points of measure zero
- Heisenberg and XXZ: triple points do not exist

In general, 3-particle effects introduce additional scattering that may spoil integrability. I will give an example in private.

Let's return to Calogero where all is well.

Scattering result plus periodicity of wavefunction gives the spectrum on a space of period 2π

$$2\pi k_i + \sum_j \ell \,\mathrm{sgn}(k_i - k_j) = 2\pi n_i$$

- n_i are integer quantum numbers, ensuring periodicity
- Constraints on their choice by continuity from $\ell = 0$
- IF $k_i \leq k_j$ then $n_i \leq n_j$
- $n_i = n_j$ represents a unique solution

For $n_1 \leq \ldots n_N$ the solution for k_i and E is

$$k_i = n_i + \ell(i - \frac{N+1}{2}) \;, \quad E = \sum_i \tfrac{1}{2} k_i^2$$

ABA momenta k_i are the pseudomomenta previously defined

- Bottom line: spectrum and degeneracies the same as those of distinguishable particles obeying generalized selection rules
- Degeneracy fixed by different ways of distributing the particles to the quantum numbers n_i

Particles with spin: spectrum and degeneracies the same as those of free particles with spin distributed among n_i

- 'Identity' of distinguishable particles becomes their spin state
- For ferro interactions: bosons: for antiferro: fermions
- Spins combine accordingly

Assume N bosons (fermions) of spin $SU(q)$, with m_i on them on quantum number n_i for a set of $n_1 \leq \cdots \leq n_N$. Total spin $SU(q)$ of this state is

$$[S] = \sum_i \oplus [m_i]_\pm$$

where $[m]_+$ ($[m]_-$) is the m-fold symmetric (antisymmetric) irrep of $SU(q)$ for bosons (fermions) respectively.

- Energy and degeneracy of this state

$$E = \sum_i m_i \frac{1}{2} \left[n_i + \ell \left(i - \frac{N+1}{2} \right) \right]^2 , \quad D = \dim[S]$$

Also works for spin-Calogero model (previously solved algebraically)

The freezing trick

Take the strength of interaction to grow large: $\ell \to \infty$

- System 'freezes' around the position of classical equilibrium
- Low-lying excitations are vibrational modes (phonons) and spin excitation modes (spinons)
- Particle distances $x_i - x_j$ have negligible fluctuations, so spin and coordinates decouple into a spinless Calogero model and a spin chain model
- Both vibrational and spin excitation energies of order ℓ
- Energy states can be found by 'modding' the spin-Calogero states by the spinless Calogero states: $Z_s = Z_{sC}/Z_C$
- From spin-Sutherland model: The Haldane-Shastry model

$$H_{HS} = \mp \sum_{i<j} \frac{\vec{S}_i \cdot \vec{S}_j}{\sin^2 \frac{\pi(i-j)}{N}}$$

From harmonic spin-Calogero model we obtain a spin chain model with spins sitting on an inhomogeneous lattice

$$H_P = \mp \sum_{i<j} \frac{\vec{S}_i \cdot \vec{S}_j}{\bar{x}_{ij}^2}$$

- $\bar{x}_i$ minimize classical potential and satisfy

$$\bar{x}_i - \sum_{j(\neq i)} \frac{1}{(\bar{x}_i - \bar{x}_j)^3} = 0 \quad \Leftrightarrow \quad \bar{x}_i - \sum_{j(\neq i)} \frac{1}{\bar{x}_i - \bar{x}_j} = 0$$

- $\bar{x}_i$ are the roots of the N-th Hermite polynomial.
- Spectrum is equidistant – spin content is nontrivial
- Antiferromagnetic end of the spectrum is a $c = 1$ conformal field theory (perturbatively exactly in $1/N$)

Generalizations to spin chains for other spin representations (e.g., $SU(p|q)$ 'supersymmetric' chain) also exist

Parting words

Exchange operator formulation gave us

- Neat way to treat the system directly at the QM domain
- Spinless and spin-particle models with arbitrary, non-quantized coupling strength
- Correspondence to free particles and generalized statistics
- Spin chain models through the freezing trick

...while Matrix formulation gave us

- Arbitrary-size spin representations

No formulation gives "everything"

Desideratum: arbitrary spin (e.g., $SU(2)$ spin 1) and arbitrary strength to use freezing trick and obtain spin-1 chain and thus validate (or disprove) Haldane's mass gap conjecture

Still an open question...

Folding the Calogero model: new systems Sorry, no time!

Hydrodynamic limit, Waves and Solitons

- Dense collection of particles: fluid description (hydro)
- Exact in $N \to \infty$ limit, can include $1/N$ corrections
- Particle system is integrable, hydro system should be as well

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Lagrange approach (body-fixed)

- Order particle coordinates $x_j < x_{j+1}$ (ordering is preserved)
- Turn particle index j into a continuous coordinate σ and describe fluid it terms of $x(\sigma, t)$ (smooth in hydro limit)
- Fluid density $1/(x_{j+1} - x_j)$ and velocity $\dot{x}_j$ are

$$\rho(\sigma, t) = \left(\frac{\partial x}{\partial \sigma}\right)^{-1} , \quad v(\sigma, t) = \frac{\partial x}{\partial t}$$

- EOM directly transfer from particles

$$\frac{\partial^2 x}{\partial t^2} = F(\sigma)$$

Force $F(\sigma)$ on particle $j = \sigma$ in general depens on full $x(\sigma)$

Euler description (space-fixed)

Thanks, Sasha!

- Use $\rho(x,t)$ and $v(x,t)$ as fundamental variables
- EOM become

$$\partial_t \rho + \partial_x(\rho v) = 0 \ , \quad \partial_t v + v\partial_x v = F(\{\rho\}, x)$$

- Continuity equation an identity in Lagrange description
- Body-fixed (material) time derivative appearing

Euler description is Hamiltonian

$$\{\rho(x), v(y)\} = \delta'(x-y) \ , \quad H = \int dx \left[\tfrac{1}{2}\rho v^2\right] + U$$

Suffices to find the potential energy of the system in terms of ρ
For external potential $V_o(x)$ and interaction $V(x-y)$ we expect

$$U = \int dx \rho(x) V_o(x) + \int dxdy \, \rho(x)\, \rho(y)\, V(x-y)$$

First part always OK. Not so for the second

Continuum formula valid if near-neighbor particles do not contribute a macroscopic fraction of the potential

- Assume $V(x-y) \sim |x-y|^{\gamma}$
- Contribution from neighborhood with density $\rho = 1/a$

$$U \sim a^{\gamma} \sum_{i \neq j} |i-j|^{\gamma} = 2a^{\gamma} \sum_{i} \sum_{k>0} k^{\gamma}$$

- If $\gamma > -1$ near neighbors do not contribute substantially $\Rightarrow$ continuum formula OK
- If $\gamma < -1$ near neighbors do contribute substantially $\Rightarrow$ continuum formula fails
- Calogero: $\gamma = -2$: a careful analysis is needed

Exercise: Consider particles interacting with nearest-neighbor harmonic potential

$$V = \sum_{j} c(x_j - x_{j+1})^2$$

Derive the Euler description of the fluid system. The particle system is obviously integrable, so the fluid system must be too. Derive the conserved integrals. (This is called the Chaplygin gas.)

QM collective field method

Basic method: perform change of variables in the quantum system from particle-specvific to collective ones

- Used historically for matrix model calculations in the singlet sector ($M \to \{\phi_n = \mathrm{Tr} M^n\}$)
- Express many-body QM in terms of "collective field" variables $\rho(x), \pi(x)$
- Change of variables in Schrödinger eq. including the proper wavefunction measure
- For the Calogero model: write the wavefunction as

$$\psi = \Delta(x)^\ell \phi$$

  and perform the procedure for function ϕ
- It gives...

Free Calogero:

$$U = \int dx \left[\frac{\pi^2 \ell^2}{6} \rho^3 + \frac{\pi \ell(\ell-1)}{2} \rho \, \partial_x \rho^H + (\ell-1)^2 \frac{(\partial_x \rho)^2}{8\rho} \right]$$

(ρ^H Hilbert transform)

- Terms $\sim \rho^3, (\partial_x \rho)^2/\rho$: short-distance contributions
- Term $\sim \rho \partial_x \rho^H$: regularized 'naive' continuum term
- Leading term: fermions with $\hbar \to \ell\hbar$: fractional statistics
- Classical limit: $\ell(\ell-1), (\ell-1)^2 \to \ell^2$

Still, can we derive it classically?

Basic identity $$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \tilde{f}(2\pi n) \quad (\tilde{f} \text{ Fourier transf.})$$

("Dual" or "instanton expansion"). $f(x)$ smooth at scale $\Delta x \sim 1$:

$$\sum_{n=-\infty}^{\infty} f(n) \simeq \tilde{f}(0) = \int_{-\infty}^{\infty} f(x) dx$$

Terms $n \neq 0$: nonperturbative corrections in $N \sim x$

Consider the Calogero potential contributed by a particle $x(0) = x_o$

$$V_o = \frac{1}{2} \sum_{n \neq 0, = -\infty}^{\infty} \frac{1}{(x_o - x_n)^2} = \frac{1}{2} \sum_{n \neq 0} \frac{1}{[x(0) - x(n)]^2}$$

Summand not a smooth function of s. Define regularized function

$$f(s) = \frac{1}{[x(0) - x(s)]^2} - \frac{1}{x'(0)^2 s^2} + \frac{x''(0)}{x'(0)^3 s + \epsilon s^3}$$

with
$$f(0) = \frac{3x''(0)^2}{4x'(0)^4} - \frac{x'''(0)}{3x'(0)^3}$$

Apply basic identity:

LHS: $$f(0) + \sum_{n \neq 0}^{\infty} f(n) = \frac{3x''(0)^2}{4x'(0)^4} - \frac{x'''(0)}{3x'(0)^3} + 2V_o - \frac{\pi^2}{3x'(0)^2}$$

RHS: $$\int \frac{ds}{[x(0) - x(s)]^2} + R \text{ ``=''} \int \frac{\rho\, dx}{[x(0) - x]^2} + R = \pi \partial_x \rho_o^H$$

Use $$\frac{d}{ds} = \frac{1}{\rho}\frac{d}{dx}\ , \quad x'(0) = \frac{1}{\rho_o}$$

to express $x'(0)$ etc. in terms of ρ_o, $\partial_x \rho_o$ etc:

$$V_o = \frac{\pi^2 \rho_o^2}{6} + \frac{(\partial_x \rho_o)^2}{8\rho_o^2} + \frac{\partial_x^2 \rho_o}{\rho_o} + \frac{\pi}{2}\partial_x \rho_o^H$$

Total potential, upon dropping a total derivative

$$U = \int dx_o\, \rho_o V_o = \int dx \left[\frac{\pi^2}{6}\rho^3 + \frac{\pi}{2}\rho\, \partial_x \rho^H + \frac{(\partial_x \rho)^2}{8\rho} \right]$$

- Same as the QM result (restoring coupling g)
- Perturbatively exact. Nonperturbative effects: depletion
- No gradient catastrophe (subleading terms regulate ρ^3)
- Trigonometric or hyperbolic potentials: change kernel of Hilbert transform

$$\rho^H(x) = \fint K(x-y)\rho(y)dy\ , \quad K(x) = \frac{1}{x}, \cot x, \coth x$$

- Trigonometric amounts to choosing $\rho(x)$ periodic

Sound waves

$$\partial_t \rho = \{\rho, H\} = -\partial_x \frac{\delta H}{\delta v} \ , \quad \partial_t v = \{v, H\} = -\partial_x \frac{\delta H}{\delta \rho}$$

Yield

$$\partial_t \rho + \partial_x(\rho v) = 0$$

$$\partial_t v + \partial_x \left[\frac{1}{2} v^2 + \frac{\pi^2 g}{2} \rho^2 + \pi g \partial_x \rho^H - \frac{g}{8} (\partial_x \ln \rho)^2 - \frac{g}{4} \partial_x^2 \ln \rho \right] = 0$$

Small amplitude waves: linearized equation over $\rho = \rho_o$

- Dispersion relation

$$\omega = \pi \sqrt{g} \rho_o |k| - \frac{\sqrt{g}}{2} k^2 \ , \quad \mathrm{v}_s = \pi \sqrt{g} \rho_o$$

- v_s is long-wavelength speed of sound over ρ_o
- In terms of group velocity v_g dispersion becomes

$$\omega = \frac{\mathrm{v}_s^2 - v_g^2}{2\sqrt{g}}$$

- Sound waves can only travel at $v_g < \mathrm{v}_s$
- $\omega(k) = \omega(2\pi\rho - k)$ (umklapp)

Solitons/waves: $\rho(x,t) = \rho(x - \mathrm{v}t)$, $v(x,t) = v(x - \mathrm{v}t)$

- EOM admit solitons of rational type

$$\rho(x) = \rho_o \left[1 + \frac{g(\mathrm{v}^2 - \mathrm{v}_s^2)}{g\mathrm{v}_s^2 + (\mathrm{v}^2 - \mathrm{v}_s^2)^2 x^2} \right]$$

- Solitons can only have speed $|\mathrm{v}| > \mathrm{v}_s$
- Become highly peaked as $\mathrm{v} \gg \mathrm{v}_s$
- Particle number Q, momentum P and energy E of soliton are

$$Q = 1 \ , \quad P = \mathrm{v} \ , \quad E = \tfrac{1}{2}\mathrm{v}^2$$

- Same as those of a free particle at speed v
- Soliton can be identified as a particle 'going through' the system in a 'Newton's cradle' fashion

Summing periodic copies of (modified) soliton solutions: waves

- Can also be considered as solitons of Sutherland model
- Wave speed v can be both above and below v_s
- Nonlinear, amplitude-dependent dispersion relation
- Small amplitude: sound waves; large amplitude: solitons

Description in terms of classical pseudo-Fermi sea

- QM πFermi sea of k_j: $\hbar \to 0$, $\hbar\ell \to g$
- Pseudo-Fermi level $k_F = \mathrm{v}_s$
- Sound waves: holes – small gaps inside the πFermi sea
- Solitons: particles flying over the πFermi sea
- Large amplitude waves: finite gaps inside πFermi sea

Waves and solitons are dual descriptions of the system

Dual formulation: the Generating Function approach

Consider $x_a,\ a = 1, \ldots, n$ particle coordinates with first order EOM

$$m_a \dot{x}_a = \partial_a \Phi$$

Φ a function of the x_a; m_a a set of constant "masses"

$$\begin{aligned} m_a \ddot{x}_a &= \sum_b \partial_b \partial_a \Phi \, \dot{x}_b = \sum_b \partial_a \partial_b \Phi \frac{1}{m_b} \partial_b \Phi \\ &= -\frac{\partial V}{\partial x_a} \ , \quad V = -\sum_a \frac{1}{2m_a} (\partial_a \Phi)^2 \end{aligned}$$

Choose $\quad \Phi = \frac{1}{2} \sum_{a \neq b} m_a m_b F_{ab}(x_a - x_b) + \sum_a m_a W_a(x_a)$

(factors of m_a are for later convenience)

$$\partial_a \Phi = \sum_b m_a m_b f_{ab} + m_a w_a$$

where $\quad f_{ab} = F'_{ab}(x_a - x_b) \ , \quad w_a = W'_a(x_a)$

After some algebra and symmetrizations of indices we obtain

$$\begin{aligned} -V &= \frac{1}{4}\sum_{b\neq c} m_b m_c (m_b + m_c) f_{bc}^2 \\ &+ \frac{1}{6}\sum_{b\neq c\neq d} m_b m_c m_d \left[f_{bc}f_{bd} + f_{cb}f_{cd} + f_{db}f_{dc}\right] \\ &+ \tfrac{1}{2}\sum_{b\neq c} m_b m_c (w_b - w_c) f_{bc} + \tfrac{1}{2}\sum_{b} m_b w_b^2 \end{aligned}$$

Look for one- and two-body relative potentials
Conditions:

$$f_{bc}f_{bd} + f_{cb}f_{cd} + f_{db}f_{dc} = g_{bc} + g_{bd} + g_{cd}$$

for all distinct b, c, d, for some functions $g_{ab}(x_a - x_b)$, and

$$(w_b - w_c)\, f_{bc} = u_{bc} + v_b + v_c$$

for some functions $u_{ab}(x_a - x_b)$ and $v_a(x_a)$

Functional equations admitting families of solutions

Table: Solutions to functional equations for relative potentialsl

C_{abc}	$f_{ab}(x_{ab})$	$F_{ab}(x_{ab})$	$V_{ab}(x_{ab})$
0	g/x_{ab}	$g\, m_a m_b \log\lvert x_{ab}\rvert$	$-\frac{g^2\, m_a m_b (m_a+m_b)}{4\, x_{ab}^2}$
$-g^2$	$g\cot x_{ab}$	$g\, m_a m_b \log\lvert\sin x_{ab}\rvert$	$-\frac{g^2 m_a m_b (m_a+m_b)}{4\sin^2 x_{ab}}$
$+g^2$	$g\coth x_{ab}$	$g\, m_a m_b \log\lvert\sinh x_{ab}\rvert$	$-\frac{g^2 m_a m_b (m_a+m_b)}{4\sinh^2 x_{ab}}$

Table: Solutions to functional equations for external potential

$f_{ab}(x_{ab})$	$w_a(x_a)$	$2V_a(x_a)$
g/x_{ab}	$c_0 + c_1 x_a + c_2\, x_a^2$	$-m_a w_a^2 + g(m_a - m_{tot})m_a(c_2\, x_a + \frac{3}{2} c_3\, x_a^2)$
$g\cot x_{ab}$	$c_0 + c_1 \cos 2x_a + c_2 \sin 2x_a$	$-m_a w_a^2 + g(m_a - m_{tot})m_a(c_2 \cos 2x_a - c_1 \sin 2x_a)$
$g\coth x_{ab}$	$c_0 + c_1 \cosh 2x_a + c_2 \sinh 2x_a$	$-m_a w_a^2 + g(m_a - m_{tot})m_a(c_2 \cosh 2x_a + c_1 \sinh 2x_a)$

- Recover standard Calogero interactions (rational, trigo, hyper)
- Arbitrary masses, external potentials quartic (in rational case) or trigonometric (in Sutherland case)

However

- Potential $-\sum_a (\partial\Phi/\partial_a)^2/2m_a$ is negative definite: instability
- To cure it: make Φ, and thus f_{ab}, w_a imaginary
- First-order equations become complex

To have a real system

- A subset of particles x_j, $j = 1, \dots N$, must be real
- The remaining z_α, $\alpha = 1, \dots M$ $(M + N = n)$ can be complex: they 'guide' the motion of x_j and we will call them solitons
- x_j and z_α must decouple in the second-order equations
- This imposes: $m_j m_\alpha (m_j + m_\alpha) = 0 \Rightarrow m_j = -m_\alpha = m \; (= 1)$
- Solitons in this formulation are negative mass particles!

The dual equations

$$
\begin{aligned}
\dot{x}_j &= i \sum_{k(\neq j)} f(x_j - x_k) - i \sum_{\alpha} f(x_j - z_\alpha) + i w(x_j) \\
\dot{z}_\alpha &= i \sum_{\beta(\neq \alpha)} f(z_\alpha - z_\beta) - i \sum_{k} f(z_\alpha - x_k) + i w(z_\alpha)
\end{aligned}
$$

- Coupled, complex equations
- By construction, second order equations decouple
- Choosing x_j, $\dot{x}_j$ real at $t = 0$ they will remain real
- This implies

$$
\sum_{k(\neq j)} f(x_j - x_k) + w(x_j) = \mathrm{Re}\left(\sum_{\alpha} f(x_j - z_\alpha) \right)
$$

$$
\dot{x}_j = \mathrm{Im}\left(\sum_{\alpha} f(x_j - z_\alpha) \right)
$$

The values of z_α determine x_j, $\dot{x}_j$

For the simplest case of harmonic Calogero we have the dual system

$$\dot{x}_j - i\omega x_j = -ig \sum_{k \neq j}^{N} \frac{1}{x_j - x_k} + ig \sum_{\alpha=1}^{M} \frac{1}{x_j - z_\alpha}$$

$$\dot{z}_\alpha - i\omega z_\alpha = ig \sum_{\beta \neq \alpha}^{M} \frac{1}{z_\alpha - z_\beta} - ig \sum_{j=1}^{N} \frac{1}{z_\alpha - x_j}$$

- For *any* M, N the above equations imply:

$$\ddot{x}_j = -g^2 \sum_{k \neq j}^{N} \frac{1}{(x_j - x_k)^3} - \omega^2 x_j$$

$$\ddot{z}_\alpha = -g^2 \sum_{\beta \neq \alpha}^{M} \frac{1}{(z_\alpha - z_\beta)^3} - \omega^2 x_\alpha$$

- For $M \geq N$ system reproduces full Calogero dynamics
- For $M < N$ the above reproduces restricted Calogero solutions
- Similarly for other Calogero systems and external potentials

(Motivating the construction of Sasha!)

Take the limit of $N \to \infty$ (but M can remain small)
We need a way to "monitor" the positions and velocities of particles

Device: Introduce one more spectator particle $x_0 = x$ with mass m_0

- Full system of $N + M + 1$ particles remains Calogero-like
- Spectator particle introduces an additional term $m_0 f(x_a - x)$ in the equations for $\dot{x}_a$ and $\dot{z}_\alpha$ of the remaining particles
- Must take $m_0 \to 0$ in order not to disturb the system
- Spectator particle is a "pilot fish" for the remaining particles: it becomes "trapped" by them and follows them as they move

- For $m_0 \to 0$ spectator particle's velocity and acceleration satisfy

$$\dot{x} \equiv u = i \sum_{j=1}^{N} f(x - x_j) - i \sum_{\alpha=1}^{M} f(x - z_\alpha) + i\, w(x)$$

$$\frac{du}{dt} = -\partial_x \quad \Big[\sum_j \tfrac{1}{2} f(x - x_j)^2 + \sum_\alpha \tfrac{1}{2} f(x - z_\alpha)^2 + (N - M) v(x) + \tfrac{1}{2} w(x)^2\Big]$$

Final twist: “Sprincle” the system with many spectator particles at various positions x

- They monitor the system at every x ("sawdust on a stream")
- The spectator velocity $u(t)$ is promoted to a field $u(x,t)$ (x dependence arises from its dependence on initial position x)
- $\frac{du}{dt}$ is thus the total time variation of u arising booth from its dependence on $x_j(t)$ and $z_\alpha(t)$ and on x

Therefore

$$\frac{\partial u}{\partial t} = \frac{du}{dt} - \dot{x}\frac{\partial u}{\partial x} = \frac{du}{dt} - u\frac{\partial u}{\partial x} = \frac{du}{dt} - \partial_x\left(\tfrac{1}{2}u^2\right)$$

or

$$\frac{du}{dt} = \partial_t u + \partial_x\left(\tfrac{1}{2}u^2\right)$$

- Need to express terms in $\frac{du}{dt}$ involving x_j and z_α in terms of u
- Take advantage of $f(x)^2 = gf(x) + C$ for all cases
- Different signs in u and $\dot{u}$ necessitate splitting $u = u^+ + u^-$

$$\begin{aligned} u^+(x) &= -i\sum_\alpha f(x - z_\alpha) + iw(x) \\ u^-(x) &= i\sum_j f(x - x_j) \end{aligned}$$

- Potential $w(x)$ could be split between u^+ and u^-

Eventually, for external potential $V(x)$, we obtain

$$\partial_t u + \partial_x \left[\frac{1}{2} u^2 + \frac{ig}{2} \partial_x (u^+ - u^-) + V \right] = 0$$

- This is a bi-chiral version of the Benjamin-Ono equation
- Equation valid for all N and M (even before $N \to \infty$)

$u^+(x)$ essentially determines everything:

- From its definition its poles determine z_α
- From EOM of particles we have

$$u^+(x_j) = \dot{x}_j + i \sum_{k(\neq j)} f(x_j - x_k)$$

- *If we know* there are N particles, we can solve (imaginary part of) the above to find x_j
- These in turn fix the form of $u^-(x)$

Go to hydro limit

- Hydro obtained by expressing u^+ and u^- in terms of ρ and v
- Can use the Fourier approach we used for finding U for Calogero. It results to

$$u^+(x) = v(x) - i\pi g \rho^H(x) + ig\, \partial_x \ln \sqrt{\rho(x)}$$
$$u^-(x \pm i0) = \mp \pi g \rho(x) + i\pi g \rho^H(x)$$

- The result of Sasha et al. with the logarithmic terms
- The rest as in Sasha's talk: inserting in EOM for u^+, u^- we obtain Calogero hydro equations

Finally some generalized solitons in external potentials

One-soliton solution: a single $z_\alpha = Z$ satisfying the equation

$$\ddot{z} + V'(z) = 0$$

Fixes both $x_j(t)$ in N-body system and $u^{\pm}(x,t)$ in hydro limit

Many-body:

$$\begin{aligned} \sum_{k(\neq j)} f(x_j - x_k) + w(x_j) &= \mathrm{Re}\,(f(x_j - z)) \\ \dot{x}_j &= \mathrm{Im}\,(f(x_j - z)) \end{aligned}$$

Hydro:

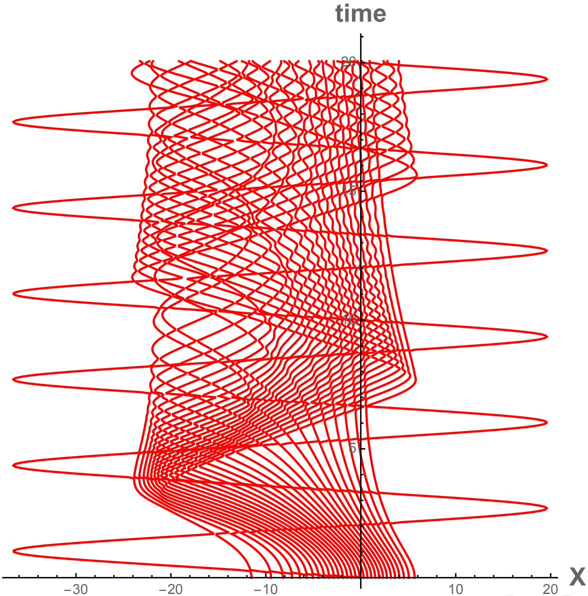
$$u^{+}(x,t) = \frac{ig}{x - z(t)} + iw(x)$$

or

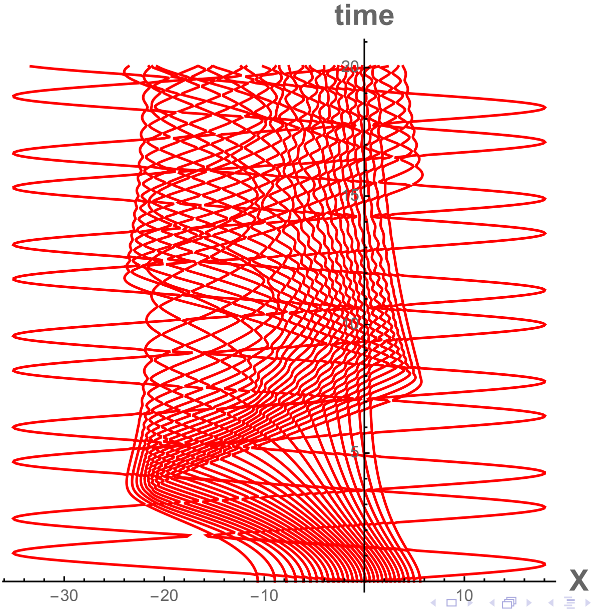
$$v - ig(\pi\rho^H - \partial_x \log\sqrt{\rho}) = \frac{ig}{x - z} + iw(x)$$

- Above eqiations can in principle be solved for x_j, ρ and v
- Numerical solutions easily obtaonable
- Et voilà some pictures:

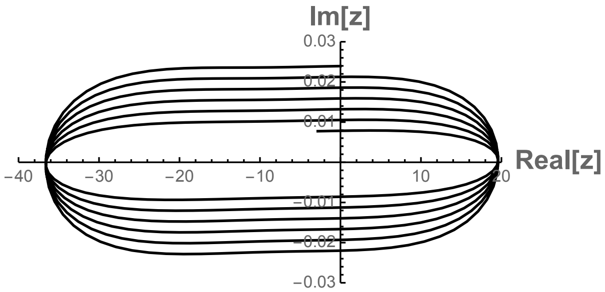
One soliton solution for quartic potential

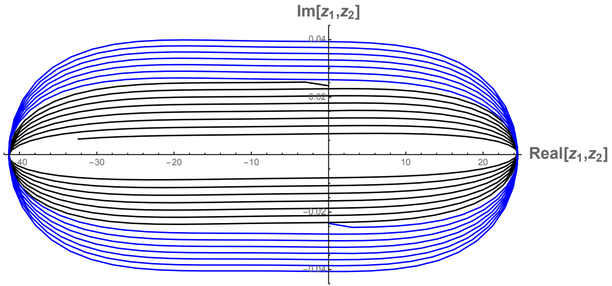


Two soliton solution for quartic potential



$z(t)$ on the complex plane for one and two solitons





Conclusions and Outlook

A grand tour in Calogero land, but we left out many sights:

- Foldings and new systems obtained through them (anisotropic, noncommutative etc)
- Density correlations and their analytic challenges (Jack polynomials, integral expressions etc.)
- Connection with Yang-Mills and Chern-Simons field theories
- "Relativistic" Calogero systems (Ruijsenaars-Schneider model)
- Duality and its manifestations; etc. etc.

Many open questions remaining

- Formulation encompassing Matrix and Operator ones
- Correlations for irrational values of ℓ
- "Continuous" expression of density correlations
- Full solution of Elliptic system
- Anything at all on the noncommutative model
- (Use your imagination...)

...sometimes fun is in the dark, unexplored corners of an old subject

Thank You!

Next page includes some basic references for further study

For further study

- Alexios P. Polychronakos, Journal of Physics A: Mathematical and General, Volume 39, Number 41, *The physics and mathematics of Calogero particles*; Les Houches 1998 Lectures, https://arxiv.org/abs/hep-th/9902157, *Generalized statistics in one dimension*.
- M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **71**, pp. 313-400, (1981), *Classical integrable finite-dimensional systems related to Lie algebras*; Phys. Rep. **94**, Issue 6, pp. 313-404 (1983), *Quantum Integrable Systems Related to Lie Algebras*.

Some original papers:

- Alexios P; Polychronakos, Nucl.Phys. B324 (1989) 597-622, *Nonrelativistic Bosonization and Fractional Statistics*; Phys.Rev.Lett. 69 (1992) 703-705, *Exchange operator formalism for integrable systems of particles*; Nucl.Phys. B419 (1994) 553-566, *Exact Spectrum of SU(n) Spin Chain with Inverse-Square Exchange*