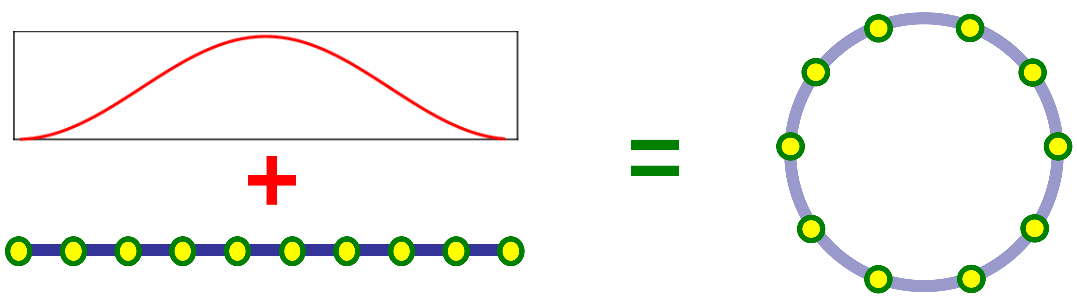
Sine-square deformations of one-dimensional critical systems

Hosho Katsura
(Dept. Phys., UTokyo)

東京大学 THE UNIVERSITY OF TOKYO



- H.K., *J. Phys. A: Math. Theor*. **44**, 252001; **45**, 115003 (2011).
- I. Maruyama, H.K., & T. Hikihara*, Phys. Rev. B* **84**, 165132 (2011).
- S. Tamura and H. Katsura, Prog. Theor. Exp. Phys, 113A01 (2017).

Outline

1. Introduction

- What is SSD (sine-square deformation)?
- What is special about SSD?

2. Ground state of solvable models with SSD

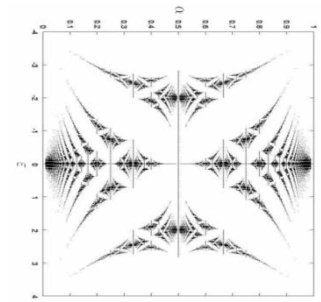
3. Excited states of solvable models with SSD

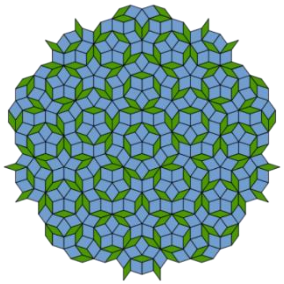
4. Summary

Many-body problems with inhomogeneities

■ Disorder and inhomogeneity in cond-mat

- Hofstadter butterfly, Wannier-Stark
- Quasi-periodic systems
- Impurity and boundary





The presence of inhomogeneity and/or boundary usually breaks solvability/integrability…

Main difficulty: Single-particle problem is already nontrivial.
What happens when the interaction is switched on?

■ Today's talk

- A new class of inhomogeneous ***but*** solvable models
  Accidentally found by numerics. Hidden CFT structure.
- Abandon "from few to many" approach!
  Solve many-body problem ***without*** using single-particle solutions.

What is SSD (sine-square deformation)?

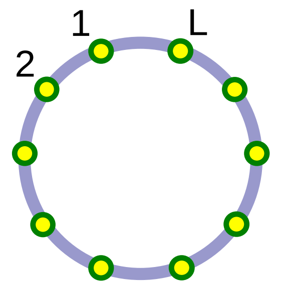
Chat with Tolya Kirrilov and Nishino

Workshop "From DMRG to TNF"@YITP (Oct. 2010)

N: *"Hikihara-san found some interesting system."*
K&K: *"Is that solvable or integrable?"*

- Two `conventional' boundary conditions
  - Periodic chain

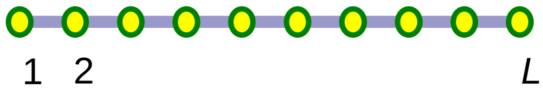
$$\mathcal{H}_0 = \sum_{j=1}^{L} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

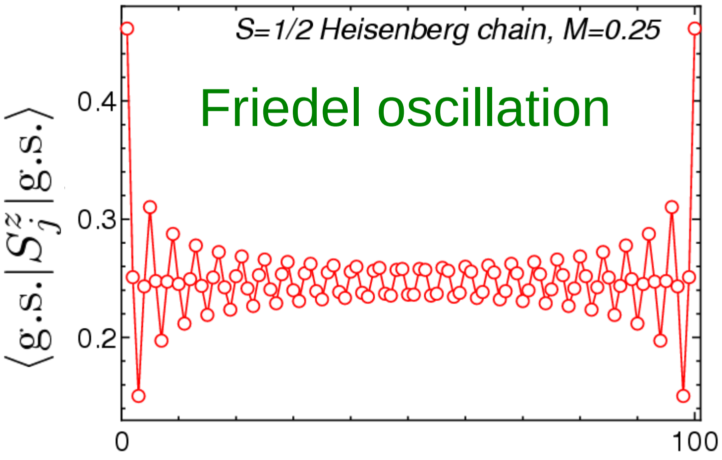


Any observable is translation invariant

  - Open chain

$$\mathcal{H}_{\rm open} = \sum_{j=1}^{L-1} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$



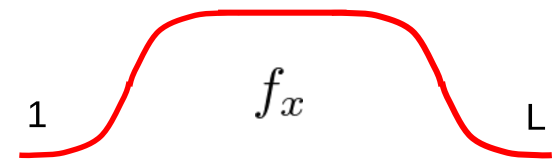


What is SSD? (contd.)

- Smooth boundary condition

Vekic & White, *PRL* **71** (1993); *PRB* **53** (1996).

$$\mathcal{H}_f = \sum_{j=1}^{L-1} \underline{f_{j+1/2}}\, \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$



Energy scale of local Hamiltonian at *x* is modified by *fx*.
This b.c. reduces the boundary effect (to some extent).

- Sine-square deformation

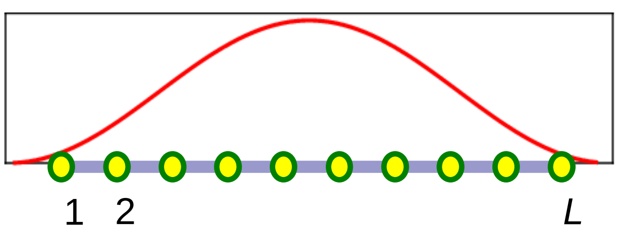
= Smooth b.c. with a specific *fx*.

A. Gendiar *et al.*, *PTP* **122**; **123** (2009-2010)

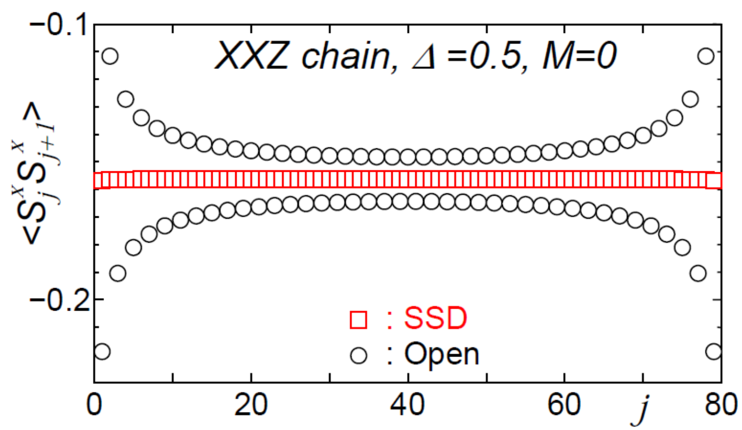
Ex.) Heisenberg chain with SSD

Hikihara & Nishino, *PRB* **83** (2011)

$$\mathcal{H}_{\rm SSD} = \sum_{j=1}^{L-1} \sin^2(\pi j/L)\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$



No boundary effect in g.s.!



What is special about SSD?

- Suppression of boundary effects
  - Negligible Friedel oscillation, uniform g.s. correlations
  - Observed in ***1D critical systems***
    XXZ, Hubbard, Kondo-lattice (Shibata-Hotta, *PRB* (2011)), …
- Scaling of entanglement entropy

$$\mathcal{S}^{\mathrm{PBC}}(\ell, L) = \frac{c}{3}\ln\left[\frac{L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)\right] + s_1$$

$$\mathcal{S}^{\mathrm{OBC}}(\ell, L) = \frac{c}{6}\ln\left[\frac{2L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)\right] + \frac{s_1}{2} + \ln(g)$$

$$\mathcal{S}^{\mathrm{SSD}} \simeq \mathcal{S}^{\mathrm{PBC}}$$

- Wavefunction overlap
  Overlap between the g.s. of systems with PBC and SSD is almost 1.

$$\langle \Psi_{\mathrm{SSD}} | \Psi_{\mathrm{PBC}} \rangle \simeq 1$$

"Conjecture"
G.S. of $\mathcal{H}_{\mathrm{SSD}}$ = G.S. of $\mathcal{H}_{\mathrm{PBC}}$

Main results
☑ XY chain, Ising chain
☑ Massless Dirac, CFTs, …

Outline

1. Introduction

2. **Ground state of solvable models with SSD**

- Definitions
- Free fermion chain with SSD
- Other examples (spin chains, Dirac fermions, CFT, …)

3. Excited states of solvable models with SSD

4. Summary

Definitions

■ Uniform and chiral Hamiltonians

Consider a lattice model on a chain of length *L*, or
a continuous field theory on a ring of length ℓ. (PBC imposed)

	Lattice model	Field theory
Uniform	$\mathcal{H}_0 = \sum_{j=1}^{L} h_j + \sum_{j=1}^{L} h_{j,j+1}$	$\mathcal{H}_0 = \int_0^\ell h(x)\,dx$
Chiral	$\mathcal{H}_\pm = \sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} h_j + \sum_{j=1}^{L} e^{\pm i\delta j} h_{j,j+1}$ ← $\delta = \frac{2\pi}{L}$	$\mathcal{H}_\pm = \int_0^\ell e^{\pm i\delta x} h(x)\,dx$ ← $\delta = \frac{2\pi}{\ell}$

■ Sine-square deformed (SSD) Hamiltonian

$$\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

Sites 1 and L do not couple!

Ex)

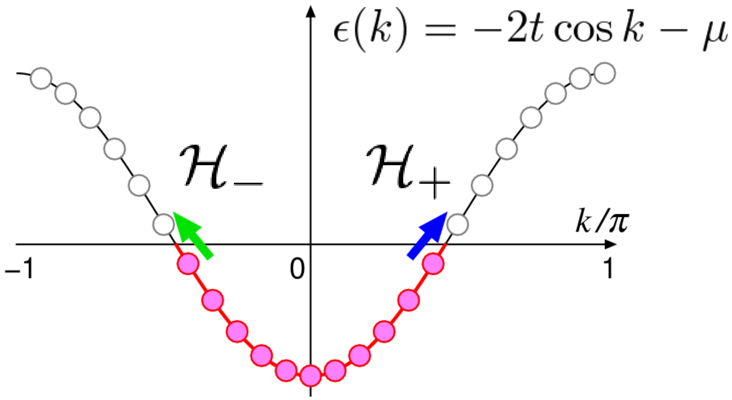
$$\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\sum_{j=1}^{L}\left(1 - \frac{1}{2}e^{i\delta j} - \frac{1}{2}e^{-i\delta j}\right)\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1} = \sum_{j=1}^{L}\sin^2\left(\frac{\pi}{L}j\right)\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}$$

Free fermion chain with SSD (1)

■ Uniform and chiral Hamiltonians

$$\mathcal{H}_0 = -t\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L} c_j^\dagger c_j$$

$c_j/c_j^\dagger$: annihilation/creation of fermion at *j*.



Fourier.tr. $$\mathcal{H}_0 = \sum_k \epsilon(k) c_k^\dagger c_k$$

Ground state of $\mathcal{H}_0$:
Fermi sea ($\epsilon(k) < 0$ occupied)

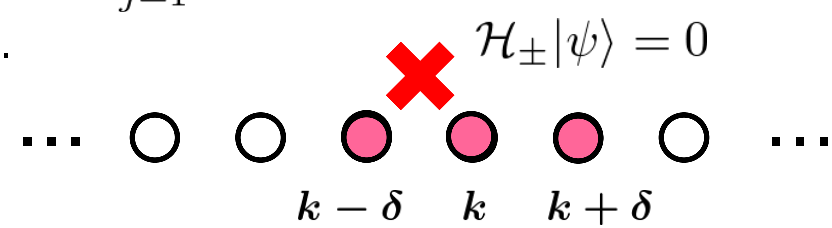
$$\mathcal{H}_0|\mathrm{FS}\rangle = E_g|\mathrm{FS}\rangle$$

Chiral Hamiltonian ($\delta = \frac{2\pi}{L}$)

$$\mathcal{H}_\pm = -t\sum_{j=1}^{L} e^{\pm i\delta j}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L} e^{\pm i\delta(j-1/2)} c_j^\dagger c_j$$

Momentum rep.

$$\mathcal{H}_\pm = e^{\mp i\delta/2}\sum_k \epsilon(k\mp\delta/2) c_k^\dagger c_{k\mp\delta}$$



If $\epsilon(k_\mathrm{F} + \delta/2) = \epsilon(-k_\mathrm{F} - \delta/2) = 0$, then $\mathcal{H}_\pm|\mathrm{FS}\rangle = 0$. ($\because$ $(c_k^\dagger)^2 = 0$)

Free fermion chain with SSD (2)

■ SSD Hamiltonian

$$\mathcal{H}_{\rm SSD} = -t\sum_{j=1}^{L-1}\sin^2\left(\frac{\pi}{L}j\right)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\sum_{j=1}^{L-1}\sin^2\left[\frac{\pi}{L}\left(j-\frac{1}{2}\right)\right]c_j^\dagger c_j$$

In terms of $\mathcal{H}_0$ & $\mathcal{H}_\pm$, $\mathcal{H}_{\rm SSD} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$

Fermi sea is annihilated by chiral Hamiltonians! $\mathcal{H}_\pm|{\rm FS}\rangle = 0$

$$\mathcal{H}_{\rm SSD}|{\rm FS}\rangle = \left[\frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)\right]|{\rm FS}\rangle = \frac{E_g}{2}|{\rm FS}\rangle$$

Fermi sea is an exact eigenstate of $\mathcal{H}_{\rm SSD}$!

■ Uniqueness of the ground state

Fermi sea is ***the unique*** g.s. of $\mathcal{H}_{\rm SSD}$. $\mathcal{H}_0$ & $\mathcal{H}_{\rm SSD}$ share the same g.s.

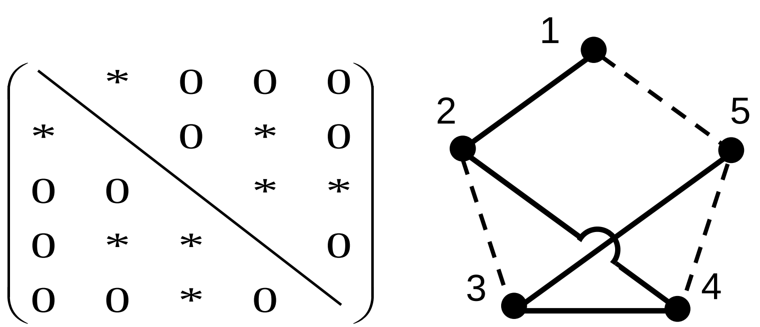
Proof. Free-fermion chain → XY spin chain (via Jordan-Wigner tr.)

Perron-Frobenius theorem tells: (i) the ground state of $\mathcal{H}_{\rm SSD}$ is unique.
(ii) it has nonvanishing overlap with $|{\rm FS}\rangle$, the ground state of $\mathcal{H}_0$.

Application of Perron-Frobenius theorem

Theorem (Perron-Frobenius).
Let M be an $N \times N$ real symmetric matrix with the properties
(i) $m_{i,j} \leq 0$ for any $i \neq j$,
(ii) all $i \neq j$ are connected via nonzero matrix elements of M.
Then the lowest eigenvalue of M is nondegenerate and the corresponding eigenvector $\boldsymbol{v} = (v_1, \ldots, v_N)$ can be taken to satisfy $v_i > 0$ for all i.

$\mathcal{H}_{\mathrm{SSD}}$ satisfy both (i) and (ii)
→ g.s. is unique



$|\mathrm{FS}\rangle$ (in spin reps.) which is an eignstate of $\mathcal{H}_{\mathrm{SSD}}$ can also be taken to satisfy $v_i > 0$ for all i.
This state cannot be orthogonal to the SSD g.s.
→ $|\mathrm{FS}\rangle$ is the unique ground state of $\mathcal{H}_{\mathrm{SSD}}$.

Real-space picture

■ Determinant identity

1-particle eigenstates of $\mathcal{H}_0$: $\phi_k(j) = e^{\mathbf{i}kj}$ (plane waves)
1-particle eigenstates of $\mathcal{H}_{\mathrm{SSD}}$: $\psi_k(j) = ?$

When the states are occupied up to the Fermi level E_F.

$$\det[\psi_k(j)]_{k,j=1,\ldots,N} = \det[\phi_k(j)]_{k,j=1,\ldots,N}$$

Solved many-body problem ***without*** using 1-particle solutions!

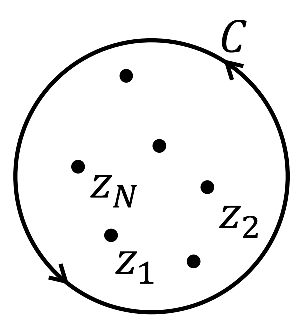
■ Curious identity

First quantization picture: Fermi sea = Vandermonde det.

$$\Delta(z_1, \ldots, z_N) = \prod_{1 \le i < j \le N} (z_i - z_j) \qquad z_j = \exp\left(i \frac{2\pi}{L} x_j\right)$$

$\mathcal{H}_\pm |\mathrm{FS}\rangle = 0$ implies the following identity:

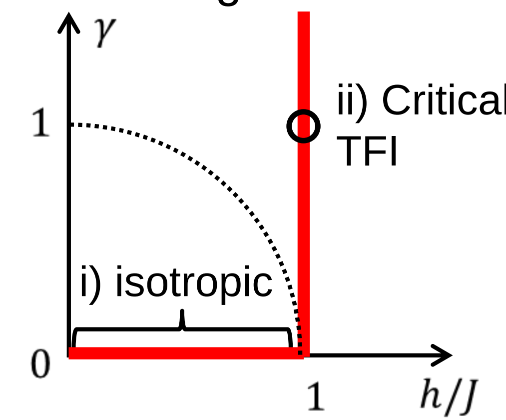
$$\sum_{j=1}^{N} z_j \prod_{k(\neq j)} \frac{z_j - t z_k}{z_j - z_k} = \sum_{j=1}^{N} z_j$$



$$f(z) = \prod_{1 \le j \le N} \frac{z - t z_j}{z - z_j}$$

for any set of $\{z_1, \ldots, z_N\}$ and t.

Anisotropic XY chain



■ Uniform Hamiltonian

$$\mathcal{H}_0 = -J\sum_{j=1}^{L}[(1+\gamma)S_j^x S_{j+1}^x + (1-\gamma)S_j^y S_{j+1}^y] - h\sum_{j=1}^{L} S_j^z,$$

Jordan-Wigner, Fourier, Bogoliubov. tr.

$$\mathcal{H}_0 = \sum_{k\in\mathcal{K}} \epsilon_0(k)\left(d_k^\dagger d_k - \frac{1}{2}\right)$$

Ground state of $\mathcal{H}_0$: $d_k|0\rangle = 0$ for all *k*.

■ Chiral and SSD Hamiltonians

Chiral Hamiltonian ($\delta = \frac{2\pi}{L}$) in momentum space

$$\mathcal{H}_\pm = \frac{1}{2}e^{\mp i\delta/2}\sum_{k\in\mathcal{K}}\left[\epsilon_\pm(k)d_k^\dagger d_{k\mp\delta} - i\eta_\pm(k)d_k^\dagger d_{-k\pm\delta}^\dagger + i\eta_\pm(k)d_{-k}d_{k\mp\delta} - \epsilon_\pm(k)d_{-k}d_{k\mp\delta}^\dagger,\right]$$

$\eta_\pm(k) = 0$ for all *k* when i) $\gamma = 0$, ii) $\gamma = 1,\ h/J = 1$, in which case $\mathcal{H}_\pm|0\rangle = 0$

$$\mathcal{H}_{\rm SSD}|0\rangle = \left[\frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)\right]|0\rangle = \frac{E_g}{2}|0\rangle$$

$|0\rangle$ is ***the unique*** ground state of $\mathcal{H}_{\rm SSD}$ (Perron-Frobenius theorem).

Critical Potts chain

Uniform Hamiltonian

$\mathbf{Z}_3$ Pauli operators

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$H_0 = -f\sum_{j=1}^{L}(\tau_j + \tau_j^\dagger) - J\sum_{j=1}^{L}(\sigma_j^\dagger\sigma_{j+1} + \sigma_j\sigma_{j+1}^\dagger)$$

PBC: $\sigma_{L+1} = \sigma_1$

The model is critical (c=4/5), self-dual, and integrable when f=J.
However, it is not reducible to free fermions. (parafermions?)

SSD Hamiltonian

$$H_{\mathrm{SSD}} = -f\sum_{j=1}^{L}\sin^2\left(\frac{\pi(j-1/2)}{L}\right)(\tau_j + \tau_j^\dagger) - J\sum_{j=1}^{L-1}\sin^2\left(\frac{\pi j}{L}\right)(\sigma_j^\dagger\sigma_{j+1} + \sigma_j\sigma_{j+1}^\dagger)$$

Numerical result

At the critical point (f=J), the overlap between the g.s. of uniform and SSD Hamiltonians is remarkably close to 1!

$$1 - \langle\psi_{\mathrm{SSD}}|\psi_{\mathrm{PBC}}\rangle \sim 10^{-5}$$

Numerical diagonalization by *Mathematica* up to 16 sites (3^{16}= 43,046,721).

Massless Dirac fermions

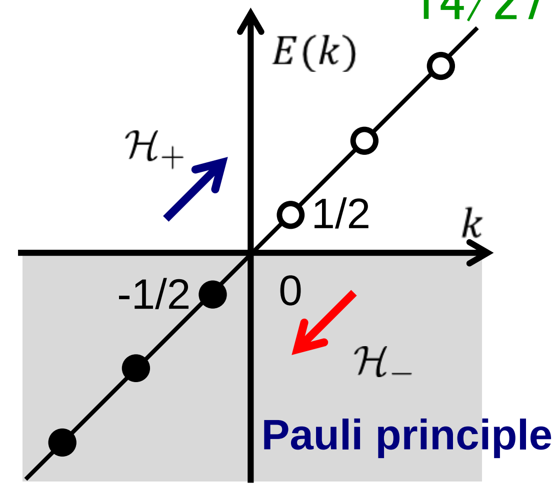
■ Uniform Hamiltonian

Ring of length ℓ. APBC: $\psi_R(x+\ell) = -\psi_R(x)$

$$\mathcal{H}_0 = -i\frac{v_\mathrm{F}}{2\pi}\int_0^\ell dx\ :\psi_R^\dagger(x)\frac{d}{dx}\psi_R(x):$$

Fourier.tr.

$$\mathcal{H}_0 = \frac{2\pi}{\ell}v_\mathrm{F}\sum_{n\in\mathbb{Z}+\frac{1}{2}} n :\psi_{R,n}^\dagger\psi_{R,n}:$$



Ground state of $\mathcal{H}_0$: Dira sea (E<0 occupied) $\mathcal{H}_0|\mathrm{DS}\rangle = 0$

■ Chiral and SSD Hamiltonians

$$(\mathcal{H}_\pm)^\dagger = \mathcal{H}_\mp$$

$$\mathcal{H}_\pm = -i\frac{v_\mathrm{F}}{2\pi}\int_0^\ell dx\, e^{\pm i\delta x} :\psi_R^\dagger(x)\frac{d}{dx}\psi_R(x): \pm\frac{\pi v_\mathrm{F}}{2\ell}\frac{1}{2\pi}\int_0^\ell dx\, e^{\pm i\delta x} :\psi_R^\dagger(x)\psi_R(x):$$

$$\Rightarrow \mathcal{H}_\pm = \frac{2\pi}{\ell}v_\mathrm{F}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\left(n\pm\frac{1}{2}\right)\psi_{R,n\pm1}^\dagger\psi_{R,n}$$

$$\mathcal{H}_\pm|0\rangle = 0$$

Dirac sea $|\mathrm{DS}\rangle$ is ***a*** ground state of $\mathcal{H}_\mathrm{SSD}$.

Proof. $\mathcal{H}_\mathrm{SSD}$ is positive semi-definite ($\langle\Psi|\mathcal{H}_\mathrm{SSD}|\Psi\rangle \geq 0$ for any state),

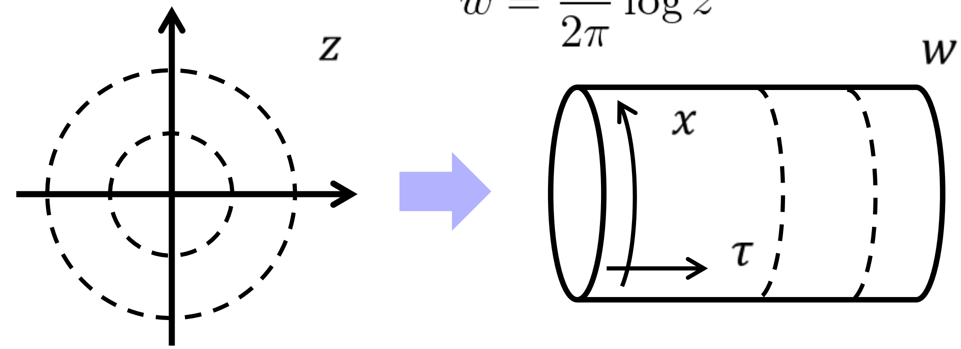
which follows from $\mathcal{H}_\mathrm{SSD} = \sum_{n\in\mathbb{Z}+\frac{1}{2},n>0}(\alpha_{R,n}^\dagger\alpha_{R,n} + \beta_{R,n}\beta_{R,n}^\dagger)$

$$\alpha_{R,n} = \psi_{R,n} - \psi_{R,n+1}$$
$$\beta_{R,n} = \psi_{R,-n} - \psi_{R,-n-1}$$

(1+1) d Conformal field theories

■ Uniform Hamiltonian

$$w = \frac{\ell}{2\pi}\log z$$



$$\mathcal{H}_0 = \int_0^{\ell} \frac{dx}{2\pi}(T_{\rm cyl}(w) + \overline{T}_{\rm cyl}(\overline{w}))$$

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

Ground state: vacuum state, $|{\rm vac}\rangle = |0\rangle \otimes |\overline{0}\rangle \quad (L_n|0\rangle = 0,\ n \geq -1)$

■ Chiral and SSD Hamiltonians ($\delta = \frac{2\pi}{\ell}$)

$$\mathcal{H}_{\pm} = \int_0^{\ell} \frac{dx}{2\pi}(e^{\pm\delta w}T_{\rm cyl}(w) + e^{\mp\delta\overline{w}}\overline{T}_{\rm cyl}(\overline{w}))$$

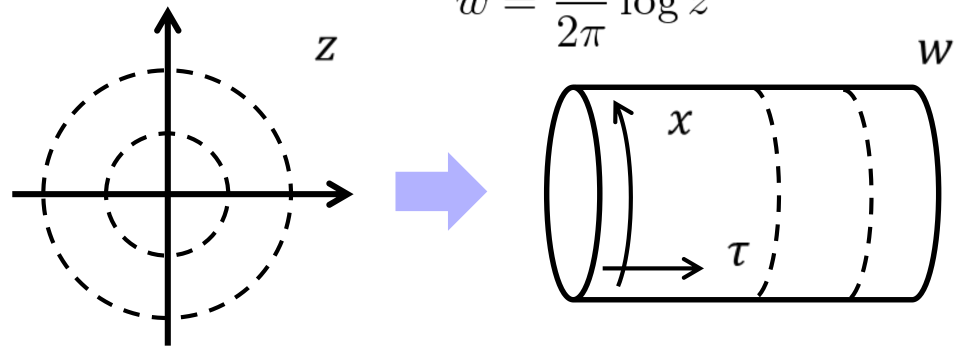
$$\mathcal{H}_{\rm SSD} = \mathcal{H}_L + \mathcal{H}_R - \frac{\pi c}{12\ell} \qquad \mathcal{H}_L = \frac{\pi}{\ell}\left(L_0 - \frac{L_{+1} + L_{-1}}{2}\right)$$

■ Vacuum state

$|0\rangle$ is ***the unique normalizable*** *E*=0 state of $\mathcal{H}_{\rm SSD}$ for unitary CFTs. (*SL*(2,**C**) invariance: $L_0|0\rangle = L_{\pm 1}|0\rangle = 0$)

(1+1) d Conformal field theories

$$w = \frac{\ell}{2\pi}\log z$$



■ Uniform Hamiltonian

$$\mathcal{H}_0 = \frac{2\pi}{\ell}(L_0 + \overline{L}_0) - \frac{\pi c}{6\ell}$$

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

Ground state: vacuum state, $|\mathrm{vac}\rangle = |0\rangle \otimes |\overline{0}\rangle \quad (L_n|0\rangle = 0,\ n \geq -1)$

■ Chiral and SSD Hamiltonians ($\delta = \frac{2\pi}{\ell}$)

$$\mathcal{H}_\pm = \frac{2\pi}{\ell}(L_{\pm 1} + \overline{L}_{\mp 1})$$

$$\mathcal{H}_{\mathrm{SSD}} = \mathcal{H}_L + \mathcal{H}_R - \frac{\pi c}{12\ell} \qquad \mathcal{H}_L = \frac{\pi}{\ell}\left(L_0 - \frac{L_{+1} + L_{-1}}{2}\right)$$

■ Vacuum state

$|0\rangle$ is ***the unique normalizable*** *E*=0 state of $\mathcal{H}_{\mathrm{SSD}}$ for unitary CFTs. (*SL*(2,**C**) invariance: $L_0|0\rangle = L_{\pm 1}|0\rangle = 0$)

Uniqueness of the vacuum (1)

■ Invariant subspaces (unitary CFT, $c, h \geq 0$)

Verma module $V(c, h)$ $\cong$ Direct sum of $SL(2, \mathbf{R})$ invariant subspaces

$|\psi_s^{(n)}\rangle$: Highest-weight state of subspace *s*.

$$L_0|\psi_s^{(n)}\rangle = (h+n)|\psi_s^{(n)}\rangle, \quad L_1|\psi_s^{(n)}\rangle = 0, \qquad \langle\psi_s^{(n)}|\psi_{s'}^{(n')}\rangle = \delta^{n,n'}\delta_{s,s'}$$

Descendants:

$$|\psi_{s,0}^{(n)}\rangle = |\psi_s^{(n)}\rangle, \quad |\psi_{s,m+1}^{(n)}\rangle = \frac{1}{\sqrt{f_{m+1}}} L_{-1}|\psi_{s,m}^{(n)}\rangle \qquad (f_m = m(2h+2n+m-1))$$

They form an orthonormal basis.

$$\langle\psi_{s,m}^{(n)}|\psi_{s,m'}^{(n)}\rangle = \delta_{m,m'}$$

Ex.)

$$|\psi_{1,0}^{(0)}\rangle = |h\rangle$$

$$|\psi_{1,1}^{(0)}\rangle \propto L_{-1}\,|\psi_{1,0}^{(0)}\rangle$$

$$|\psi_{1,2}^{(0)}\rangle \propto L_{-1}\,|\psi_{1,1}^{(0)}\rangle \quad \overset{\perp}{\longleftrightarrow} \quad |\psi_{2,0}^{(0)}\rangle \propto 3(L_{-1})^2|h\rangle - (2+4h)L_{-2}|h\rangle$$

$$|\psi_{1,3}^{(0)}\rangle \propto L_{-1}\,|\psi_{1,2}^{(0)}\rangle \quad \overset{\perp}{\longleftrightarrow} \quad |\psi_{2,1}^{(0)}\rangle \propto L_{-1}\,|\psi_{2,0}^{0}\rangle$$

$$\vdots \qquad\qquad\qquad \vdots$$

NOTE) Need to discard null states. $\{|0\rangle\}$ is a one-dimensional subspace.
All the other subspaces are infinite dimensional.

Uniqueness of the vacuum (2)

■ Tridiagonal expression

In each subspace (≠ $\{|0\rangle\}$), $\langle\psi^{(n)}_{s,m}|\,\mathcal{H}_{\rm SSD}\,|\psi^{(n)}_{s,m'}\rangle$ takes the form:

$$\left(\begin{array}{ccccc} h+n & -\frac{\sqrt{f_1}}{2} & & & \\ -\frac{\sqrt{f_1}}{2} & h+n+1 & -\frac{\sqrt{f_2}}{2} & & \\ & -\frac{\sqrt{f_2}}{2} & h+n+2 & -\frac{\sqrt{f_3}}{2} & \\ & & -\frac{\sqrt{f_3}}{2} & h+n+3 & \ddots \\ & & & \ddots & \ddots \end{array}\right)$$

$(f_m = m(2h+2n+m-1))$

Looks like corner Hamiltonian!

(i) The matrix is positive semi-definite if $h \geqq 0$

∵ It is a direct sum of p.s.d. 2x2 matrices.

NOTE) Need an appropriate truncation.

$$\left(\begin{array}{cc} h+n+\frac{m}{2} & -\frac{\sqrt{f_{m+1}}}{2} \\ -\frac{\sqrt{f_{m+1}}}{2} & \frac{m+1}{2} \end{array}\right)_{m,m+1}$$

(ii) The (normalizable) vacuum is unique

Proof by contradiction. Suppose $|\psi_0\rangle \neq |0\rangle$ is another E=0 state of $\mathcal{H}_{\rm SSD}$.

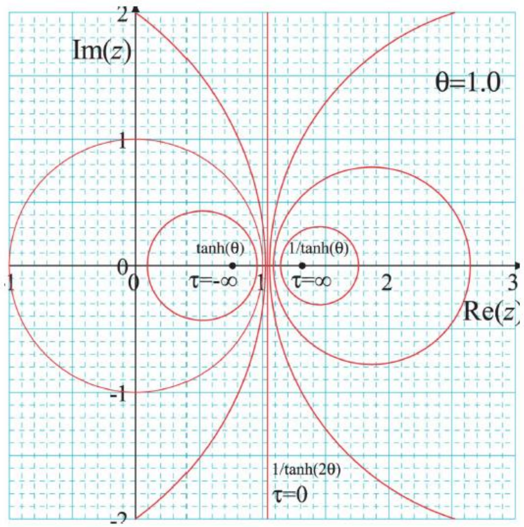
Then we can determine the coefficients c_m recursively.

But $|\psi_0\rangle$ so obtained is unnormalizable.

$$|\psi_0\rangle = \sum_{m=0}^{\infty} c_m |\psi^{(n)}_{s,m}\rangle$$

Relation to Mobius quantization

■ Mobius coordinate (Okunishi, *PTP*, 063A02 (2016))



Mobius transformatin: $u = -\dfrac{\sinh\theta - z\cosh\theta}{\cosh\theta - z\sinh\theta}$

Virasoro generators

$$\mathcal{L}_n(\theta) = \oint_t \frac{\mathrm{d}u}{2\pi \mathrm{i}} u^{n+1} T^{(u)}(u) = \oint_t \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{(z-\tanh(\theta))^{n+1}}{(z-1/\tanh(\theta))^{n-1}} T(z)$$

Their commutation relations are the same as those of the original Virasoro.

$$\mathcal{L}_0(\theta) = \cosh(2\theta)\, L_0 - \sinh(2\theta)\, \frac{L_1 + L_{-1}}{2}$$

$$\lim_{\theta\to\infty} \frac{\mathcal{L}_0(\theta)}{\cosh 2\theta} = L_0 - \frac{L_1 + L_{-1}}{2}$$ SSD Hamiltonian!

Cf.) dipolar quantization, Ishibashi-Tada, *JPA* **48**; IJMPA **31** (2016).

■ Unitary equivalence (Tamura-Katsura, *PTP*, 113A01 (2017))

$$\mathcal{L}_n(\theta) = e^{-\theta(L_1 - L_{-1})} L_n e^{\theta(L_1 - L_{-1})}$$

Easy to guess from the Mobius tr., but not so easy to prove…
Explain unnormalizable zero-energy states. $e^{L_{-1}}|h\rangle, \quad \sum_{n>1} L_{-n}|0\rangle$

N=2 SCFT

$J(z)$: Δ=1 U(1) current, $G^{\pm}(z)$: Δ=3/2 fermionic fields

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},$$
$$[L_m, G_r^{\pm}] = \left(\frac{m}{2} - r\right) G_{m+r}^{\pm},$$
$$[L_m, J_n] = -nJ_{m+n},$$
$$[J_m, G_r^{\pm}] = \pm G_{m+r}^{\pm},$$
$$\{G_r^{\pm}, G_s^{\mp}\} = 2L_{r+s} \pm (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0},$$
$$\{G_r^{+}, G_s^{+}\} = \{G_r^{-}, G_s^{-}\} = 0,$$
$$[J_m, J_n] = \frac{c}{3} m \delta_{m+n,0},$$

$m, n \in \mathbb{Z},$

$r, s \in \mathbb{Z} + \alpha$

$\alpha = 0 \quad (\mathrm{Ramond})$

$\alpha = \frac{1}{2} \quad (\mathrm{Neveu-Schwarz})$

- H_0 in Ramond sector

$$\frac{2\pi}{\ell}\left(L_0 - \frac{c}{24}\right) = \frac{\pi}{\ell}\{G_0^{+}, G_0^{-}\}$$

- H_{SSD} in Neveu-Schwarz sector

$$\mathcal{H}_{\mathrm{L}} = \frac{\pi}{\ell}\left(L_0 - \frac{L_1 + L_{-1}}{2}\right) = \frac{\pi}{2\ell}\{Q, Q^{\dagger}\}$$

- 1-parameter family connecting R and NS (spectral flow)?

Supersymmetry

$$Q^{\dagger} = \frac{G_{\frac{1}{2}}^{+} - G_{-\frac{1}{2}}^{+}}{\sqrt{2}}$$
$$Q^2 = (Q^{\dagger})^2 = 0$$

Outline

1. Introduction

2. Ground state of solvable models with SSD

3. **Excited states of solvable models with SSD**

- What about excited states of SSD?
- Free-fermion chain with SSD
- Further steps towards exact solution

4. Summary

What about excited states of SSD?

■ Gapped or gapless?

- Lieb-Schultz-Mattis argument (*Ann. Phys*. **16** (1961)

G.S. of $H_{\rm SSD}$ (XY spin chain) Trial state

$$|\Psi_0\rangle \quad \Rightarrow \quad |\Psi_1\rangle := U|\Psi_0\rangle, \quad U = \exp\left(i\sum_{j=1}^{L}\frac{2\pi}{L}jS_j^z\right)$$

Orthogonality

$|\Psi_0\rangle$ is translation invariant.

$$\langle\Psi_0|\Psi_1\rangle = \langle\Psi_0|U|\Psi_0\rangle$$
$$= \langle\Psi_0|TUT^{-1}|\Psi_0\rangle = -e^{-2\pi i M/L}\langle\Psi_0|\Psi_1\rangle$$

$$\langle\Psi_0|\Psi_1\rangle = 0$$

unless M (total S^z) is $\pm L/2$.

Upper bound on the gap

$$\Delta E = \langle\Psi_1|H_{\rm SSD}|\Psi_1\rangle - \langle\Psi_0|H_{\rm SSD}|\psi_0\rangle$$
$$= \langle\Psi_0|U^\dagger H_{\rm SSD}U - H_{\rm SSD}|\Psi_0\rangle$$
$$\le \frac{\pi^2 J}{L} + O(1/L^2)$$

because $\sin^2(\pi j/L)$ is $O(1)$ for all j.

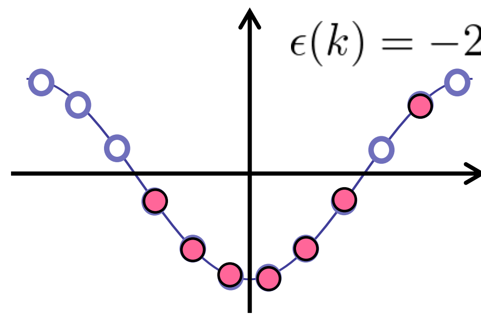
Suggests that 1d critical system with SSD is still critical.
But is the upper bound optimal? → **NO!**

Free-fermion chain with SSD

■ Hamiltonian (reminder)

$$\mathcal{H}_0 = -t\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)\,.$$

$$\mathcal{H}_{\mathrm{SSD}} = -t\sum_{j=1}^{L-1}\sin^2\left(\frac{\pi}{L}j\right)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)\,.$$



Fermi sea
+ one extra fermion

Ansatz state: $|\Psi\rangle = \left(\sum_{k\in\circ}\psi_k c_k^\dagger\right)|\mathrm{FS}\rangle$

Using $(\mathcal{H}_{\mathrm{SSD}} - E_g/2)|\mathrm{FS}\rangle = 0,$
we get Harper-like eq. in *k*-space (*m*=0,1,…,*L*/2-1):

$$-\sin\left(\frac{2\pi}{L}m\right)\psi_{m-1} + 2\sin\left[\frac{2\pi}{L}\left(m+\frac{1}{2}\right)\right]\psi_m - \sin\left[\frac{2\pi}{L}(m+1)\right]\psi_{m+1} = \varepsilon\psi_m$$

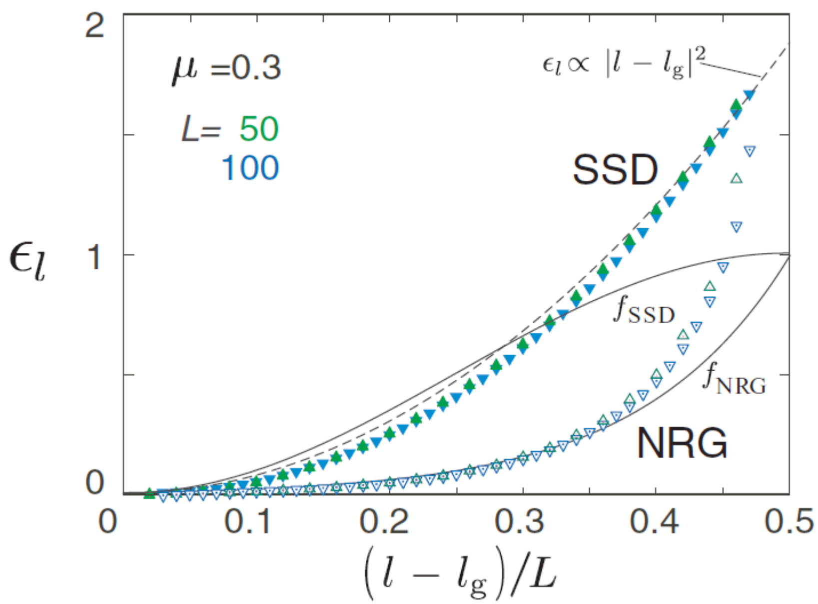
- Scaling of excitation energy

  A simple variational ansatz shows $\varepsilon \leq \frac{2\pi}{L^2} + O\left(\frac{1}{L^3}\right)$ $\quad \forall\psi_m = \sqrt{\frac{2}{L}}$

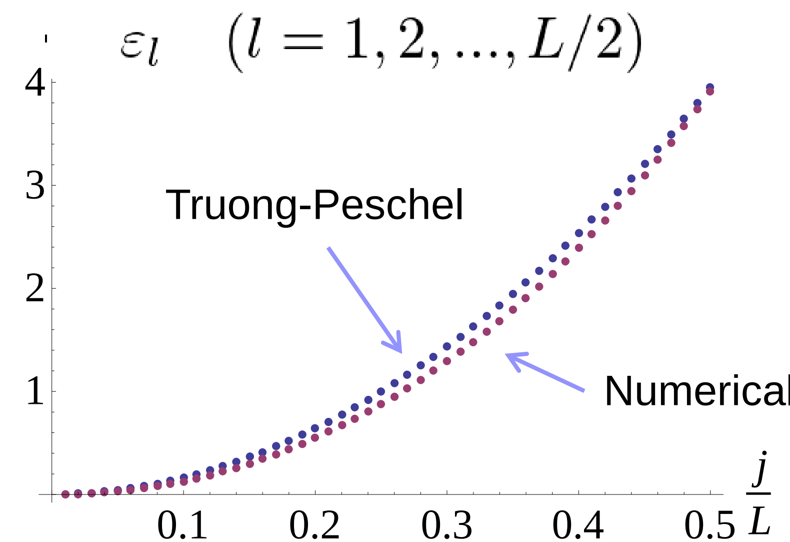
  Very low-energy states! Breakdown of CFT scaling (1/*L*)

Scaling property of excitation energy

Hotta *et al.*, *PRB* (2013)
μ=0.3, L=50 & 100.



Numerical v.s. analytical
μ=0, L=100.

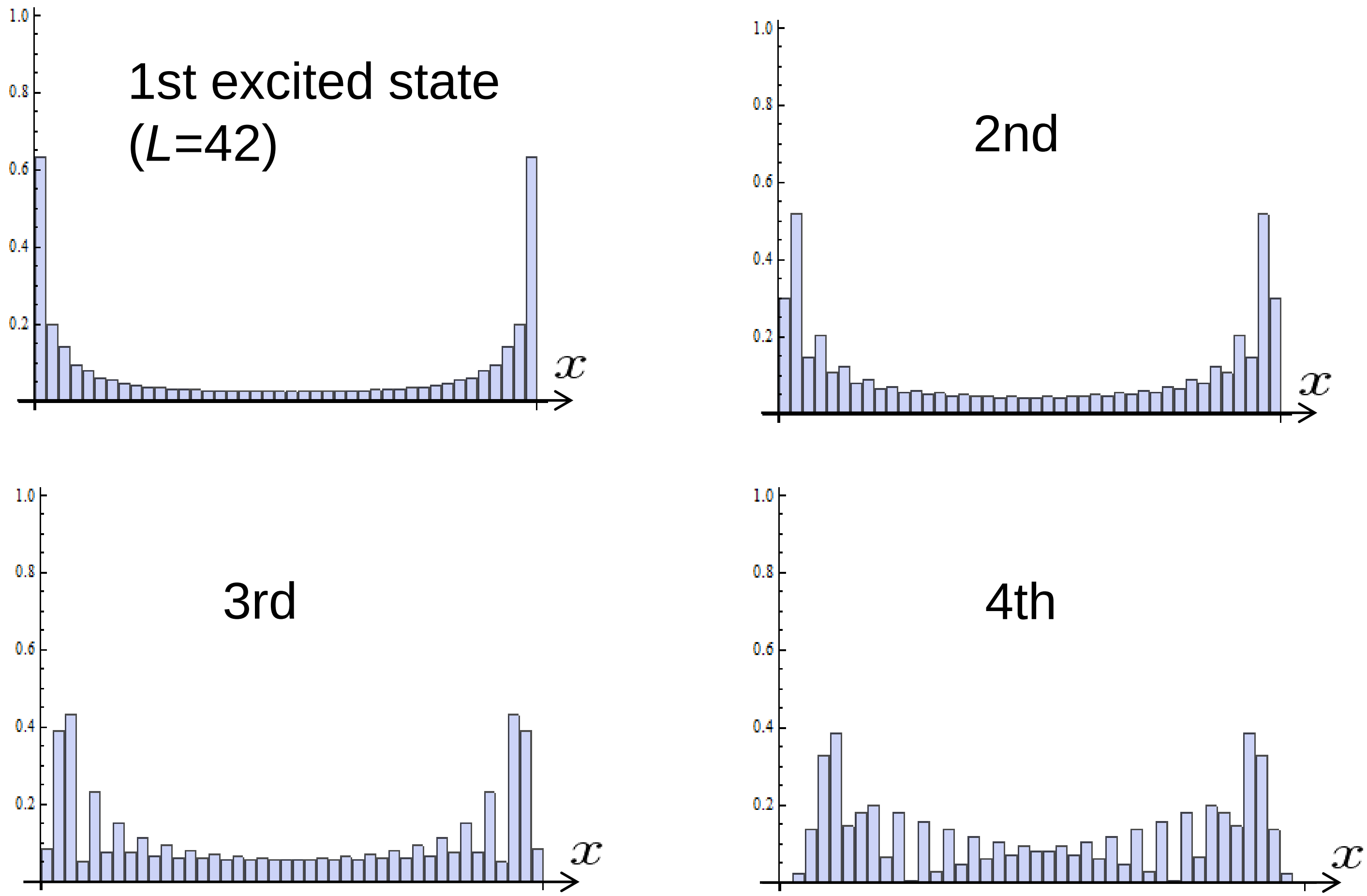


In Truong-Peschel, *IJMPB* **4** (‘90), they studied a linearized model (corner Hamiltonian) and obtained

$$\epsilon_{\mathrm{TP}}(j, L) = \frac{\pi^3}{2L(N-1)}\left(j + \frac{1}{4}\right)^2$$

Here N is the truncation number.
N~L gives the best fit to the result.
Variational estimate seems optimal.

Spatial profile of ‘extra’ states



Low-energy states ~ edge states
Can we get exact solutions?

Further steps towards exact solution (1)

- (Functional) Bethe ansatz

Hofstadter problem:
Wiegmann-Zabrodin, *PRL* (1994).

$$-\sin\left(\frac{2\pi}{L}m\right)\psi_{m-1}+2\sin\left[\frac{2\pi}{L}\left(m+\frac{1}{2}\right)\right]\psi_m-\sin\left[\frac{2\pi}{L}(m+1)\right]\psi_{m+1}=\varepsilon\psi_m$$

Generating function

$$\Psi(z)=\sum_{m=0}^{Q-1}z^m\psi_m=\prod_{m=1}^{Q-1}(z-z_m),\quad (Q=L/2)$$

Functional relation (T-Q relation)

$$q=e^{2\pi i/L}$$

$$\frac{i}{2}(z^{-1}-2q^{1/2}+qz)\Psi(qz)-\frac{i}{2}(z^{-1}-2q^{-1/2}+q^{-1}z)\Psi(q^{-1}z)=\varepsilon\Psi(z)$$

Pole-free condition

$(\text{L.H.S.})/\Psi(z)=\varepsilon$ does not have poles. → Residues at $z=z_m$ are zero!

Bethe eq.:

$$\frac{z_m^2-2q^{1/2}z_m+q}{qz_m^2-2q^{1/2}z_m+1}=-\prod_{m=1}^{Q-1}\frac{qz_m-z_n}{z_m-qz_n},\qquad m=1,2,\ldots,Q-1$$

$$\varepsilon=-\frac{i}{2}(q-q^{-1})\sum_{m=1}^{Q-1}z_m-i(q^{1/2}-q^{-1/2})$$

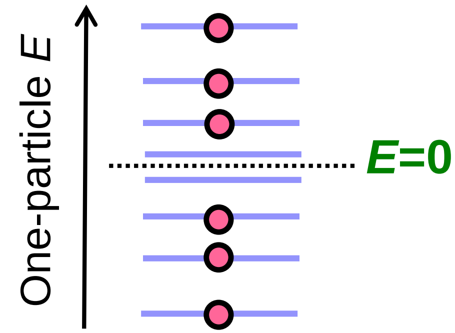
Asymptotic (large *L*) behavior??

Further steps towards exact solution (2)

Huge degeneracy at E=0

$$\mathcal{H}_0 = -t\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$

$$\mathcal{H}_{\rm SSD} = -t\sum_{j=1}^{L-1}\sin^2\left(\frac{\pi}{L}j\right)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$



(# many-body states) = 2^L

For both H_0 & $H_{\rm SSD}$, (Deg. at E=0) = $2^{L/2}$

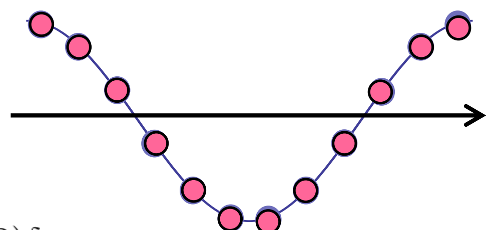
→ A number of operators that commute with Hamiltonian

- Pair-operator approach

Operators commuting with $\mathcal{H}_{\rm SSD}$

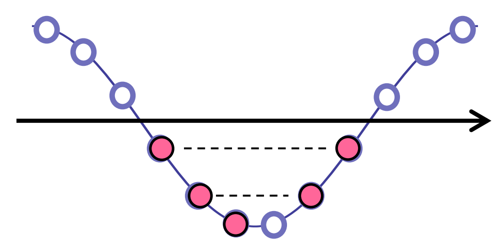
$$P_1^\dagger = \sum_{j=1}^{L/2} c^\dagger_{k_{\rm F}+(j-1/2)\delta}\, c^\dagger_{k_{\rm F}-(j-1/2)\delta}$$

$$P_2^\dagger = \sum_{j=1}^{L/2} e^{-\mathbf{i}j\delta}\, c^\dagger_{k_{\rm F}+(j-1/2)\delta}\, c^\dagger_{-k_{\rm F}+(j-1/2)\delta}$$



E=0 states of $\mathcal{H}_{\rm SSD}$

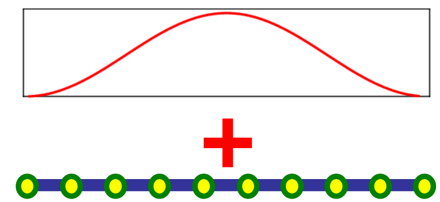
$$(P_1^\dagger)^m (P_2^\dagger)^n |\varphi\rangle$$



Exact many-body eigenstates appear when $N=L/4$.
They are also eigenstates of uniform Hamiltonian.

Summary

Hamiltonian with Sine-Square Deformation (SSD) shares the same ground state with periodic chain.

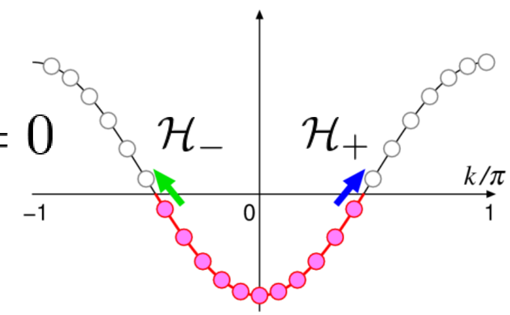


Mechanism of SSD

Chiral Hamiltonians annihilate the periodic g.s.

$$\mathcal{H}_{\mathrm{SSD}} = \frac{1}{2}\mathcal{H}_0 - \frac{1}{4}(\mathcal{H}_+ + \mathcal{H}_-)$$

$$\mathcal{H}_\pm |0\rangle = 0$$



Ex) Free-fermion chain, anisotropic XY, Dirac fermions

CFT interpretation:

- Chiral Hamiltonians are $L_{\pm 1}$ in CFT
- *SL*(2, **C**) invariance → $L_0|0\rangle = L_{\pm 1}|0\rangle = 0$
- The vacuum state $|0\rangle$ is the unique (normalizable) *E*=0 state of $\mathcal{H}_{\mathrm{SSD}}$.

Future directions

- Exact results for lattice SSD not reducible to free fermions
- Excited states of free fermions with SSD

  Related work: SUSY QM approach (Okunishi-H.K., *JPA* **48** ('15))