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Calogero Particles and Fluids
A Review Lecture 2

August 3, 2018

The Casimirs of L and R are $I_n = \mathrm{Tr} L^n$, $\tilde{I}_n = \mathrm{Tr} R^n$, thus

$$[I_n, I_m] = [\tilde{I}_n, \tilde{I}_m] = [I_n, \tilde{I}_m] = 0$$

- We recovered Quantum Integrals of the matrix model
- I_n and $\tilde{I}_n$ are related:

$$\mathrm{Tr} L^n = \mathrm{Tr}(N - R)^n - \sum_{k=0}^{n-1} \mathrm{Tr}(N - R)^k \, \mathrm{Tr} L^{n-k-1}$$

The Hamiltonian can be written

$$H = \mathrm{Tr}\, \mathbf{H} \;, \quad \mathbf{H} = \frac{1}{2}\omega(AA^\dagger + A^\dagger A) = \frac{1}{2}\omega(L - R + N)$$

and therefore

$$H = \omega I_1 + \omega \frac{N^2}{2}$$

so it is a member of the I_N. It satisfies

$$[H, A^\dagger] = \omega A^\dagger \;, \quad [H, A] = -\omega A$$

We obtain N^2 creation operators producing all states

Rreduction to sectors and spectrum

Quantum conserved angular momentum

$$J = i :[M, P]_{\text{matrix}}: = :[A^\dagger, A]_{\text{matrix}}: = L + R \ , \quad \mathrm{Tr} J = 0$$

$SU(N)$ algebra generating unitary conjugations $M \to UMU^{-1}$

- Fix a representation of $SU(N)$ (choose the 'form' of J)
- Identify states related through J (use residual symmetry)

Identifying irreps of J

- L and R are Jordan-Schwinger realizations of $U(N)$ on N (anti)fundamentals

$$L_{jk} = \sum_s A_{js}^\dagger A_{sk} = \sum_s L_{s,jk} \ , \ R_{jk} = -\sum_s A_{sk}^\dagger A_{js} = \sum_s R_{s,jk}$$

$L_{s,jk}$, $R_{s,jk}$ are N mutually commuting $U(N)$ algebras

$$[L_{s1}, L_{s'2}] = \delta_{ss'}(L_{s1} - L_{s2})T_{12} \ , \quad [R_{s1}, R_{s'2}] = \delta_{ss'}(R_{s1} - R_{s2})T_{12}$$

$$[L_{s1}, R_{s'2}] = 0$$

Choosing an irrep for J:

$$J = \sum_s L_s + \sum_s R_s$$

- Each L_s and R_s contains all symmetric products of (anti)fundamentals
- Full L and R contain all irreps (with multiplicities)
- Reps carried by L and R are conjugate (Casimirs are related)
- Therefore, $J = L + R$ contains reps of the form $r \times \bar{r}$
- For such irreps the $U(1)$ charge and the Z_N charge of J vanish

Young tableaux boxes of J are multiple of N

We must construct oscillator states belonging to such irreps

Constructing the states

From ground state $|0\rangle$, $A_{jk}|0\rangle = 0$, act with any number of creation operators:

$$|j_1, k_1; j_2, k_2; \dots j_n k_n\rangle = A^\dagger_{j_1 k_1} A^\dagger_{j_2 k_2} \dots A^\dagger_{j_n k_n} |0\rangle$$

- j_s indices: fundamental of L; k_s indices: antifundamental of R
- Rep of J: $(F \times \cdots \times F) \times (\bar{F} \times \cdots \times \bar{F})$
- Sectors: Clebsch-Gordan decomposition into irreps of J
- Calogero sector: J rank one $\Rightarrow$ ℓN-symmetric irrep, ℓ integer

Trick: put $\quad J_{jk} = \psi^\dagger_k \psi_j - \ell \delta_{jk} \ , \quad [\psi_j, \psi^\dagger_k] = \delta_{jk}$

That is, form singlets of $J + J_\psi + \ell$, with $(J_\psi)_{jk} = -\psi^\dagger_k \psi_j$

Tracing the above relation gives

$$\mathrm{Tr} J_\psi + \ell N = 0 \ \Rightarrow \ \sum_j \psi^\dagger_j \psi_j = \ell N$$

The generic form of the states is

$$\prod_{n=1}^{N} \left[\mathrm{Tr}(A^\dagger)^n \right]^{m_n} \left[\epsilon_{j_1 \dots j_N} \psi^\dagger_{j_1} (\psi^\dagger A^\dagger)_{j_2} \dots \left(\psi^\dagger (A^\dagger)^{N-1} \right)_{j_N} \right]^\ell |0\rangle$$

Energy given by the total number of $A^\dagger$ oscillators appearing in the state plus $\omega N^2/2$

$$E = \omega \left(\sum_{k=1}^{N} m_k + \ell \frac{N(N-1)}{2} + \frac{N^2}{2} \right)$$

- Energies of N noninteracting bosons in oscillator potential
- m_k: gap between the top k bosons and the next lower one
- Bosonization: same spectrum as N noninteracting fermions
- m_k+1: gap between the top k fermions and the next lower one

So we may rewrite the above spectrum as

$$E = \omega \left(\sum_{i=1}^{N} n_i + \ell \frac{N(N-1)}{2} + \frac{N^2}{2} \right) = \omega \left(\sum_{i_1}^{N} \bar{n}_i + \frac{N}{2} \right)$$

$n_1 \leq n_2 \leq \ldots n_N$ are single-particle bosonic excitation numbers
'Pseudoexcitation' numbers $\bar{n}_i$ have been defined as

$$\bar{n}_i = n_i + (\ell+1)(i-1) \ , \quad \bar{n}_i \leq \bar{n}_{i+1} + \ell$$

Parting remarks

We recovered Integrability and energy spectrum of quantum harmonic Calogero model

- One question: where is the QM particle model? What is the potential strength g?

Turns out to be $C_{2,J} = \ell(\ell + 1)$. More on this and the particle correspondence in the unitary matrix model

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The larger gauge symmetry may have anomalies!
A 'monopole-like' quantization is needed to avoid anomaly

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  Where is $\bar{\ell} = 0$ (bosons)?

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Gone! Matrix model fermionizes particles (renormalizes ℓ to $\ell + 1$)

The unitary matrix model: Classical analysis

Particles on unit circle (periodic): phases of eigenvalues of a unitary matrix U

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(U^{-1}\dot{U})^2 \;\Rightarrow\; \frac{d}{dt}\left(U^{-1}\dot{U}\right) = 0$$

- Invariant under $U \to VUW^{-1}$, V, W unitary
- Two conserved matrix angular momenta L and R:

$$\begin{aligned} U \to VU : \qquad L &= i\dot{U}U^{-1} \\ U \to UW^{-1} : \quad R &= -iU^{-1}\dot{U} \end{aligned}$$

- Unitary conjugation corresponds to $W = V$ with generator

$$J = L + R = i[\dot{U}, U^{-1}]$$

As in Hermitian case, parametrize

$$U = V\Lambda V^{-1} \;\; \text{with} \;\; \Lambda = diag\{e^{ix_i}, \ldots e^{ix_N}\}$$

Hamiltonian becomes, after a few steps,

$$H = \sum_i \frac{1}{2}p_i^2 + \frac{1}{2}\sum_{i\neq j} \frac{K_{ij}K_{ji}}{4\sin^2\frac{x_i - x_j}{2}} \;, \quad K = V^{-1}JV$$

- $J = K = 0$ reproduces free particles on the circle
- $J = \ell(uu^\dagger - 1) \;\Rightarrow\; K_{ij}K_{ji} = \ell^2$ recovers the Sutherland inverse-sine-square model

$$H = \sum_i \tfrac{1}{2}\dot{x}_i^2 + \tfrac{1}{2}\sum_{i\neq j} \frac{\ell^2}{4\sin^2\frac{x_i - x_j}{2}}$$

Integrable and solvable by the same techniques as the Hermitian model. The conserved invariant quantities are

$$I_n = \mathrm{Tr} L^n = \mathrm{Tr}(-R)^n = \mathrm{Tr}(iU^{-1}\dot{U})^n$$

and the solution is $$U = B\, e^{iCt}$$

with B a unitary and C a hermitian matrix satisfying

$$BCB^{-1} - C = J$$

For the Sutherland case with $J = \ell(uu^\dagger - 1)$, $u_i = 1$, B, C become

$$B_{jk} = \delta_{jk} e^{iq_j} \;, \quad C_{jk} = \delta_{jk}\, p_j + (1 - \delta_{jk}) \frac{i\ell}{e^{i(q_j - q_k)} - 1}$$

q_j and p_j are initial positions and momenta

- $q = x/a$ and limit $a \to 0$ recovers (free) Calogero system

Unitary matrix model: quantization

- Matrices L and R similar to those in Hermitian model
- Hamiltonian proportional to $I_2 = \mathrm{Tr} L^2$ rather than I_1
- Diagonalize I_2 (I_1 is degenerate)
- Find quantum states in J sectors

Canonical momentum matrix conjugate to the 'coordinate' U

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{U}} = -U^{-1} \dot{U} U^{-1} \;, \quad [U_1, \Pi_2] = i T_{12}$$

Π is neither unitary nor hermitian. Better to work in terms of hermitian L and R defined previously

$$L = i \dot{U} U^{-1} = -iU\Pi \;, \quad R = -iU^{-1}\dot{U} = i\Pi U$$

$$[L_1, L_2] = (L_1 - L_2) T_{12} \;, \; [R_1, R_2] = (R_1 - R_2) T_{12} \;, \; [L_1, R_2] = 0$$

Again two $U(N)$ algebras. $U(1)$ charges

$$I_1 = L^o = -R^o = \mathrm{Tr}(-iU^{-1}\dot{U}) = \sum_i p_i$$

are essentially the total momentum of the system.

Construction of states in U-representation:

- States become functions of U
- Π becomes the matrix derivative $\Pi = -i\delta_U$ acting as

$$\delta_U \mathrm{Tr}(UB) = B \ , \quad \delta_U \mathrm{Tr}(U^{-1}B) = -U^{-1}BU^{-1}$$

  and similarly on expressions containing more Us.
- L and R, upon proper ordering, are represented as

$$L = -U\delta_U \ , \quad R = \delta_U \cdot U$$

and thus

$$L\,\mathrm{Tr}(UB) = -UB \ , \quad L\,\mathrm{Tr}(U^{-1}B) = BU^{-1}$$

$$R\,\mathrm{Tr}(UB) = BU \ , \quad R\,\mathrm{Tr}(U^{-1}B) = -U^{-1}B$$

or

$$\mathrm{Tr}(i\epsilon L)f(U) \quad = f((1-i\epsilon)U) - f(U)$$

$$\mathrm{Tr}(i\epsilon R)f(U) \quad = f(U(1+i\epsilon)) - f(U)$$

for an infinitesimal matrix ϵ

Hamiltonian: Laplacian operator on the manifold

$$H = \frac{1}{2}\mathrm{Tr}L^2 = \frac{1}{2}\mathrm{Tr}R^2$$

Common quadratic Casimir of the left- and right-$U(N)$ algebra

- Irreps of L and R $U(N)$ are degenerate energy eigenstates.
- $U(1)$ (center of mass) part trivially separates: boost by multiplying the wavefunction by $(\det U)^p$
- Will focus on the $SU(N)$ part

$R_{\alpha\beta}(U)$, $\alpha,\beta = 1,\dots d_R$: matrix element of U in irrep R
Complete orthonormal basis of wavefunctions for U:

$$\text{(Haar measure)} \quad \int [dU]\, R_{\alpha\beta}(U)\, R'_{\gamma\delta}(U)^* = \delta_{RR'}\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}$$

- Group property: $R_{\alpha\beta}(UV) = \sum_\gamma R_{\alpha\gamma}(U) R_{\gamma\beta}(V)$

Let us show explicitly that $R_{\alpha\beta}(U)$ is an energy eigenstate:

$$\begin{aligned}\mathrm{Tr}(i\epsilon L)R_{\alpha\beta}(U) &= R_{\alpha\beta}((1-i\epsilon)U)-R(U)\\ &= \sum_{\gamma} R_{\alpha\gamma}(1-i\epsilon)R_{\gamma\beta}(U)-R_{\alpha\beta}(U)\\ &= -i\sum_{\gamma} R_{\alpha\gamma}(\epsilon)R_{\gamma\beta}(U)\end{aligned}$$

and thus

$$\mathrm{Tr}(i\epsilon_1 L)\mathrm{Tr}(i\epsilon_2 L)R_{\alpha\beta}(U) = -\sum_{\gamma\delta} R_{\alpha\gamma}(\epsilon_2)R_{\gamma\delta}(\epsilon_1)R_{\delta\beta}(U)$$

Choosing $\epsilon_1 = \epsilon_2 = T^a$ and summing over a

$$\sum_a \mathrm{Tr}(T^a L)\mathrm{Tr}(T^a L)R_{\alpha\beta}(U) = \sum_a\sum_\gamma (R^a R^a)_{\alpha\gamma}R_{\gamma\beta}(U)$$

From completeness $$\sum_a \mathrm{Tr}(T^a L)\mathrm{Tr}(T^a L) = \mathrm{Tr}L^2$$

From irreducibility of R $$\sum_a (R^a R^a)_{\alpha\gamma} = C_{2,R}\,\delta_{\alpha\gamma}$$

with $C_{2,R}$ the quadratic Casimir of $U(N)$, giving

$$\mathrm{Tr}L^2 R_{\alpha\beta}(U) = C_{2,R}\,R_{\alpha\beta}(U)$$

$$H\,R_{\alpha\beta}(U) = E_R\,R_{\alpha\beta}(U)\ ,\qquad E_R = \frac{1}{2}C_{2,R}$$

- Energy E_R is that of N free fermions on the circle with the ground state energy subtracted
- Lengths of rows R_i of the Young tableau of R correspond to the "bosonized" fermion momenta $p_i = R_i - i + 1$
- The condition $R_i \geq R_{i+1}$ for the rows amounts to the fermionic condition $p_i > p_{i+1}$
- The spectrum of the full matrix model, then, is identical to the free fermion one but with degeneracies d_R^2

Reduce to sectors (representations) of J

- Identify the corresponding reduced quantum model
- Identify the subspace of states belonging to the reduced model
- $R_{\alpha\beta}(U)$ carries irrep R for L and $\bar{R}$ for R
- Rep of $J = L + R$ is $R \times \bar{R}$ (as in Hermitian model)

States for irrep r of J become

$$\Psi_{R,r}(U) = \sum_{\alpha,\beta} C[R,\bar{R};r]_{\gamma}^{\alpha\beta} R_{\alpha\beta}(U)$$

$C[R,\bar{R};r]_{\gamma}^{\alpha\beta}$ projects states α of R and β of $\bar{R}$ to state γ of r

- d_r states corresponding to various γ are gauge copies
- Degeneracy corresponds to number of irreps r in $R \times \bar{R}$r
- Call this integer $D(R,r)$; spectrum and degeneracies are

$$E_R = \frac{1}{2} C_R \ , \quad D_R = D(R,r)$$

If $D_R = 0$ the corresponding energy level is absent

We have determined the spectrum for any irrep r of J

- $r = 0$ should correspond to free fermions
- $r = \ell N$-symmetric should correspond to Sutherland particles

Before finding the spectrum let us derive the QM particle model

Reduction to spin-particle systems

Classically we found

$$H = \sum_i \tfrac{1}{2} p_i^2 + \tfrac{1}{2} \sum_{i \neq j} \frac{K_{ij} K_{ji}}{4 \sin^2 \frac{x_i - x_j}{2}} - E_o$$

To extend it to the QM domain we must

- Determine appropriate ordering of operators
- Take into account correct measure [cf. Cartesian $\to$ polar]

Write metric $-\mathrm{Tr}(U^{-1} dU)^2$ in terms of $U = V \Lambda V^{-1}$

$$-\mathrm{Tr}(U^{-1} dU)^2 = \sum_i dx_i^2 + \sum_{i,j} 4 \sin^2 \frac{x_i - x_j}{2} \left| (V^{-1} dV)_{ij} \right|^2$$

Diagonal in dx_i and $(V^{-1} dV)_{ij}$; so volume form is

$$[dU] = \Delta(x)^2 [dx]\, [dV] \,, \quad \Delta(x) = \prod_{i<j} 2 \sin \frac{x_i - x_j}{2} \,, \;\; [dx] = \prod_j dx_j$$

$[dU]$,$[dV]$ are Haar measures and $\Delta(x)$ is the unitary Vandermonde

To have 'flat' x_j-Laplacian ($\sum_j \partial_j^2$), redefine wavefunction

$$\psi(x, V) = \Delta(x)\Psi(U)$$

- Produces an additive constant E_o (cf. $-1/4r^2$ in 2d polar)
- $\Delta(x)$ is ground state of N fermions $\Rightarrow E_o = \frac{N(N^2-1)}{24}$
- $\psi(x, V)$ becomes fermionic in x: fermionization of eigenvalues

Ordering and properties of K: generator of transformations

- J generates $U \to WUW^{-1}$
- Since $U = V\Lambda V^{-1}$, J generates $V \to WV$
- Therefore, $K = V^{-1}JV$ generates $V \to VW^{-1}$

K is also an $SU(N)$ algebra. Some constraints:

- K carries the same irrep r as J
- $V \to VW^{-1}$ with W diagonal leaves $U = V\Lambda V^{-1}$ invariant
- Therefore states must be invariant $\Rightarrow K_{ii}\psi = 0$ (no sum)
- K does not commute with H, but above constraint does

Let there be spins

Realize K in a Jordan-Wigner construction with Nq oscillators

$$[a_{mj}, a^{\dagger}_{nk}] = \delta_{mn}\delta_{jk} \ , \quad j,k = 1,\dots N \ , \ \ m,n = 1,\dots q$$

$$K_{jk} = \sum_{m=1}^{q} a^{\dagger}_{mj} a_{mk} - \ell\delta_{jk}$$

The constraint $K_{jj} = 0$ on physical states ψ means

$$\sum_{m=1}^{q} a^{\dagger}_{mj} a_{mj} \ \psi = \ell \ \psi$$

requiring, again, that ℓ be integer
The coefficient of the particle potential becomes ($j \neq k$)

$$K_{jk} K_{kj} = \sum_{m,n} a^{\dagger}_{mj} a_{nj} \ a^{\dagger}_{nk} a_{mk} + \ell$$

DOF of K redistributed into DOF for each particle

Define independent $SU(q)$ spins for each particle

$$S_{j,mn} = a_{mj}^{\dagger} a_{nj} - \frac{\ell}{q}\delta_{mn}$$

In terms of spins

$$K_{ij}K_{ji} = \sum_{mn} S_{i,mn}S_{j,nm} + \frac{\ell(\ell+q)}{q} = \vec{S}_i \cdot \vec{S}_j + \frac{\ell(\ell+q)}{q}$$

$\vec{S}_i \cdot \vec{S}_i = \mathrm{Tr}(S_i S_j)$ is the $SU(q)$-scalar product of the two vectors
The QM Hamiltonian becomes

$$H = \sum_j \frac{1}{2} p_j^2 + \sum_{j<k} \frac{\vec{S}_i \cdot \vec{S}_j + \frac{\ell(\ell+q)}{q}}{4\sin^2 \frac{x_i - x_j}{2}}$$

- Particles carry $SU(q)$ spins in the ℓ-fold symmetrix irrep
- Sutherland potential contains an antiferromagnetic interaction between the spins and a constant
- Coefficient and constant fixed – not free parameters
- $q = 1$: no spins, Sutherland model with $g = \ell(\ell + 1)$

Spectrum of the particle-spin model

Basic idea:

- Decompose total spin $\vec{S} = \sum_j \vec{S}_j$ into irreps of $SU(q)$: Young tableaux up to q rows with ℓN boxes
- Interpret them as $SU(N)$ irreps r of K or J
- Read off spectrum and degeneracy as derived before

Let us reproduce the two simplest cases

a) Free particles: $q = 0$ or $\ell = 0$, $J = K = 0$

- $r = 0$: No spin, no potential, free particles
- $D(R, 0) = 1$, free fermion spectrum (fermionization)

b) Spinless Sutherland particles: $q = 1$

- $r = [\ell N]$, single-row, ℓN boxes: no spin, Sutherland particles
- $D(R, r) = 1$ if each row at least ℓ boxes longer than next, else 0
- Free particles with momentum selection rule $p_j \geq p_{j+1} + \ell + 1$

  Particles with generalized statistic of order $\ell + 1$

Parting remarks

Other realizations of K lead to more general spin models

- A realization in terms of *fermionic* oscillators leads to ℓ-fold *antisymmetric* $SU(q)$ spins with *ferromagnetic* interactions

$$H = \sum_j \tfrac{1}{2} p_j^2 - \sum_{j<k} \frac{\vec{S}_j \cdot \vec{S}_k + \frac{\ell(\ell-q)}{q}}{4 \sin^2 \frac{x_i - x_j}{2}}$$

with spins in the ℓ-fold *antisymmetric* irrep of $SU(q)$

- A single bosonic and a single fermionic oscillator reproduces the so-called supersymmetric Calogero model

Other matrix choices possible

- Positive definite matrices (constant negative curvature spaces) lead to the hyperbolic Calogero model
- Also obtained through analytic continuation $x \to ix$

Farewell to Matrices

The Matrix Model has provided us with:

- An augmentation of S_N to $SU(N)$ and the corresponding possibility to define statistics through irreps of $SU(N)$.
- A realization of generalized scalar statistics but with a *quantized* Calogero statistics parameter $\ell + 1$
- A realization of generalized 'non-abelian statistics' in terms of spin degrees of freedom in the Calogero potential.
- 4. A systematic way of solving the above models.

What the matrix model has *not* provided is

- A realization of the Calogero model for fractional values of ℓ
- A realization of spin-Calogero systems with the spins in arbitrary (non-symmetric or antisymmetric) representations.
- A control of the coupling strength of the potential for the spin-Calogero models.

A new approach is needed!

Exchange operator formalism

Operators M_{ij} permute the *coordinate* DOF of N particles in one dimension. They satisfy the permutation algebra (symmetric group)

$$M_{ij} = M_{ij}^{-1} = M_{ij}^{\dagger} = M_{ji}$$

$$[M_{ij}, M_{kl}] = 0 \quad \text{if } i, j, k, l \text{ distinct}$$

$$M_{ij} M_{jk} = M_{ik} M_{ij} \quad \text{if } i, j, k \text{ distinct}$$

One-particle operators: any A_i satisfying

$$\begin{aligned} M_{ij} A_k &= A_k M_{ij} \quad \text{if } i, j, k \text{ distinct} \\ M_{ij} A_i &= A_j M_{ij} \end{aligned}$$

Construct the exchange-momenta one-particle operators

$$\pi_j = p_j + \sum_{k(\neq j)} i\, W(x_j - x_k) M_{jk} := p_j + \sum_{k(\neq j)} i\, W_{jk} M_{jk}$$

For π_i to be Hermitian the prepotential $W(x)$ should satisfy

$$W(-x) = -W(x)^*$$

'Free' Hamiltonian in π_i would be $H = \sum_j \frac{1}{2}\pi_j^2$
Contains terms linear in p_j: to eliminate them

$$W(-x) = -W(x) = \text{real}$$

Commutators of π_i and Hamiltonian become

$$[\pi_i, \pi_j] = \sum_k W_{ijk}(M_{ijk} - M_{jik})$$

$$H = \sum_i \frac{1}{2}p_i^2 + \sum_{i<j} (W_{ij}^2 + W_{ij}' M_{ij}) + \sum_{i<j<k} W_{ijk} M_{ijk}$$

where $M_{ijk} = M_{ij} M_{jk}$ is cyclic permutation of (i, j, k) and

$$W_{ijk} = W_{ij} W_{jk} + W_{jk} W_{ki} + W_{ki} W_{ij}$$

- Goal: commutator zero or a constant
- This leads to functional equation for $W(x)$:

$$W(x)W(y) - W(x+y)[W(x) + W(y)] = \text{const}(= W_{ijk})$$

Can be solved and we will list its solutions (up to scaling of x)

a) $W_{ijk} = 0 \quad \Rightarrow \quad W(x) = \ell / x$
b) $W_{ijk} = -\ell^2 < 0 \quad \Rightarrow \quad W(x) = \ell \cot x$
c) $W_{ijk} = +\ell^2 > 0 \quad \Rightarrow \quad W(x) = \ell \coth x$

Case a)

$$\pi_j = p_j + \sum_{k \neq j} \frac{i\ell}{x_{jk}} M_{jk} \ , \quad [\pi_j, \pi_k] = 0$$

$$H = \sum_i \tfrac{1}{2} p_i^2 + \sum_{i<j} \frac{\ell(\ell - M_{ij})}{x_{ij}^2}$$

- Calogero-like model with exchange interactions
- Trivially integrable: $I_n = \sum_i \pi_i^n$

Assume particles are bosons or fermions: $M_{ij} = \pm 1$ on states

- The model becomes the standard Calogero model
- Projected integrals $I_{n,\pm}$ commute
- Commutativity implied by $[I_n, I_m] = 0$ and locality of $I_{n,\pm}$

We proved quantum integrability of Calogero model in one sweep!

Construct 'harmonic oscillator' operators

$$a_j = \frac{1}{\sqrt{2}}(\pi_j - i\omega x_j) \ , \quad a_j^\dagger = \frac{1}{\sqrt{2}}(\pi_j + i\omega x_i)$$

Their commutators are calculated as

$$[a_i, a_j^\dagger] = \omega\left(1 + \ell \sum_{k\neq i} M_{ik}\right)\delta_{ij} - \omega\ell M_{ij}\,(1 - \delta_{ij})$$

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$$

Oscillator Hamiltonian reads

$$H = \sum_i \tfrac{1}{2}(a_i^\dagger a_i + a_i a_i^\dagger) = \sum_i \tfrac{1}{2}p_i^2 + \sum_i \tfrac{1}{2}\omega^2 x_i + \sum_{i<j} \frac{\ell(\ell - M_{ij})}{x_{ij}^2}$$

and satisfies $\quad [H, a_i] = \omega a_i \ , \quad [H, a_i^\dagger] = \omega a_i^\dagger$

- Harmonic Calogero-like model with exchange interactions
- Becomes standard harmonic model on bosonic or fermionic subspaces
- Can use ladder operators to construct states

$$H = \sum_i a_i^\dagger a_i + \frac{1}{2} N\omega + \frac{1}{2}\ell\omega \sum_{i\neq j} M_{ij}$$

On bosonic or fermionic states the state annihilated by all a_i (if it exists) will be the ground state. Solving $a_i \psi = 0$

$$\psi_B = \prod_{i<j} |x_{ij}|^\ell e^{-\frac{1}{2}\omega \sum_i x_i^2}$$

$$\psi_F = \prod_{i<j} \left\{ sgn(x_{ij}) |x_{ij}|^{-\ell} \right\} e^{-\frac{1}{2}\omega \sum_i x_i^2}$$

- Bose state for $\ell > 0$ and Fermi state for $\ell < 0$ are acceptable
- In the "wrong" combinations GS involves $|\ell| + 1$ and is annihilated by permutation-invariant combinations of the a_i
- Spectrum obtained by acting on ground state with all possible permutation-symmetric homogeneous polynomials in $a_i^\dagger$

A basis is $$A_n = \sum_i (a_i^\dagger)^n$$

Spectrum identical to non-interacting fermions or bosons, but with ground state energy shift

$$E_o = \frac{N}{2}\omega + \frac{N(N-1)}{2}|\ell|\omega$$

The one-particle operators $h_i = a_i^\dagger a_i$ obey

$$[h_i, h_j] = -\omega\ell(h_i - h_j)M_{ij}$$

and this leads to commuting invariant quantities

$$I_n = \sum_i (a_i^\dagger a_i)^n \ , \quad [I_n, I_m] = 0$$

proving integrability. Similar results hold for

$$\tilde{h}_i = a_i a_i^\dagger \ , \quad \tilde{I}_n = \sum_i (a_i a_i^\dagger)^n$$

Case b)

$$\pi_j = p_i + i\ell \sum_{k \neq j} \cot x_{ij} M_{ij}$$

$$[\pi_i, \pi_j] = -\ell^2 \sum_k (M_{ijk} - M_{jik})$$

The Hamiltonian becomes

$$H = \sum_i \frac{1}{2} p_i^2 + \sum_{i<j} \frac{\ell(\ell - M_{ij})}{\sin^2 x_{ij}} - \ell^2 \left(\frac{N(N-1)}{2} + \sum_{i<j<k} M_{ijk} \right)$$

- Sutherland-like model with exchange interactions
- On Bose or Fermi states it becomes standard Sutherland model
- $H = \sum_i \pi_i^2$, so if a state $\pi_i \psi = 0$ exists it is the ground state

We obtain

$$\psi_B = \prod_{i<j} |\sin x_{ij}|^\ell$$

$$\psi_F = \prod_{i<j} sgn(x_{ij}) |\sin x_{ij}|^\ell$$

- Acceptable for Bose, $\ell > 0$ or Fermi, $\ell < 0$
- For both cases $M_{ijk} = 1$ so

$$E_o = \ell^2 \frac{N(N^2-1)}{24}$$

The quantities

$$\tilde{\pi}_i = \pi_i + \ell \sum_{j \neq i} M_{ij}$$

have the same commutation relations as the h_i defined previously for the harmonic system. Therefore, the integrals

$$I_n = \sum_i \tilde{\pi}_i^n \ , \quad [I_n, I_m] = 0$$

commute and the model is integrable.

Case c)

$$H = \sum_i \tfrac{1}{2} p_i^2 + \sum_{i<j} \frac{\ell(\ell - M_{ij})}{\sinh^2 x_{ij}} + \ell^2 \left(\frac{N(N-1)}{2} + \sum_{i<j<k} M_{ijk} \right)$$

- Hyperbolic model with exchange interactions
- Supports only scattering states
- Integrability similary as for the trigonometric model, or through analytic continuation $x \to ix$

All the above work directly, and only, for the quantum case

Let there be spin (again!)

Assume particles carry a number q of discrete internal states

- σ_{ij} exchanges the internal states of particles i and j
- Total particle permutation operator is $T_{ij} = M_{ij}\,\sigma_{ij}$

Assume states bosonic or fermionic under total particle exchange

$$T_{ij}\psi_{B,F} = \pm\psi_{B,F} \quad \Rightarrow \quad M_{ij}\psi_{B,F} = \pm\sigma_{ij}\psi_{B,F}$$

Exchange Calogero and Sutherland Hamiltonians become

$$H_c = \sum_i \tfrac{1}{2}p_i^2 + \sum_i \tfrac{1}{2}\omega^2 x_i + \sum_{i<j} \frac{\ell(\ell \mp \sigma_{ij})}{x_{ij}^2}$$

$$H_s = \sum_i \tfrac{1}{2}p_i^2 + \sum_{i<j} \frac{\ell(\ell \mp \sigma_{ij})}{\sin^2 x_{ij}} - \ell^2 \left(\frac{N(N-1)}{2} + \sum_{i<j<k} \sigma_{ijk} \right)$$

Calogero-Sutherland models with spin-exchange interactions

Fundamental $SU(q)$ generators T^a satisfy completeness relation

$$\sum_{a=1}^{q^2-1} T^a_{\alpha\beta} T^a_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\gamma\beta} - \frac{1}{q}\delta_{\alpha\beta}\delta_{\gamma\delta} \quad \text{or} \quad \sum_a T_1 T_2 = T_{12} - \frac{1}{q}$$

Therefore

$$\sigma_{ij} = \vec{S}_i \cdot \vec{S}_j + \frac{1}{q}$$

where S_i^a is T^a acting on internal states of particle i

Calogero-Sutherland interaction coefficient becomes

$$\ell(\ell \mp \sigma_{ij}) = \ell\left(\mp \vec{S}_i \cdot \vec{S}_j + \ell \mp \frac{1}{q}\right)$$

- Ferro/Antiferromagnetic spin interaction models (as in Matrix)
- Arbitrary coefficient strength (ℓ)
- Spins necessarily in the fundamental of $SU(q)$
- Ferro $\to$ Antiferro: B $\to$ F or $\ell \to -\ell$
- $\ell = 1$: Matrix and Exchange models agree
  (B $\leftrightarrow$ fundamental, F $\leftrightarrow$ antifundamental)

We are now ready to determine the spectrum of these spin-Calogero models. This will be done in Lecture 3.