Collegium Urbis Nov Eborac

ಸೈದ್ಧಾಂತಿಕ ವಿಜ್ಞಾನಗಳ ಅಂತರರಾಷ್ಟ್ರೀಯ ಕೇಂದ್ರ

Calogero Particles and Fluids
A Review

August 2, 2018

Introduction

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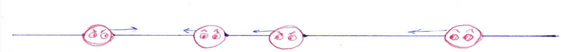
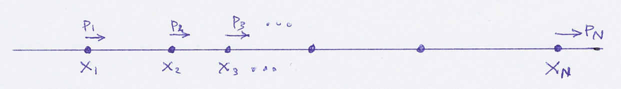
Case in point: Long-range one-dimensional billiards



...aka Calogero particles (and their fluids, spins, ghosts and avatars)

A system of nonrelativistic identical particles on the line ($m = 1$)



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Calogero 1969

How it all started...

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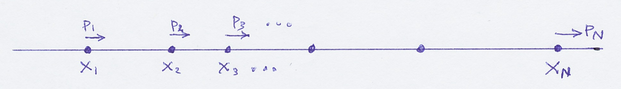

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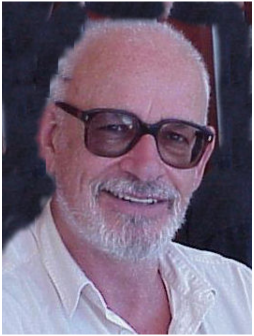

Francesco Calogero, 1969:

- Looking for solvable many-body problems
- Solved 3 particle problem Quantum Mechanically
- Solved N particle problem
- Others generalized it (Sutherland, Moser, etc.)
- Seminal review by Olshanetskii & Perelomov

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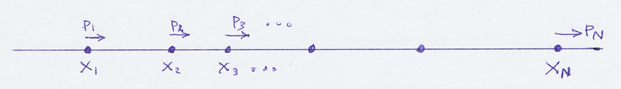

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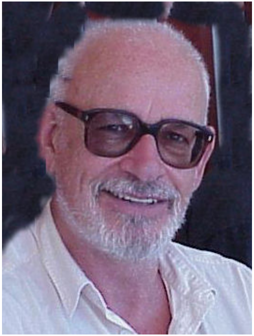

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(Καλόγερος = Monk)

• Calogero: 1989-1997 Secretary General of Pugwash Conferences on Science and World Affairs, Peace Nobel Prize 1995

Other Physicists Pugwash Chairs and Secretaries:

- Bertrand Russel: founder and first President (CCNY Ex.Prof.)
- Joseph Rotblat (nuclear): co-founder, Hippocratic Oath
- John Cockroft (nuclear, Nobel 1951 with Walton)
- Bernard Feld (CCNY alumnus, MIT professor)
- Francis Perrin (nuclear, son of Jean Baptiste Perrin, Nobel 1926 for Brownian motion)
- Mikhail Millionshchikov (turbulence)
- Eugene Rabinowitch (biophysics)
- Hannes Alfvén (Nobel 1970, magnetohydrodynamics)
- Michael Atiyah (1966 Fields medal)
- Paolo Cotta-Ramusino (mathematical and particle physics)

The Calogero system: a physical look

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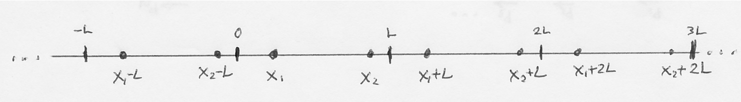
- $p^2 \sim \hbar^2/x^2$: the only scale-free potential in QM. If we put $g = \hbar^2\alpha$, then α is a *pure number*
- Classically $g \geq 0$ for stability. QM a bit subtler: $\alpha > -\frac{1}{2}$

We shall put $\hbar = 1$ from now on

We can also 'confine' the particles in a box:

Either add an external harmonic trap: $\sum_i \frac{1}{2}\omega^2 x_i^2$ (Calogero 1971)

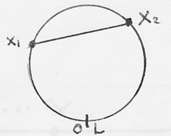
...or put the particles on a periodic space of length L



Each particle interacts with the images of all other particles, so the potential becomes

$$V(x) = \sum_{n=-\infty}^{\infty} \frac{g}{(x+nL)^2} = \frac{g}{\left(\frac{L}{\pi}\sin\frac{\pi x}{L}\right)^2}$$ (Sutherland 1972)

Same as interacting through the chord length on a circle.



By analytically continuing $L = iL'$ we can also consider a model with hyperbolic interactions

$$V(x) = \sum_{n=-\infty}^{\infty} \frac{g}{(x + inL')^2} = \frac{g}{\left(\frac{L'}{\pi} \sinh \frac{\pi x}{L'}\right)^2}$$

Unconfined model, interaction falls of exponentially with distance

Finally, we can periodize the hyperbolic model in $x \to x + L$. This gives the elliptic model with a Weierstrass elliptic function potential

$$V(x) = g\mathscr{P}(x; L, iL')$$

Very little is known about the elliptic system's classical or quantum solution and we will give it short shrift in these lectures.

Why is this system interesting?

Arises in various situations of physical interest: a 'Jack in the box'!

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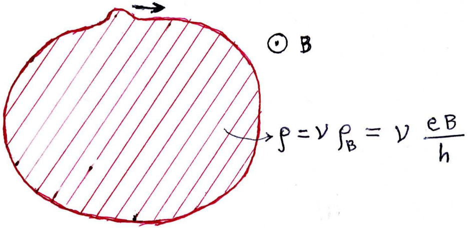
- Effective descriptions of fractional quantum Hall states



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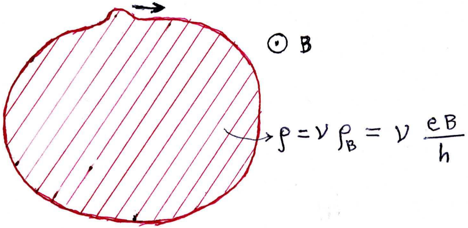

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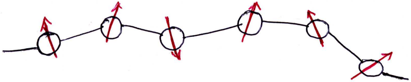
- Spin chains in macromolecules



Spin states behave like Calogero particles

- Interface dynamics in stratified fluids

Benjamin-(Davis-Acrivos)-Ono model
1967 - (1967) - 1975

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^2}{\partial x^2} \fint \frac{u(y)}{x-y} dy = 0$$

(Hilbert transform)

Admits 'soliton' solutions whose guiding centers behave like Calogero particles

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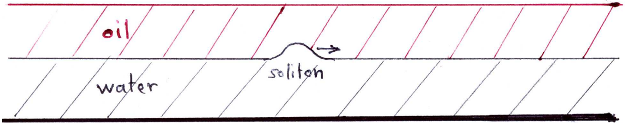

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- Matrix model descriptions of quantum gravity and string theory
- Two-dimensional gauge field theories; etc.

Stability, Hermiticity, Statistics

- Classically g should be positive to ensure stability
- Quantum uncertainty principle improves the minimum to $g = -\frac{1}{4}$ (with $\hbar = 1$) (Exercise in Landau & Lifshitz)
- Convenient to parametrize $g = \ell(\ell - 1)$ (ℓ real)
- Hermitian extensions to all values of g. Will keep $g \geq -\frac{1}{4}$
- Two values, ℓ and $\ell' = 1 - \ell$, give the same g
- Wavefunction behaves as $\psi \sim x^\ell$ at coincidence points
- We will keep $\ell \geq 0$. For $g \leq 0$ **two** values of ℓ survive
- Represent two different quantizations of the $g \leq 0$ model
- ℓ is dimensionless; reinstating $\hbar$: $g = \ell(\ell - \hbar)$

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What about statistics?

- $1/x^2$ potential is QM impenetrable; therefore ordinary statistics (symmetry of wavefunction) **irrelevant**
- All physical observables of the system independent of statistics
- Dimensionless parameter ℓ subsumes all statistics of the model
- $\ell = 0$: free bosons; $\ell = 1$: free fermions; other values?

Integrability

A classical Hamiltonian system possessing **conserved quantities** $I_n,\ n = 1, 2, \ldots N$ functions of coordinates and momenta not involving time explicitly is called integrable.

The I_n also must be (and always are) in involution: their Poisson brackets must vanish

$$\{I_n, I_m\} = 0$$

Obvious conserved quantities like total energy H and total momentum P are part of I_n but for $N > 2$ there must be more.

What are those for the Calogero system?
Equations of motion:

$$\dot{x}_i = p_i \ , \quad \dot{p}_i = 2 \sum_{j(\neq i)} \frac{g}{(x_i - x_j)^3}$$

Consider $N \times N$ matrices

$$L_{jk} = p_j \delta_{jk} + (1 - \delta_{jk}) \frac{i\ell}{x_{jk}} \ , \quad A_{jk} = \ell \delta_{jk} \sum_{s \neq j} \frac{1}{x_{js}^2} + \ell(\delta_{jk} - 1) \frac{1}{x_{jk}^2}$$

where we defined the shorthand $x_{ij} = x_i - x_j$
Upon use of the equations of motion and some algebra we find

$$\frac{dL}{dt} = i[L, A]$$

and thus

$$L(t + dt) = L(t) + idt(LA - AL) = e^{-iAdt} L(t) e^{iAdt} + O(dt^2)$$

or

$$L(t) = U(t)^{-1} L(0) U(t) \ , \quad U(t) = P e^{i \int_0^t A(\tau) d\tau}$$

So $L(t)$ evolves by unitary conjugation and its eigenvalues are constants of motion. Alternatively:

$$I_n = \mathrm{Tr} L^n$$

We get $I_1 = P$, $I_2 = 2H$ and $I_3, \ldots$ new conserved quantities.

Properties of the classical system

As time goes to $\pm\infty$ particles fly away from each other and all coordinate terms in L and A vanish. Therefore

$$I_n = \sum_{i=1}^{N} k_i^n$$

with k_i the asymptotic momenta.

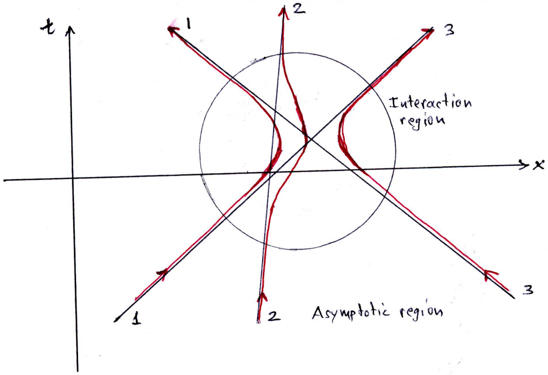
- Asymptotic monenta k_i are conserved

Some more work reveals that the angle variables θ_n conjugate to I_n asymptotically become g-independent functions of the coordinates x_i and momenta k_i. Therefore, at $t \to \pm\infty$

$$x_i(t) = k_i t + a_i$$

- Impact parameters a_i are also conserved!

(Will show that explicitly later on)



- After scattering, all particles resume their initial asymptotic momenta *and impact parameters*
- There is no time delay, just a permutation; therefore...
- System looks asymptotically free!
- Suggestive fact: action increases w.r.t. free system by $\Delta S = \pi\sqrt{g}$ per particle exchange

Quantum properties of the model

- Parametrize $g = \hbar^2 \ell(\ell - 1)$ (and take $\hbar = 1$)
- Classical limit is now $\ell \to \infty$

Lax pair formulation extends quantum mechanically

- Particles scatter and preserve asymptotic momenta
- Scattering phase shift is a constant:

$$\theta_{sc} = \frac{N(N-1)}{2} \pi \ell$$

(Compare with classical result $\Delta S = \pi \sqrt{g}$ per scattering)

- Since $\partial \theta_{sc} / \partial k_i = 0$ there is no time delay (again, compare with classical result)

Calogero particles can be interpreted as free particles with an extra phase shift of $\pi \ell$ when they exchange, that is,

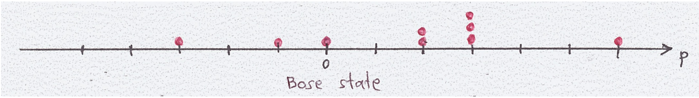
particles with exchange statistics of order ℓ (my entry, 1988)

The periodic model: particles, holes and duality

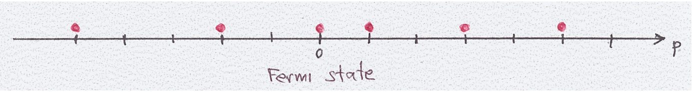
Particles on a circle of length L have discrete momenta $p = \frac{2\pi\hbar}{L} n = 0, \pm1, \pm2, \dots$ upon choosing $\hbar = 1$ and $L = 2\pi$
Free particles would have energy eigenvalues

$$E = \sum_{i=1}^{N} \frac{1}{2} p_i^2$$

If the particles are indistinguishable only the set of values $\{p_1, p_2, \dots, p_N\}$ matters, not their arrangement
Bose statistics: $p_1 \leq \cdots \leq p_N$ or $p_{i+1} - p_i \geq 0$



Fermi statistics: $p_1 < \cdots < p_N$ or $p_{i+1} - p_i \geq 1$ (no spin!)



For Calogero particles the energies turn out to be

$$E = \sum_{i=1}^{N} \tfrac{1}{2} p_i^2 + \ell \sum_{i<j} (p_j - p_i) + \ell^2 \frac{N(N^2-1)}{24}$$

with $p_i \leq p_{i+1}$ (boson rule)
"True" statistics (symmetry or antisymmetry of wavefunction) is irrelevant since again potential is impenetrable

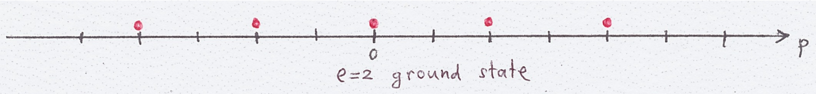
Defining 'pseudomomenta' $\bar{p}_i = p_i + (\frac{N}{2} - i)\ell$ ('open the fan': a partial bosonization) the energy becomes *free*

$$E = \sum_{i=1}^{N} \tfrac{1}{2} \bar{p}_i^2$$

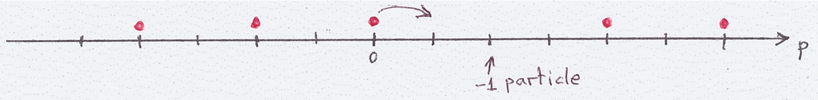
but now the $\bar{p}_i$ satisfy $\bar{p}_{i+1} - \bar{p}_i \geq \ell$

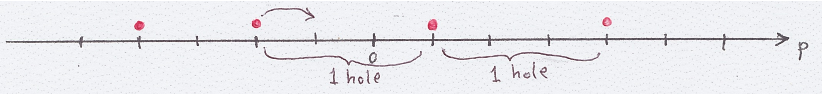
Fractional exclusion statistics (Haldane 1991)

- The ground state forms a ‘pseudo-Fermi sea’

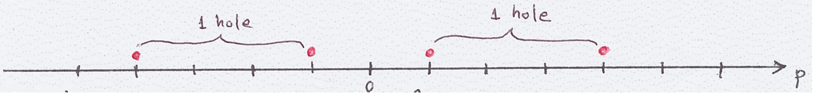


- There are particle and hole excitations:





$1 \text{ particle} = -\ell \text{ holes}$



Particles	Holes
• Pack one every ℓ units	• Pack one every one unit
• Excite by one unit	• Excite by ℓ units

There is a particle-hole duality generalizing the one in free Fermi systems:

$$1 \text{ particle} + \ell \text{ holes} = 0$$

$$\text{particle} \leftrightarrow \text{hole} \ , \quad \ell \leftrightarrow \frac{1}{\ell} \ , \quad p \leftrightarrow \ell p$$

In the classical ($\ell \to \infty$) continuum ($N \to \infty$) limit:

Particles: Solitons

Holes: Waves

Particle symmetries and Statistics

We will 'derive' the model, obtaining it from a set of principles.
Consider a set of indistinguishable particles

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Many-body (first quantized) point of view:

- Identical particles admit the permutation group S_N as a dynamical symmetry commuting with the Hamiltonian
- indistinguishable particles elevate the permutation group to a (discrete) gauge symmetry
- Configurations in which particles are permuted correspong to the same physical state and constitute 'gauge' copies of the system.
- All physical operators commute with S_N

This means that

- Irreps of S_N are superselection sectors, that is,
- Irreps are independent quantum systems

The two-fold way

There are two distinct ways to deal with a gauge system:

- Reduce the system to a set of gauge invariant observables (eliminate gauge freedom)
- Realize the gauge symmetry as a symmetry of the Hilbert space and impose gauge constraints on the states

In the latter case, we also have the option to

- Augment the system and its gauge symmetry as long as the gauge invariant observables remain the same
- This can lead to simplifications, but also may cause problems ('anomalies')

We shall explain in the sequel how each of the above will lead to (versions of) the Calogero model.

Reducing to gauge-invariant observables

- Phase space particle coordinates x_i, p_i are not physical observables
- A set of invariants can be constructed in terms of symmetric functions of the above coordinates:

$$I_{n,m} = \sum_{i=1}^{N} : x_i^n p_i^m :$$

with $n, m \geq 0$ and $: \cdot :$ denoting an ordering

- E.g., the symmetric (Weyl) ordering between x_i and p_i can be adopted, which also ensures the hermiticity of $I_{m,n}$

However

- $I_{n,m}$ are, in general, overcomplete even if $n, m < N$
- N^2 observables for $2N$ independent operators
- Classically: algebraic identities between various $I_{m,n}$
- QM: presence of Casimirs in the algebra of $I_{m,n}$

The Algebra

- $I_{m,n}$ satisfy a version of the so-called W_N algebra
- In the $N \to \infty$ limit this becomes the W_∞ algebra
- In that limit their commutation relations can be repackaged into the 'sine algebra' by defining

$$I(k,q) = \sum_{m,n=0}^{\infty} \frac{k^m q^n}{m!n!} I_{m,n}$$

  with k, q continuous 'Fourier' variables
- Assuming Weyl ordering, the $I_{m,n}$, the $I(k,q)$ satisfy

$$[I(k,q), I(k',q')] = 2i \sin \frac{kq' - k'q}{2} I(k+k', q+q')$$

- Commutators of $I_{m,n}$ can be obtained by Taylor expanding $I(k,q)$ and matching coefficients of $k^m q^n$

$$[I_{m,n}, I_{m'n'}] = i(mn' - nm') I_{m+m'-1,n+n'-1} + O(\hbar^2)$$

- This is the 'classical' W_∞ algebra. (We have put $\hbar = 1$, but lowest order in $\hbar$ corresponds to lowest order in m, n.)
- For finite N the algebra becomes nonlinear due to the presence of identities
- We can keep the full tower of $I_{m,n}$ and impose the identities as relations for the Casimirs of the algebra

The above algebra admits a host of representations, corresponding to the underlying particles being bosons, fermions, parabosons, parafermions and various other possibilities. Where is Calogero?

Concentrate on the special case $N = 2$

- Center of mass variables: gauge unvariant ($I_{1,0}$ and $I_{0,1}$)

$$X = \frac{x_1 + x_2}{2} \ , \quad P = p_1 + p_2$$

- Relative variables: not gauge invariant (odd under perm)

$$x = x_1 - x_2 \ , \quad p = \frac{p_1 - p_2}{2}$$

- We can form quadratic invariants as

$$A = x^2 \ , \quad B = \frac{xp + px}{2} \ , \quad C = p^2$$

corresponding to

$$A = 2I_{2,0} - I_{1,0}^2 \ , \quad B = I_{1,1} - \frac{1}{4}(I_{1,0}I_{0,1} + I_{0,1}I_{1,0}) \ , \quad C = \frac{1}{2}I_{0,2} - \frac{1}{4}I_{0,1}^2$$

A, B, C commute with the center of mass variables and close to the $SL(2, R)$ algebra

$$[A, B] = 2iA \ , \quad [B, C] = 2iB \ , \quad [A, C] = 4iB$$

- Classically, they satisfy the constraint $AC = B^2$
- Quantum mechanically this translates to the Casimir

$$G = \frac{AC + CA}{2} - B^2$$

- Physical Hilbert spaces correspond to irreducible representations of the algebra of A, B, C, along with X, P

- Unitarity of $SL(2,R)$ representations mandates $G \geq -\frac{1}{2}$, so we may parametrize in direct analogy to the Calogero case

$$G = \ell(\ell - 1)$$

- A realization of the representation with the above G is

$$A = x^2 \ , \quad B = \frac{xp + px}{2} \ , \quad C = p^2 + \frac{\ell(\ell-1)}{x^2}$$

- The relative kinetic energy of the particles has acquired an inverse-square potential part. The free particle hamiltonian for this system would become

$$H = \frac{P^2}{2} = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{\ell(\ell-1)}{(x_1 - x_2)^2}$$

  This is the Calogero model!

- The Calogero coupling appears through the Casimir
- The Hamiltonian and other observables of the system in terms of physical variables $X, P; A, B, C$ are ℓ-independent

For $N > 2$: similar approach:

- Scalar representations of the W_N or W_∞ algebra
- Calogero obtains as one particular realization of the indistinguishabe particle system
- A Hilbert space realization in terms of x_i, p_i contains $N!$ copies (the defining $N!$-dimensional realization of S_N)
- This is inconsequential: 'normal' statistics of the particles have become immaterial, supplanted by the Calogero dynamics
- For $G \leq 0$ $(0 \leq \ell \leq 1)$ there are two inequivalent irreps
- For $G = 0$ we get $\ell = 0$ (Bose irrep) and $\ell = 1$ (Fermi irrep)

Statistics $\leftrightarrow$ Superselection sectors $\leftrightarrow$ Calogero dynamics

Augmenting the symmetry: Matrix Model

Standard approach: keep the original $\{x_i, p_i\}$ formulation and impose S_N gauge invariance on the states ('Gauss' law' – irreps). This leads to fermions, bosons and parastatistics generalizations.

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A different approach: start with an augmented system, with both the dynamical variables and the gauge symmetry expanded but with the same gauge invariant degrees of freedom:

- Formulate the particle coordinates as the eigenvalues of an $N \times N$ matrix
- No a priori ordering of these eigenvalues: identical particles
- Permutation symmetry promoted to the continuous symmetry of unitary conjugations U_N of the matrix
- Indistinguishable particles: U_N promoted to gauge symmetry
- Irreps of U_N correspond to physical states
- States within irrep related throught U_N mapped to a unique physical state

The hermitian matrix model

M: Hermitian $N \times N$ matrix with Lagrangian

$$\mathcal{L} = \mathrm{Tr}\left\{\frac{1}{2}\dot{M}^2 - V(M)\right\}$$

$V(x)$ is a scalar potential evaluated for the matrix variable M.

- Time-translation invariance leads to conserved energy

$$H = \mathrm{Tr}\left\{\frac{1}{2}\dot{M}^2 + V(M)\right\}$$

- Invariance under time-independent $SU(N)$ conjugations $M \to UMU^{-1}$ leads to conserved Hermitian traceless matrix

$$J = i[M, \dot{M}]$$

  where [,] denotes ordinary matrix commutator
- J are 'gauge charges' that will determine the realization ('statistics') of the indistinguishable particle system

Prametrize M as

$$M = U \Lambda U^{-1}$$

$U(t)$ is the unitary 'angular' part of the matrix
$\Lambda(t) = diag\{x_1, \ldots x_N\}$ are the eigenvalues.
Conserved matrix J is the 'angular momentum' of $U(t)$
Define the 'gauge potential'

$$A = -U^{-1}\dot{U}$$

$\dot{M}$, J and the lagrangian $\mathcal{L}$ become, in this parametrization,

$$\begin{aligned}
\dot{M} &= U\left(\dot{\Lambda} + [\Lambda, A]\right)U^{-1} := ULU^{-1} \\
J &= iU\left([\Lambda, [\Lambda, A]]\right)U^{-1} := UKU^{-1} \\
\mathcal{L} &= \mathrm{Tr}\left\{\tfrac{1}{2}\dot{\Lambda}^2 + [\Lambda, A]^2 - V(\Lambda)\right\} \\
&= \tfrac{1}{2}\sum_{i=1}^{N}\dot{x}_i - \tfrac{1}{2}\sum_{i,j=1}^{N}(x_i - x_j)^2 A_{ij}A_{ji}
\end{aligned}$$

The matrix elements of A and K are related

$$K_{jk} = i\,[\Lambda, [\Lambda, A]]_{jk} = i(x_j - x_k)^2 A_{jk}\ , \quad K_{ii} = 0$$

Solving for A_{jk} and putting into $\mathcal{L}$ we obtain

$$\mathcal{L} = \sum_i \tfrac{1}{2}\dot{x}_i^2 + \tfrac{1}{2}\sum_{i\neq j} \frac{K_{ij}K_{ji}}{(x_i - x_j)^2} - \sum_i V(x_i)$$

First two terms: kinetic (from $\dot{M}^2$); last term: potential
The Hamiltonian H is

$$H = \sum_i \tfrac{1}{2}p_i^2 + \tfrac{1}{2}\sum_{i\neq j} \frac{K_{ij}K_{ji}}{(x_i - x_j)^2} + \sum_i V(x_i)$$

- The K_{ij} are still dynamical
- Models obtain by choosing ‘sectors’ for J and thus K
- $J = 0 \Rightarrow K = 0$: free particles

The plain vanilla Calogero

Choose J as simple as possible: rank 1 minus trace

and thus

$$J = \ell(vv^\dagger - 1)\ , \quad v^\dagger v = N$$

$$K = \ell(uu^\dagger - 1)\ , \quad u = U^{-1} v$$

From $K_{ii} = 0$

$$u_i u_i^* = 1 \quad \text{(no sum)}$$

So the coefficient of the inverse-square potential becomes

$$K_{ij} K_{ji} = \ell u_i u_j^* \ \ell u_j u_i^* = \ell^2 \quad (i \neq j)$$

and H becomes

$$H = \sum_i \tfrac{1}{2} p_i + \sum_{i<j} \frac{\ell^2}{(x_i - x_j)^2} + \sum_i V(x_i)$$

The Calogero model!

- The strength $g = \ell^2$ is related to the conserved charge ℓ
- External potential $V(x)$ arbitrary at this stage
- Other choices of J and K: postpone to quantum case

Consider first the free matrix model: $V(x) = 0$

- Free motion on matrix space
- Matrix momentum $\dot{M}$ is conserved: $\dot{M} = 0$

Therefore $$I_n = \mathrm{Tr}\,\dot{M}^n = \mathrm{Tr}(U\dot{M}U^{-1})^n = \mathrm{Tr}\,L^n$$

are constants of motion

- $\dot{M}$ is the canonical momentum matrix
- Its elements have vanishing Poisson brackets

Therefore, the I_n are in involution: $\{I_m, I_n\} = 0$

$$L_{jk} = (U\dot{M}U^{-1})_{jk} = \delta_{jk}\dot{x}_j - (1-\delta_{jk})\frac{iK_{jk}}{x_j - x_k}$$

The Lax matrix! $$= \delta_{jk}\,\dot{x}_j - (1-\delta_{jk})\frac{iu_j u_k^*}{x_j - x_k}$$

- In $\mathrm{Tr}\,L^n$ products $u_{i_1}u_{i_2}^* u_{i_2}u_{i_3}^* \dots u_{i_1}^* = \ell^m$ will appear
- I_n will involve only x_i, $\dot{x}_i = p_i$ and the constant ℓ
- I_n are the conserved integrals of the Calogero model

The actual motion of the Calogero model can be obtained:

$$M = B + Ct \ , \quad J = i[M, \dot{M}] = i[B, C] = \ell(uu^\dagger - 1)$$

- Time independent unitary transformation: put $u_i = 1$
- Have to find matrices B and C satisfying the constraint

$$[B, C]_{jk} = -i\ell(1 - \delta_{jk})$$

Initial value choice: B diagonal

$$B_{jk} = \delta_{jk}\, q_j \ , \quad C_{jk} = \delta_{jk} p_j - (1 - \delta_{jk}) \frac{i\ell}{q_j - q_k}$$

q_i, p_i: initial coordinates and momenta of particles

Scattering choice: C diagonal

$$B_{jk} = \delta_{jk} a_j + (1 - \delta_{jk}) \frac{i\ell}{k_j - k_k} \ , \quad C_{jk} = \delta_{jk} k_j$$

k_i, a_i: asymptotic momenta and impact parameters

$$x_i = k_i t + a_i + \sum_j \ell^2/(tk_{ij}^3) + O(1/t^2)$$

Add harmonic oscillator potential $V(x) = \frac{1}{2}\omega^2 x^2$

- Use similar technique as in ordinary harmonic oscillator
- Non-Hermitian matrix $Q = \dot{M} + i\omega M$ satisfies

$$\dot{Q} = i\omega Q \ , \quad [Q, Q^\dagger] = 2\omega J = 2\omega\ell(uu^\dagger - 1)$$

and evolves as

$$Q(t) = e^{i\omega t} Q(0) \ , \quad Q(0) = C + i\omega B$$

with B, C as in the free case. It follows that

$$I_n = \mathrm{Tr}(Q^\dagger Q)^n = \mathrm{Tr}\left[(L + i\omega\Lambda)(L - i\omega\Lambda)\right]^n$$

are conserved. Involution of I_n follows from

$$\{Q_{jk}, Q_{lm}\} = \{Q^\dagger_{jk}, Q^\dagger_{lm}\} = 0 \ , \quad \{Q_{jk}, Q^\dagger_{lm}\} = 2i\omega\delta_{jm}\delta_{lk}$$

and some algebra (more on this in the QM case)

- Up to quartic external potentials $V = c_2 x^2 + c_3 x^3 + c_4 x^4$ are also integrable
- Are there any more? (Open question)

The Quantum Hermitian matrix model

Hermitian model with quadratic potential

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}(\dot{M}^2 - \omega^2 M^2) = \sum_{jk}\left(\frac{1}{2}|\dot{M}_{jk}|^2 - \frac{\omega^2}{2}|M_{jk}|^2\right)$$

- N^2 harmonic oscillators! In principle trivial and solvable
- All nontrivial features emerge from reduction to 'angular momentum' sectors J

Canonical momentum matrix P conjugate to M

$$P = \frac{\partial \mathcal{L}}{\partial \dot{M}} = \dot{M}$$

In terms of M and P the Hamiltonian becomes

$$H = \mathrm{Tr}\left(\frac{1}{2}P^2 + \frac{1}{2}\omega^2 M^2\right)$$

The QM commutator implied by the Poisson brackets is

$$[M_{j_1 k_1}, P_{j_2 k_2}] = i\delta_{j_1 k_2}\delta_{j_2 k_1}$$

or

$$[M_1, P_2] = iT_{12}$$

Operator $T_{12} = T_{21}$ exchanges spaces 1 and 2

$$X_1 T_{12} = T_{12} X_2 \ , \quad X_2 T_{12} = T_{12} X_1 \ , \quad \mathrm{Tr}_2 T_{12} = I_1$$

Matrix ladder ops $A = \frac{1}{\sqrt{2\omega}}(P - i\omega M)$, $A^\dagger = \frac{1}{\sqrt{2\omega}}(P + i\omega M)$

$$[A_1, A_2^\dagger] = T_{12} \ , \quad [A_1, A_2] = [A_1^\dagger, A_2^\dagger] = 0$$

Define the matrices $L = A^\dagger A \ , \quad R = - :AA^\dagger: = -AA^\dagger + N$

Ladder commutators imply

$$[L_1, A_2^\dagger] = T_{12} A_2^\dagger \ , \quad [R_1, A_2^\dagger] = -A_2^\dagger T_{12} \ \ \text{(plus herm. conj.)}$$

Using these, we obtain for L and R

$$\begin{aligned} [L_1, L_2] &= (L_1 - L_2) T_{12} \\ [R_1, R_2] &= (R_1 - R_2) T_{12} \\ [L_1, R_2] &= 0 \end{aligned}$$

L, R are commuting $U(N)$ algebras in disguise! To see, define

$$L^a = \mathrm{Tr}(T^a L) \ , \quad R^a = \mathrm{Tr}(T^a R)$$

T^a, $a = 0, \ldots N^2 - 1$ are fundamental $U(N)$ matrices

$$[T^a, T^b]_{\mathrm{matrix}} = i\, f^{abc} T^c \ , \quad \mathrm{Tr}(T^a T^b) = \delta^{ab}$$

Then scalar operators L^a, R^a satisfy

$$\begin{aligned} [L^a, L^b] &= i\, f^{abc} L^c \\ [R^a, R^b] &= i\, f^{abc} R^c \\ [L^a, R^b] &= 0 \end{aligned}$$

Now we have laid all the algebraic groundwork.
We will continue quantization in lecture 2.