

Statistical physics of 2d gas mixtures from symplectic volumes of vortex moduli

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Some *mantras* of Statistical Physics:

- ▶ A subject driven by examples (e.g. “integrable” cases)
- ▶ Models are often built from a minimal toolkit
- ▶ Macroscopic features: the thermodynamic limit
- ▶ Information on particle interactions out of global data



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Today:

Another case-study with explicit results, from gauge theory.
Particle interactions are built in **geometrically**.

Outline

Gauged sigma-models and the vortex equations

The L^2 metric on vortex moduli spaces

Vortices with toric targets

Asymptotic geometry of L^2 metrics

Gibbs' partition functions and thermodynamics

By-product of on-going interdisciplinary project:

Supersymmetric quantum mechanics on vortex moduli spaces

Main collaborators:

- ▶ Marcel Bökstedt (Aarhus)
- ▶ Ákos Nagy (Duke NC)
- ▶ Martin Speight (Leeds)
- ▶ Christian Wegner (Bonn)

Some references:

arXiv:1010.1488 (MB+NR)

arXiv:1410.2429 (MB+NR)

arXiv:1605.07921 (MB+NR)

arXiv:1711.04624 (MB)

arXiv:1807.00712 (NR+MS)

(in preparation) (MB+NR), (NR+CW), (MB+NR+CW)

Classical partition functions for 2-dimensional vortex gas mixtures.

Symplectic geometry of (nonlinear) gauged sigma-models

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Gauged sigma-models and the vortex equations

Gauged sigma-models

Ingredients:

- ▶ $(\Sigma, j_\Sigma, \omega_\Sigma)$ Kähler structure on an oriented surface (*base*)
- ▶ (X, j_X, ω_X) another Kähler manifold (*target*)
- ▶ G compact Lie group with invariant metric
 $\mathfrak{g} := \text{Lie}(G)$
- ▶ $\sharp : \mathfrak{g}^* \rightarrow \mathfrak{g}$ ‘musical’ isomorphism
- ▶ G -action on X : holomorphic, Hamiltonian
- ▶ $\mu^\sharp : X \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{\sharp} \mathfrak{g}$ moment map

Gauged sigma-model: energy

Fields: $(A, \phi) \in \mathcal{A}(P) \times \Gamma(\Sigma, P^X)$

- ▶ A connection in principal G -bundle $P \rightarrow \Sigma$
- ▶ ϕ section of associated bundle $P^X := P \times_G X \rightarrow \Sigma$

Topological charge:

$$[\phi]_2^G := ((\tilde{f} \times \phi)/G)_*[\Sigma] \in H_2^G(X; \mathbb{Z}) \quad \text{for } P = f^*EG$$

Yang–Mills–Higgs functional and Bogomol'nyi's trick:

$$\begin{aligned} E(A, \phi) &:= \frac{1}{2} \int_{\Sigma} \left(|F_A|^2 + |d^A \phi|^2 + |\mu \circ \phi|^2 \right) \\ &= \langle [\omega_X - \mu]_G^2, [\phi]_2^G \rangle + \frac{1}{2} \int_{\Sigma} \left(\left| *F_A + \mu^\sharp \circ \phi \right|^2 + 2|\bar{\partial}^A \phi|^2 \right) \end{aligned}$$

Vortex moduli spaces

The (symplectic) vortex equations:

$$(V1) \quad \bar{\partial}^A \phi = 0$$

$$(V2) \quad *F_A + \mu^\sharp \circ \phi = 0$$

NB: Same can be done for “antivortices” s.t. $\partial^A \phi = 0$ etc.
But they don't live with vortices in BPS configurations.

We'll see in a moment how to implement coexistence of vortices and antivortices in another sense.

Fix $\mathbf{h} \in H_2^G(X; \mathbb{Z})$. Moduli spaces defined:

$$\mathcal{M}_{\mathbf{h}}^X(\Sigma) := \left\{ (A, \phi) \left| \begin{array}{l} (V1), (V2) \\ \text{and } [\phi]_2^G = \mathbf{h} \end{array} \right. \right\} / \mathcal{G}(P)$$

The L^2 metric on vortex moduli spaces

The L^2 -metric

Can recast this quotient in terms of Kähler reduction:

- ▶ (V1) is invariant under complexification $\mathcal{G}(P)^{\mathbb{C}}$.
- ▶ RHS of (V2) interpreted as $\mathcal{G}(P)$ -moment map.

Thus (the smooth part of) $\mathcal{M}_{\mathfrak{h}}^X(\Sigma)$ receives a *Kähler structure*.

Tangent spaces:

$$T_A \mathcal{A}(P) = \Omega^1(\Sigma; P \times_{Ad} \mathfrak{g})$$

$$T_{\phi} \Gamma(\Sigma, P^X) = \Gamma(\Sigma, \phi^* TX/G)$$

Complex structure:

$$(\dot{A}, \dot{\phi}) \mapsto (*\dot{A}, (\phi^* j_{\Sigma})\dot{\phi})$$

L^2 -metric:

$$(\dot{A}_1, \dot{\phi}_1) \cdot (\dot{A}_2, \dot{\phi}_2) := \int_{\Sigma} \left(\langle \dot{A}_1, * \dot{A}_2 \rangle + (\phi^* g_X)(\dot{\phi}_1, \dot{\phi}_2) \omega_{\Sigma} \right)$$

Protypical example: vortices in line bundles

Take $X = \mathbb{C}$ with usual action of $G = \mathrm{U}(1)$,

$$\mu(x) = -\frac{1}{2}(|x|^2 - \tau)$$

Suppose Σ is closed and $\deg(P) = k$.

Then (V2) has solutions only if $\tau \mathrm{Vol}(\Sigma) \geq 4\pi k$.

THEOREM (... , Bradlow,...): Assume $\tau \mathrm{Vol}(\Sigma) > 4\pi k$; then

$$\mathcal{M}_k^{\mathbb{C}}(\Sigma) \cong \mathrm{Sym}^k(\Sigma).$$

This is a complex manifold with obvious complex structure J_{Σ} .
But describing g_{L^2} (or ω_{L^2}) is very hard.

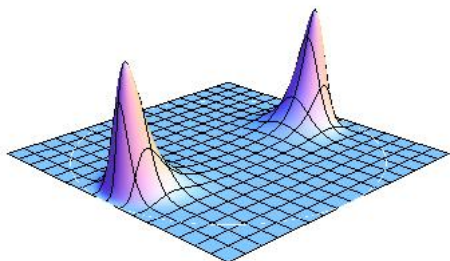
Protoptypical example: vortices in line bundles



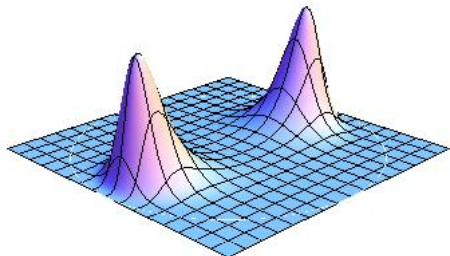
Vortice di
golosità
alla vaniglia



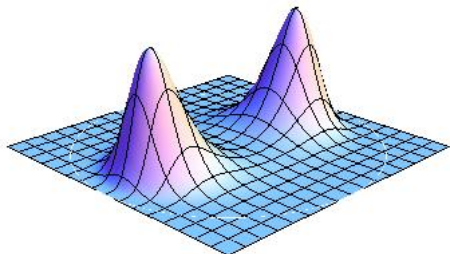
Moduli spaces: geodesic motion



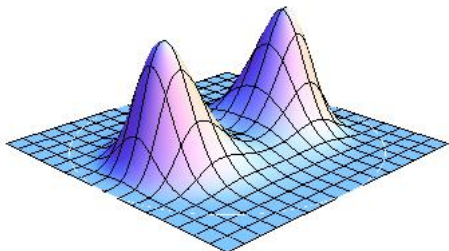
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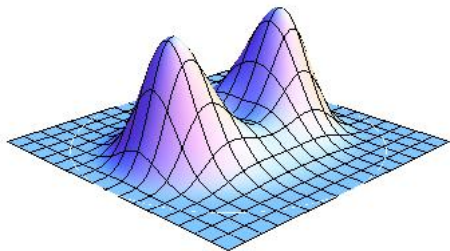
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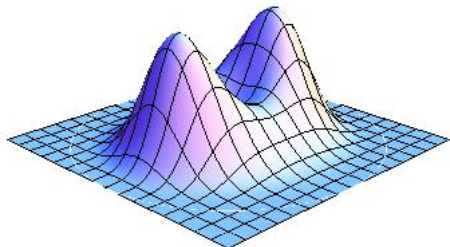
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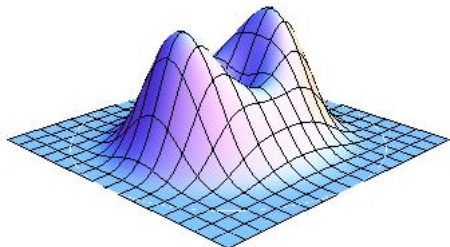
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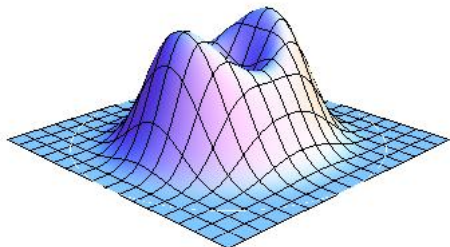
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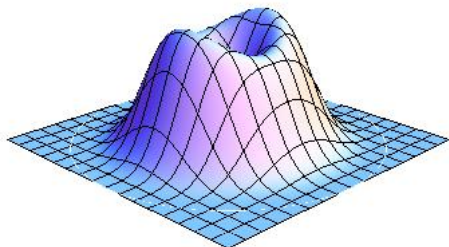
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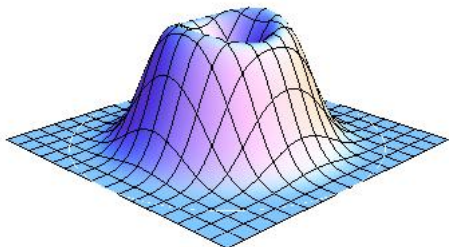
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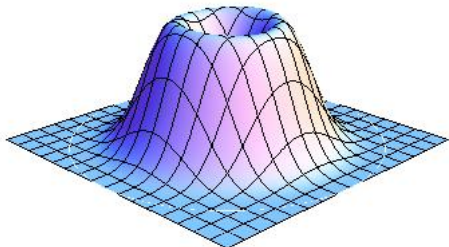
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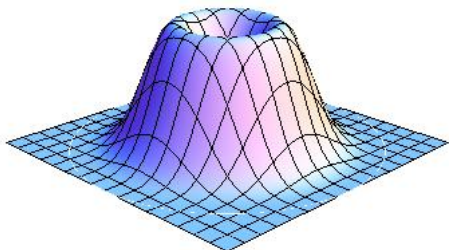
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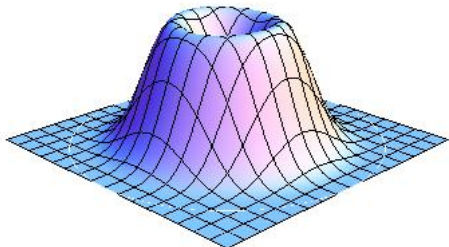
Moduli spaces: geodesic motion



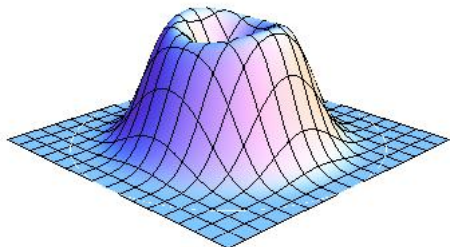
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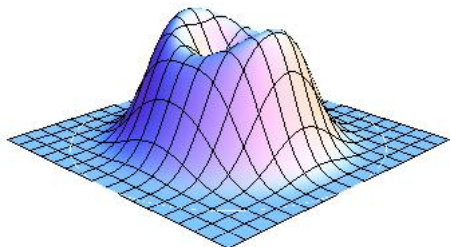
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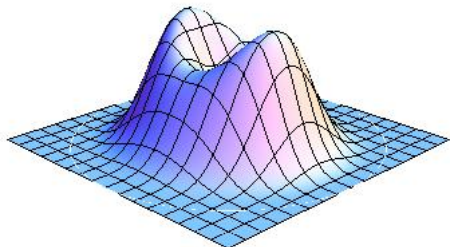
Moduli spaces: geodesic motion



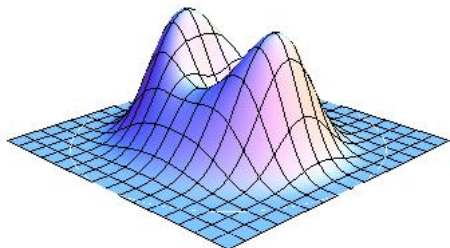
Moduli spaces: geodesic motion



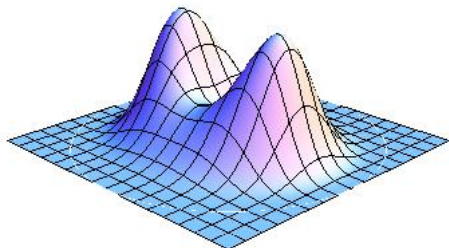
Moduli spaces: geodesic motion



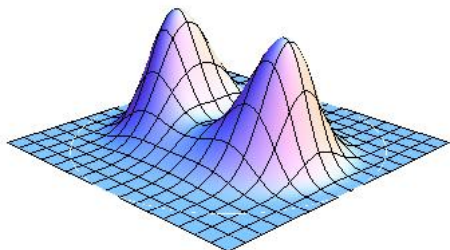
Moduli spaces: geodesic motion



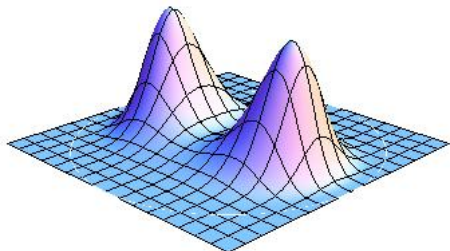
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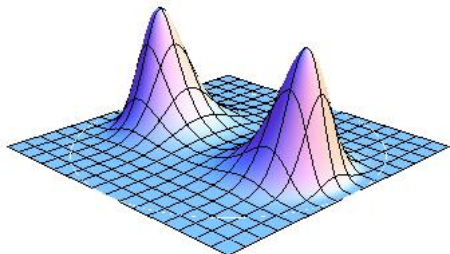
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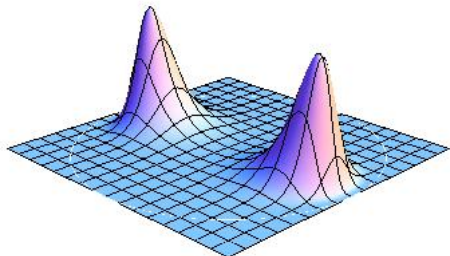
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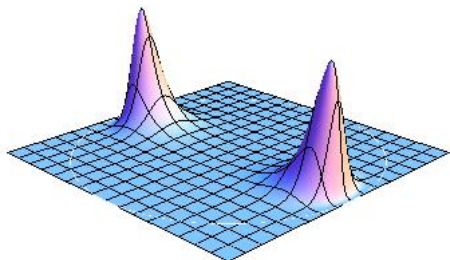
Moduli spaces: geodesic motion



Moduli spaces: geodesic motion



Moduli spaces: geodesic motion



Vortices with toric targets

Toric targets

An interesting setting:

- ▶ X Kähler toric manifold,
- ▶ $G = T \subset T^{\mathbb{C}} \subset X$ its (real) torus

Then for X, Σ compact we have a good description of $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$:

THEOREM (M Bökstedt + NR):

Suppose X is constructed as Fan_{Δ} for a Delzant polytope Δ ,
 $\mathbf{h} \in {}^T\text{BPS}_{\Sigma}^X$ with $a_{\mathbb{R}}^* \circ \mathbf{h}([\omega_{\Sigma}]^{\vee}) \in \text{int } \Delta$ and

$$k_{\rho} = \langle c_1^T(D_{\rho}), \mathbf{h} \rangle \quad \text{for } \rho \in \text{Fan}_{\Delta}(1).$$

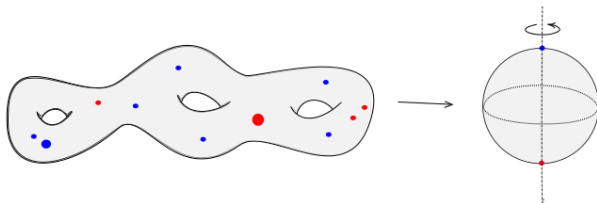
Then $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$ is the smooth manifold

$$\begin{aligned} \mathcal{M}_{\mathbf{h}}^X(\Sigma) &= \text{Div}_{+}^{\mathbf{k}}(\Sigma; (\partial\Delta)^{\vee}) \subset \prod_{\rho \in \text{Fan}_{\Delta}(1)} \text{Sym}^{k_{\rho}}(\Sigma) \\ &=: \left\{ \mathbf{d} : [\lambda_0, \dots, \lambda_{\ell}] \neq (\partial\Delta)^{\vee} \Rightarrow \bigcap_{i=0}^{\ell} \text{supp}(d_{\lambda_{\rho_i}}) = \emptyset \right\} \end{aligned}$$

Gauged $\mathbb{P}^1$ -model: BPS vortices and antivortices

- ▶ For today's talk we focus on $X = \mathbb{P}^1 \cong S^2$, $T = \mathrm{U}(1) \cong S^1$ (Schroers)
- ▶ In this situation, $\mathcal{M}_{(k_+, k_-)}^{\mathbb{P}^1}(\Sigma)$ was already well understood as a complex manifold: (Mundet, Sibner-Sibner-Yang, Baptista)

$$\mathcal{M}_{(k_+, k_-)}^{\mathbb{P}^1}(\Sigma) = \text{Sym}^{k_+}(\Sigma) \times \text{Sym}^{k_-}(\Sigma) \setminus \Delta_{(k_+, k_-)}$$



These spaces have a boundary even if Σ is compact.

Gauged $\mathbb{P}^1$ -model: BPS vortices and antivortices



Gauged $\mathbb{P}^1$ -model: BPS vortices and antivortices

- ▶ Target: Riemann sphere $\mathbb{P}^1$ of unit radius.
- ▶ Moment map: (minus) height function, possibly translated:

$$\mu(x) = -\frac{1 - |x|^2}{1 + |x|^2} + \tau$$

- ▶ For Σ compact, have “Bradlow bounds”

$$-(1 + \tau)\text{Vol}(\Sigma) \leq 2\pi(k_+ - k_-) \leq (1 - \tau)\text{Vol}(\Sigma)$$

obtained from integrating (V2). Here, $(k_+ - k_-) = \deg P$.

- ▶ Energy bound: $E(A, \phi) \geq 2\pi(1 - \tau)k_+ + 2\pi(1 + \tau)k_-$

We will later focus on example with $k_+ = k_- = 1$, and more specifically at L^2 -geometry close to ‘pair annihilation’.

Asymptotic geometry of L^2 metrics

The case $\Sigma = \mathbb{R}^2$

- ▶ Topology is different for $\Sigma = \mathbb{R}^2$: P trivial, $\partial\Sigma = S_\infty^1$

ϕ maps S_∞^1 to equator of $\mathbb{P}^1$

- ▶ THEOREM (Y Yang): ($\tau = 0$):

$$\mathcal{M}_{(k_+, k_-)}^{\mathbb{P}^1}(\mathbb{R}^2) = \text{Sym}^{k_+} \mathbb{C} \times \text{Sym}^{k_-} \mathbb{C} \setminus \Delta_{(k_+, k_-)}$$

- ▶ (V1)+(V2) $\Rightarrow$ Taubes' equation for $h := \log |\phi|^2$:

$$\nabla^2 h - 2 \tanh \frac{h}{2} = 4\pi \left(\sum_{r=1}^{k_+} \delta_{z_r^+} - \sum_{r=1}^{k_-} \delta_{z_r^-} \right)$$

AIM: Extract L^2 -metric from this elliptic PDE.

The case $\Sigma = \mathbb{R}^2$

- ▶ $h := \log |\phi|^2$
- ▶ Taubes' equation: $(k_+, k_-) = (1, 1)$

$$\nabla^2 h - 2 \tanh \frac{h}{2} = 4\pi (\delta_{z_+} - \delta_{z_-})$$

The case $\Sigma = \mathbb{R}^2$

- ▶ $h := \log |\phi|^2$
- ▶ Taubes' equation: $z_+ = -z_- = \varepsilon$

$$\nabla^2 h - 2 \tanh \frac{h}{2} = 4\pi (\delta_\varepsilon - \delta_{-\varepsilon})$$

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- ▶ $h := \log |\phi|^2$
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$$\nabla^2 h - 2 \tanh \frac{h}{2} = 4\pi (\delta_\varepsilon - \delta_{-\varepsilon})$$

- ▶ Regularize: $h(z) = \log \left| \frac{z-\varepsilon}{z+\varepsilon} \right|^2 + \tilde{h}(z)$

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$$\nabla^2 \tilde{h} - 2 \frac{|z-\varepsilon|^2 e^{\tilde{h}} - |z+\varepsilon|^2}{|z-\varepsilon|^2 e^{\tilde{h}} + |z+\varepsilon|^2} = 0$$

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$$\nabla_z^2 \tilde{h} - 2 \frac{|z-\varepsilon|^2 e^{\tilde{h}} - |z+\varepsilon|^2}{|z-\varepsilon|^2 e^{\tilde{h}} + |z+\varepsilon|^2} = 0$$

- ▶ Rescale: $z =: \varepsilon w$, $\hat{h}(w) = \tilde{h}(\varepsilon w)$

The case $\Sigma = \mathbb{R}^2$

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$$\nabla_w^2 \hat{h} - 2\varepsilon^2 \frac{|w-1|^2 e^{\hat{h}} - |w+1|^2}{|w-1|^2 e^{\hat{h}} + |w+1|^2} = 0$$

- ▶ Solve with boundary condition $\hat{h}(w) \xrightarrow{|w| \rightarrow \infty} 0$.

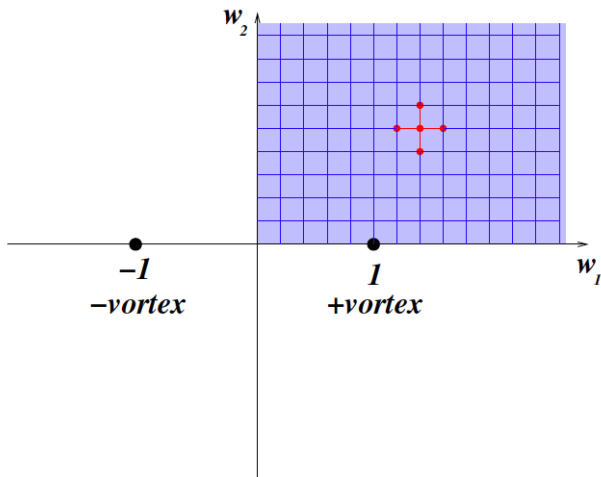
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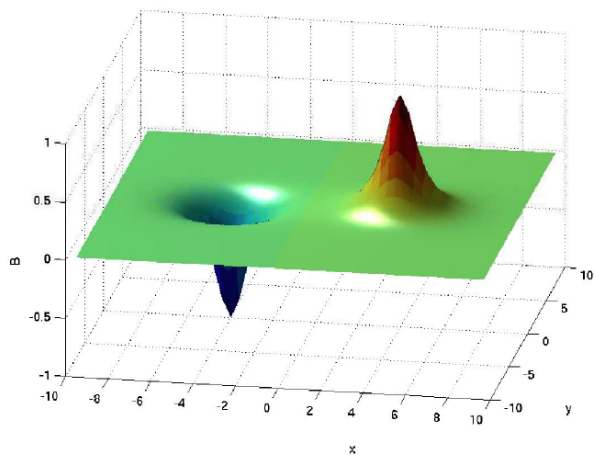
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The case $\Sigma = \mathbb{R}^2$



The case $\Sigma = \mathbb{R}^2$



$$\varepsilon = 4$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A}, \dot{\phi})\|_{L^2}^2 = \frac{1}{2} \int_{\Sigma} (|\dot{A}|^2 + |\dot{\phi}|^2)$$

- ▶ Assume all vortex positions remain distinct
- ▶ $\dot{\phi}$ solution of linearised Taubes' equation at ϕ
Impose Coulomb gauge

$\eta := \frac{\dot{\phi}}{\phi}$ provides 1-current in $\Sigma \times \mathcal{M}_{(k_+, k_-)}^{\mathbb{P}^1}(\mathbb{R}^2)$

It evaluates at $v = (z^{\pm}, \dot{z}^{\pm}) \in T\mathcal{M}_{(k_+, k_-)}^{\mathbb{P}^1}(\mathbb{R}^2)$ to yield
0-current η_v singular over $\{(z, (\phi)) : z \in \text{supp}(\phi)\}$

$$\left(\nabla^2 - \text{sech}^2 \frac{h}{2}\right) \eta_v = 4\pi \left(\sum_{r=1}^{k_+} \dot{z}_r^+ \partial_{z_r^+} \delta_{z_r^+} - \sum_{s=1}^{k_-} \dot{z}_s^- \partial_{z_s^-} \delta_{z_s^-} \right)$$

$$\Rightarrow \quad \eta_v = \sum_{r=1}^{k_+} \dot{z}_r^+ \frac{\partial h}{\partial z_r^+} + \sum_{s=1}^{k_-} \dot{z}_s^- \frac{\partial h}{\partial z_s^-} \quad \text{is (unique) solution}$$

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The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\begin{aligned}\|(\dot{A}, \dot{\phi})\|_{L^2}^2 &= \frac{1}{2} \int_{\mathbb{R}^2} (|\dot{A}|^2 + |\dot{\phi}|^2) \\ &= \frac{1}{2} \int_{\mathbb{R}^2} \left(4 \partial_z \bar{\eta}_v \partial_{\bar{z}} \eta_v + \operatorname{sech}^2 \frac{h}{2} \bar{\eta}_v \eta_v \right)\end{aligned}$$

The case $\Sigma = \mathbb{R}^2$

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$$\begin{aligned}\|(\dot{A}, \dot{\phi})\|_{L^2}^2 &= \frac{1}{2} \int_{\mathbb{C}} (|\dot{A}|^2 + |\dot{\phi}|^2) \\ &= \frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C} \setminus \cup_{r,s} B_{\epsilon}(z_{r,s}^{\pm})} \left(4 \partial_z \bar{\eta}_v \partial_{\bar{z}} \eta_v + \operatorname{sech}^2 \frac{h}{2} \bar{\eta}_v \eta_v \right)\end{aligned}$$

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The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\|(\dot{A}, \dot{\phi})\|_{L^2}^2 = \frac{1}{2} \int_{\mathbb{C}} (|\dot{A}|^2 + |\dot{\phi}|^2)$$

$$= -i \lim_{\epsilon \rightarrow 0} \sum_{r,s=1}^{k_+, k_-} \oint_{\partial B_{\epsilon}(z_{r,s}^{\pm})} \bar{\eta}_v \bar{\partial} \eta_v$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

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$$= -i \lim_{\epsilon \rightarrow 0} \sum_{r,s=1}^{k_+, k_-} \oint_{\partial B_\epsilon(z_{r,s}^\pm)} \bar{\eta}_v \bar{\partial} \eta_v$$

$$\eta_v = \frac{\mp \dot{z}_{r,s}^\pm}{z - z_{r,s}^\pm} + O(1) \quad \text{as } z \rightarrow z_{r,s}^\pm$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\begin{aligned}\|(\dot{A}, \dot{\phi})\|_{L^2}^2 &= \frac{1}{2} \int_{\mathbb{C}} \left(|\dot{A}|^2 + |\dot{\phi}|^2 \right) \\ &= \pi \left(\sum_{r,s=1}^{k_+, k_-} |\dot{z}_{r,s}^\pm|^2 + \sum_{p,q}^{+ \text{ or } -} \frac{\partial b_q}{\partial z_p} \dot{z}_p \dot{\bar{z}}_q \right)\end{aligned}$$

Asymptotics near vortex cores, $z = z_{r,s}^\pm$:

$$+h = \log |z - z_r^+|^2 + a_r^+ + \frac{\bar{b}_r^+}{2} (z - z_r^+) + \frac{b_r^+}{2} (\bar{z} - \bar{z}_r^+) + O(|z - z_r^+|^2) \quad \text{as } z \rightarrow z_r^+$$

The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\begin{aligned}\|(\dot{A}, \dot{\phi})\|_{L^2}^2 &= \frac{1}{2} \int_{\mathbb{C}} \left(|\dot{A}|^2 + |\dot{\phi}|^2 \right) \\ &= \pi \left(\sum_{r,s=1}^{k_+, k_-} |\dot{z}_{r,s}^\pm|^2 + \sum_{p,q}^{+ \text{ or } -} \frac{\partial b_q}{\partial z_p} \dot{z}_p \dot{\bar{z}}_q \right)\end{aligned}$$

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The case $\Sigma = \mathbb{R}^2$

Calculation of the metric: **Strachan–Samols localisation**

$$\begin{aligned}\|(\dot{A}, \dot{\phi})\|_{L^2}^2 &= \frac{1}{2} \int_{\mathbb{C}} (|\dot{A}|^2 + |\dot{\phi}|^2) \\ &= \pi \left(\sum_{r,s=1}^{k_+, k_-} |\dot{z}_{r,s}^\pm|^2 + \sum_{p,q}^{+ \text{ or } -} \frac{\partial b_q}{\partial z_p} \dot{z}_p \dot{\bar{z}}_q \right)\end{aligned}$$

Asymptotics near vortex cores, $z = z_{r,s}^\pm$:

$$-h = \log |z - z_s^-|^2 + a_s^- + \frac{\bar{b}_s^-}{2} (z - z_s^-) + \frac{b_s^-}{2} (\bar{z} - \bar{z}_s^-) + O(|z - z_s^-|^2) \quad \text{as } z \rightarrow z_s^-$$

The case $\Sigma = \mathbb{R}^2$

$$\mathcal{M}_{(1,1)}^{\mathbb{P}^1}(\mathbb{C}) \cong \mathbb{C}_{\text{cm}} \times \mathbb{C}^*$$

Aim: understand metric for centred $(+, -)$ -pairs (i.e. on $\mathbb{C}^*$)

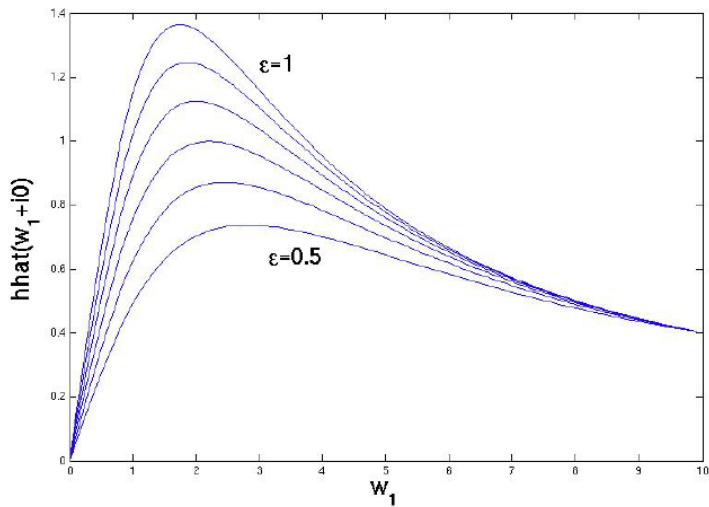
$$z_+ = -z_- = \varepsilon e^{i\vartheta}$$

$$g_{L^2}^{(0)} = F(\varepsilon)(d\varepsilon^2 + \varepsilon^2 d\vartheta^2)$$

where:

- ▶ $F(\varepsilon) = 2\pi \left(2 + \frac{1}{\varepsilon} \frac{d}{d\varepsilon} (\varepsilon b(\varepsilon)) \right)$
- ▶ $b(\varepsilon) := b_1^+(\varepsilon, -\varepsilon)$
- ▶ $\varepsilon b(\varepsilon) = \left. \frac{\partial \hat{h}}{\partial (\operatorname{Re} w)} \right|_{w=1} - 1$

The case $\Sigma = \mathbb{R}^2$



The case $\Sigma = \mathbb{R}^2$

Self-similarity conjecture: consider $\varepsilon \ll 1$.

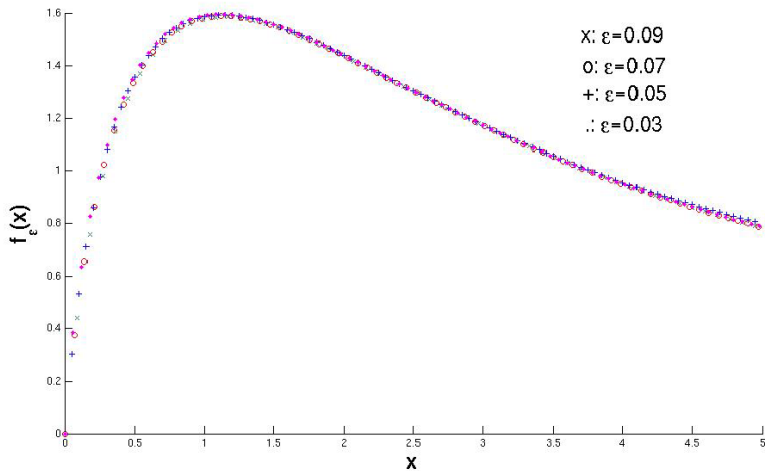
- ▶ Results suggest

$$\hat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w) \quad \text{for } f_* \text{ fixed !?}$$

- ▶ To test this: study

$$f_\varepsilon(z) := \frac{1}{\varepsilon} \hat{h}\left(\frac{z}{\varepsilon}\right)$$

The case $\Sigma = \mathbb{R}^2$



The case $\Sigma = \mathbb{R}^2$

- Results suggest

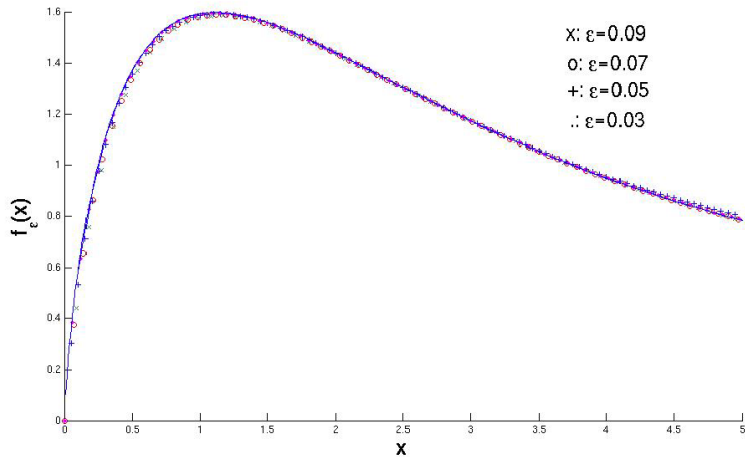
$$\hat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w) \quad \text{for } f_* \text{ fixed !?}$$

- To test this: study

$$f_\varepsilon(z) := \frac{1}{\varepsilon} \hat{h}\left(\frac{z}{\varepsilon}\right) \approx f_*(z)$$

- Now plug this into Taubes' equation to obtain PDE for f_*
- Take formal limit as $\varepsilon \rightarrow 0$;
obtain screened Poisson equation with a simple source.
- Solve equation to obtain asymptotics of metric $g_{L^2}^{(0)}$.

The case $\Sigma = \mathbb{R}^2$



The case $\Sigma = \mathbb{R}^2$

Asymptotics of conformal factor:

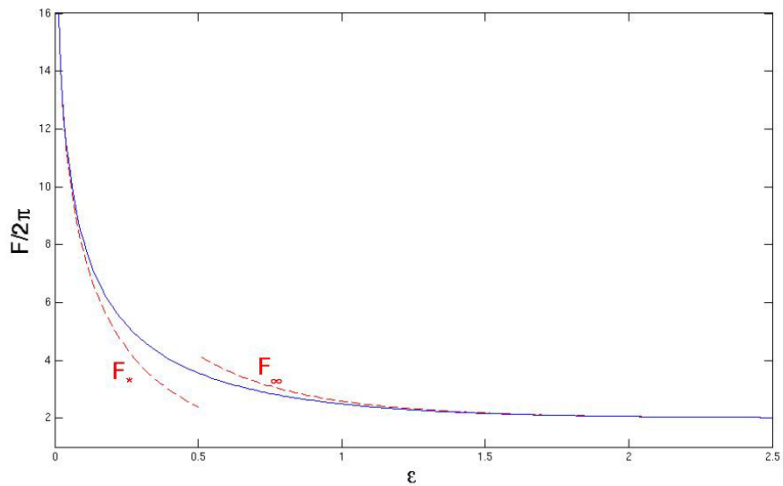
- ▶ As $\varepsilon \rightarrow 0^+$:

$$F_*(\varepsilon) = 2\pi(2 + 4K_0(\varepsilon) - 2\varepsilon K_1(\varepsilon)) \sim -8\pi \log \varepsilon$$

- ▶ As $\varepsilon \rightarrow \infty$ (different argument, cf. Manton & Speight):

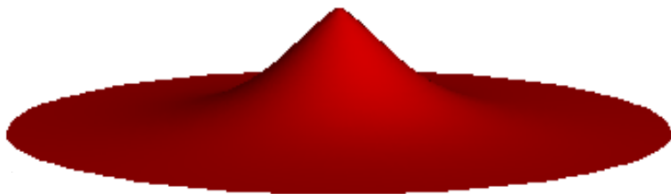
$$F_\infty(\varepsilon) = 2\pi \left(2 + \frac{q^2}{\pi^2} K_0(2\varepsilon) \right), \quad q \approx -7.1388$$

The case $\Sigma = \mathbb{R}^2$



The case $\Sigma = \mathbb{R}^2$

Our formula for F_* implies incompleteness with unbounded positive Gauß curvature as $\varepsilon \rightarrow 0$.



The case $\Sigma = S_R^2$

Regularised Taubes equation for $(k_+, k_-) = (1, 1)$:

$$\nabla_w^2 \hat{h} - \frac{8R^2 \varepsilon^2}{(1 + \varepsilon^2 |w|^2)^2} \frac{|w - 1|^2 e^{\hat{h}} - |w + 1|^2}{|w - 1|^2 e^{\hat{h}} + |w + 1|^2} = 0.$$

THEOREM (NR + M Speight): The conjectural volume formula holds for $\tau = 0$, i.e.

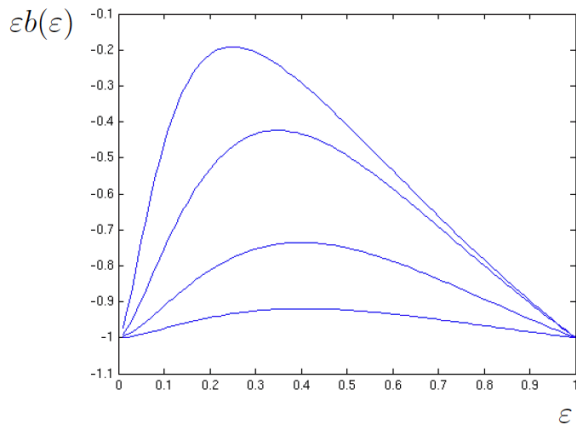
$$\text{Vol} \left(\mathcal{M}_{(1,1)}^{\mathbb{P}^1}(S_R^2) \right) = (2\pi)^2 (4\pi R^2)^2.$$

Ingredients in the proof:

- ▶ Strachan–Samols localisation (again)
- ▶ Use of $\text{SO}(3) \times \mathbb{Z}_2$ isometry
- ▶ Elliptic estimates (and various tricks) to establish

$$\|\partial_{w_1} \hat{h}\|_{C^0(\bar{B}_{1/2}(1))} \leq \|\partial_{w_1} \hat{h}\|_{H^2(\bar{B}_{1/2}(1))} < C(R) \sqrt{\varepsilon}$$

The case $\Sigma = S_R^2$ (numerics)



$R = 8, 4, 2, 1$ (top to bottom)

NB: $\frac{d}{d\varepsilon}(\varepsilon b(\varepsilon))$ bounded as $\varepsilon \rightarrow 0 \Rightarrow \mathcal{M}_{(1,1)}^{\mathbb{P}^1}(S_R^2)$ incomplete

The case $\Sigma = S_R^2$ (incompleteness)

THEOREM (NR + M Speight): $\mathcal{M}_{(1,1)}^{\mathbb{P}^1}(S_R^2)$ is incomplete.

Extra ingredients in the proof:

- ▶ Aim (rather) at weaker, but already sufficient estimate

$$\left| \frac{d}{d\varepsilon}(\varepsilon b(\varepsilon)) \right| \leq \frac{C(R)}{\sqrt{\varepsilon}} \quad \text{as } \varepsilon \rightarrow 0.$$

- ▶ Control the Sobolev norms of ε -derivatives:

$$\|\partial_\varepsilon \hat{h}\|_{H^1(S_R^2)} \leq \frac{C(R)}{\sqrt{\varepsilon}},$$

$$\|\partial_\varepsilon^2 \hat{h}\|_{H^2(\bar{B}_{1/2}(1))} \leq \frac{C(R)}{\sqrt{\varepsilon}}.$$

Gibbs' partition functions and thermodynamics

Geometry of $\mathcal{M}_{(k_+, k_-)}^{\mathbb{P}^1}(\Sigma)$ from GLSMs

- Consider a gauged sigma-model with Σ compact,

$$X = \mathbb{C}^2, \quad T^2 = \mathrm{U}(1)^2 = \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$$

- Vortex equations (for certain weights $Q_{\pm}^j$ and τ_j)

$$\bar{\partial}^{(A_1, A_2)} \varphi_{\pm} := \left(\bar{\partial} - \mathrm{i} \sum_{j=1}^2 Q_{\pm}^j A_j^{(0,1)} \right) \varphi_{\pm} = 0$$

$$*F_{A_j} = -\mu_j^{\sharp} \circ \phi = \frac{e_j^2}{2} \left(Q_+^j |\varphi_+|^2 + Q_-^j |\varphi_-|^2 - \tau_j \right) \quad j = 1, 2$$

- Gauged $\mathbb{P}^1$ -model is formal limit as $e_1^2 = e^2 \rightarrow \infty$, $e_2^2 = 1$,
i.e. $g_{T^2} = e^{-2} d\theta_1^2 + d\theta_2^2$.

Take e.g. $Q_+ = (1, 1)$, $Q_- = (1, 0)$, $(\tau_1, \tau_2) = (4, 2 - 2\tau)$.

- ▶ Matching of charges: $k_+ = k_1 + k_2$, $k_- = k_1$
- ▶ Bradlow bounds: $-(1 + \tau) + \frac{2\pi k_-}{e^2 \text{Vol}(\Sigma)} \leq \frac{2\pi(k_+ - k_-)}{\text{Vol}(\Sigma)} \leq 1 - \tau$
- ▶ PROPOSITION: (NR + M Speight):
For the strict inequalities and $k_+ \geq k_- > \max\{2g - 2, 0\}$, the moduli space of the GLSM (for any e) is

$$\mathcal{M}_{(k_+, k_-)}^{\mathbb{C}^2}(\Sigma) = \text{Sym}^{k_+}(\Sigma) \times \text{Sym}^{k_-}(\Sigma).$$

- ▶ Note the inclusion $\iota : \mathcal{M}_{(k_+, k_-)}^{\mathbb{P}^1}(\Sigma) \hookrightarrow \mathcal{M}_{(k_+, k_-)}^{\mathbb{C}^2}(\Sigma)$.
- ▶ CONJECTURE: (NR + M Speight)
The family of metrics $\iota^* g_{L^2}^e$ on $\mathcal{M}_{(k_+, k_-)}^{\mathbb{P}^1}(\Sigma)$ converges uniformly to g_{L^2} as $e \rightarrow \infty$.

Symplectic volume of $\mathcal{M}_{(k_+, k_-)}^{\mathbb{P}^1}(\Sigma)$

A little work leads to the result

$$\text{Vol} \left(\mathcal{M}_{(k_+, k_-)}^{\mathbb{C}^2, e}(\Sigma) \right) = \sum_{\ell=0}^g \frac{g!(g-\ell)!}{(-1)^\ell \ell!} \prod_{\sigma=\pm} \sum_{j_\sigma=\ell}^g \frac{(2\pi)^{2\ell} J_\sigma^{k_\sigma - j_\sigma} K_\sigma^{j_\sigma - \ell}}{(j_\sigma - \ell)!(g - j_\sigma)!(k_\sigma - j_\sigma)!}.$$

for certain $J_\pm, K_\pm$ (depending on e), as well as to the

CONJECTURE (NR + M Speight) :

$$\begin{aligned} \text{Vol} \left(\mathcal{M}_{(k_+, k_-)}^{\mathbb{P}^1}(\Sigma) \right) &= \lim_{e \rightarrow \infty} \text{Vol} \left(\mathcal{M}_{(k_+, k_-)}^{\mathbb{C}^2, e}(\Sigma) \right) \\ &= \sum_{\ell=0}^g \frac{g!(g-\ell)!}{(-1)^\ell \ell!} \prod_{\sigma=\pm} \sum_{j_\sigma=\ell}^g \frac{(2\pi)^{k_\sigma + j_\sigma} [(1 - \sigma\tau)V - 2\pi(k_\sigma - k_{-\sigma})]^{k_\sigma - j_\sigma}}{(j_\sigma - \ell)!(g - j_\sigma)!(k_\sigma - j_\sigma)!}. \end{aligned}$$

From these formulae, one can compute Gibbs' partition function and study the [thermodynamics](#) of vortex-antivortex gas mixtures.

Statistical mechanics on $T^*\mathcal{M}_{(k_+,k_-)}^{\mathbb{P}^1}(\Sigma)$

Phase space:

$$\left(T^*\mathcal{M}_{(k_+,k_-)}^{\mathbb{P}^1}(\Sigma), \omega_{\text{can}}\right), \quad \omega_{\text{can}} = -d\vartheta$$

Classical Hamiltonian:

$$H = \frac{1}{2}|\vartheta|_{g_{L^2}}^2 + 2\pi(1-\tau)k_+ + 2\pi(1+\tau)k_-$$

Thermodynamic limit:

$$k_{\pm} \rightarrow \infty,$$

$$V := \text{Vol}(\Sigma) \rightarrow \infty,$$

$$\text{with } \nu_{\pm} := \frac{k_{\pm}}{V} \text{ finite.}$$

Gibbs' partition function

$$\begin{aligned}
 \mathcal{Z} &= \frac{1}{(2\pi\hbar)^{2(k_++k_-)}} \int_{T^*\mathbb{M}_{(k_+,k_-)}^{\mathbb{P}^1}(\Sigma)} e^{-\frac{H}{KT}} \frac{\omega_{\text{can}}^{2(k_++k_-)}}{(2(k_++k_-))!} \\
 &= \left(\frac{KT}{2\pi\hbar^2}\right)^{k_++k_-} \int_{\mathbb{M}_{(k_+,k_-)}^{\mathbb{P}^1}(\Sigma)} e^{-\frac{2\pi}{KT}((1-\tau)k_++(1+\tau)k_-)} \frac{\omega_{L^2}^{k_++k_-}}{(k_++k_-)!} \\
 &= \left(\frac{KT}{2\pi\hbar^2}\right)^{k_++k_-} e^{-\frac{2\pi}{KT}((1-\tau)k_++(1+\tau)k_-)} \text{Vol}\left(\mathbb{M}_{(k_+,k_-)}^{\mathbb{P}^1}(\Sigma)\right) \\
 &= \mathcal{Z}_+ \mathcal{Z}_- \sum_{\ell=0}^g \frac{g!(g-\ell)!}{(-1)^\ell \ell!} \prod_{\sigma=\pm} \sum_{j_\sigma=\ell}^g \frac{(2\pi)^{j_\sigma} [(1-\sigma\tau)V - 2\pi(k_\sigma - k_{-\sigma})]^{k_\sigma-j_\sigma}}{(j_\sigma-\ell)!(k_\sigma-j_\sigma)!(g-j_\sigma)!}
 \end{aligned}$$

with

$$\mathcal{Z}_\pm := \left(\frac{KT}{\hbar^2}\right)^{k_\pm} e^{-\frac{2\pi}{KT}(1\mp\tau)k_\pm}.$$

Helmholtz free energy

$$\begin{aligned} F &= -KT \log \mathcal{Z} \\ &= -KT(k_+ + k_-) \log \frac{KT}{\hbar^2} + 2\pi \sum_{\sigma=\pm} (1 - \sigma\tau) k_\sigma \\ &\quad -KT \log \left[\sum_{\ell=0}^g \frac{g!(g-\ell)!}{(-1)^\ell \ell!} \times \right. \\ &\quad \left. \times \prod_{\sigma=\pm} \sum_{j_\sigma=\ell}^g \frac{(2\pi)^{j_\sigma} [(1 - \sigma\tau)V - 2\pi(k_\sigma - k_{-\sigma})]^{k_\sigma - j_\sigma}}{(j_\sigma - \ell)!(k_\sigma - j_\sigma)!(g - j_\sigma)!} \right]. \end{aligned}$$

Asymptotics:

$$\begin{aligned} \frac{k_\sigma! V^{-j_\sigma - \ell}}{(k_\sigma - j_\sigma - \ell)!} &= \nu_\sigma^{j_\sigma + \ell} + O\left(\frac{1}{V}\right) \quad \text{as } V \rightarrow \infty, \\ \log(k_\sigma!) &\approx k_\sigma \log k_\sigma - k_\sigma \quad (\text{Stirling}) \end{aligned}$$

Helmholtz free energy and entropy

$$\begin{aligned} F \approx & 2\pi \sum_{\sigma=\pm} (1 - \sigma\tau) k_{\sigma} - KT \sum_{\sigma=\pm} k_{\sigma} \log \frac{eKT}{\hbar^2 \nu_{\sigma}} \\ & - KT \sum_{\sigma=\pm} k_{\sigma} \log [1 - \sigma\tau - 2\pi(\nu_{\sigma} - \nu_{-\sigma})] \\ & - KTg \log \frac{1 - \tau^2 + 2\pi[(1 - \tau)\nu_{+} + (1 + \tau)\nu_{-}]}{1 - \tau^2 - 4\pi\tau(\nu_{+} - \nu_{-}) - (2\pi)^2(\nu_{+} - \nu_{-})^2}. \end{aligned}$$

$$\begin{aligned} S &= -\frac{\partial F}{\partial T} \\ &\approx K \sum_{\sigma=\pm} k_{\sigma} \log \frac{e^2 KT}{\hbar^2 \nu_{\sigma}} \\ &\quad + K \sum_{\sigma=\pm} (k_{\sigma} - g) \log [1 - \sigma\tau - 2\pi(\nu_{\sigma} - \nu_{-\sigma})] \\ &\quad + Kg \log [1 - \tau^2 + 2\pi[(1 - \tau)\nu_{+} + (1 + \tau)\nu_{-}]]. \end{aligned}$$

Pressure and equation of state

$$\begin{aligned} P &= -\frac{\partial F}{\partial V} \\ &\approx KT \sum_{\sigma=\pm} \nu_{\sigma} + KT \sum_{\sigma=\pm} \left(\nu_{\sigma} - \frac{g}{V} \right) \frac{2\pi(\nu_{\sigma} - \nu_{-\sigma})}{1 - \sigma\tau - 2\pi(\nu_{\sigma} - \nu_{-\sigma})} \\ &\quad + \frac{gKT}{V} \frac{2\pi[(1-\tau)\nu_{+} + (1+\tau)\nu_{-}]}{1 - \tau^2 + 2\pi[(1-\tau)\nu_{+} + (1+\tau)\nu_{-}]}. \end{aligned}$$

Further asymptotics: note that

$$\frac{g}{V} \rightarrow 0 \quad \text{in the thermodynamic limit.}$$

This leads to a genus-independent equation of state:

$$\begin{aligned} P &\approx KT \sum_{\sigma=\pm} \nu_{\sigma} + KT \sum_{\sigma=\pm} \frac{2\pi\nu_{\sigma}(\nu_{\sigma} - \nu_{-\sigma})}{1 - \sigma\tau - 2\pi(\nu_{\sigma} - \nu_{-\sigma})} \\ &= \sum_{\sigma=\pm} \frac{KT(1 - \sigma\tau)\nu_{\sigma}}{1 - \sigma\tau - 2\pi(\nu_{\sigma} - \nu_{-\sigma})}. \end{aligned}$$

Virial expansion

$$\begin{aligned}\frac{P}{KT} &= \sum_{n=0}^{\infty} \sum_{\sigma=\pm} \nu_{\sigma} \left(\frac{2\pi}{1-\sigma\tau} (\nu_{\sigma} - \nu_{-\sigma}) \right)^n \\ &= \nu_{+} + \nu_{-} + \frac{2\pi}{1-\tau} \nu_{+}^2 + \frac{2\pi}{1+\tau} \nu_{-}^2 - \frac{4\pi}{1-\tau^2} \nu_{+} \nu_{-} \\ &\quad + \frac{4\pi^2}{(1-\tau)^2} \nu_{+}^3 + \frac{4\pi^2}{(1+\tau)^2} \nu_{-}^3 - \frac{16\pi^2(1+\tau^2)}{(1-\tau^2)^2} (\nu_{+}^2 \nu_{-} + \nu_{+} \nu_{-}^2) \\ &\quad + \dots\end{aligned}$$

Convergence of this series is implied by the Bradlow bounds.

Entropy of mixing

In the thermodynamic limit:

$$S \approx K \sum_{\sigma=\pm} k_{\sigma} \log \frac{e^2 KTV}{\hbar^2 k_{\sigma}} + K \sum_{\sigma=\pm} k_{\sigma} \log \left[1 - \sigma\tau - \frac{2\pi(k_{\sigma} - k_{-\sigma})}{V} \right].$$

Thought experiment: separate $\sigma = \pm$ gas species so that

$$V = V_+ + V_- \quad \text{and} \quad P_+ = P_-.$$

Find: $V_{\pm} = \frac{V \pm \frac{4\pi\tau}{1-\tau^2} k_{\mp}}{1 + \frac{k_{\mp}}{k_{\pm}}}$, provided $\frac{V}{2\pi} \geq \frac{k}{1-\tau} + \frac{k_-}{1+\tau}$.

Entropy of mixing:

$$\begin{aligned} \Delta S^{\text{mix}} &= K \sum_{\sigma=\pm} k_{\sigma} \log \left[\left(1 + \frac{\nu_{-\sigma}}{\nu_{\sigma}} \right) \frac{1 - \sigma\tau - 2\pi(\nu_{\sigma} - \nu_{-\sigma})}{1 - \sigma\tau - 2\pi[\nu_{\sigma} + (\frac{1-\sigma\tau}{1+\sigma\tau})\nu_{-\sigma}]} \right] \\ &> 0. \end{aligned}$$