Theme

Conventional thermodynamics rests on the assumption that initial states thermalize to a "Gibbs ensemble", determined by the conserved quantities (e.g., energy and particle number).

What about systems with infinitely many conservation laws?

Simple guesses:
MBL-type systems: particles move a localization length, then stop
Yang-Baxter type systems: particles move freely

Both simple guesses miss some interesting behavior.

Will focus here on cases with *spatial inhomogeneity* and connections to transport.

Outline of this lecture

0. Entanglement and numerical methods with tensor networks.

1. For non-interacting systems, we understand essentially completely the effects of disorder, at least away from transitions.

Review of one-particle localization, where numerics are relatively easy.

For the simplest symmetries (orthogonal and unitary ensembles), disorder is localizing for essentially all states in 1D and 2D.

2. The combination of interactions and disorder in closed systems ("many-body localization", Basko et al.) is not nearly as well understood, even in 1D.

Different properties of the MBL phase lead to different possible numerical experiments. (Until very recently, "numerical experiments" were the only experiments! no longer. But the current experiments on atomic systems are possibly not in the long-time limit.)

Examples: level statistics; entanglement of eigenstates; dynamics after a quench;…

Numerics
Mostly will talk about 2 numerical methods:

Most numerics on MBL so far were done with "exact diagonalization" (ED): find all eigenvalues, or a subset, of the Hamiltonian matrix.

ED is great for small systems as it gives essentially complete information and its implementation and convergence are well understood.

It doesn't scale very well: cost for all eigenvalues goes as the cube of the matrix dimension, so beyond 20 spin-half sites becomes expensive.

Good news: there has been enormous progress 1992-present in DMRG/"matrix product state" methods to solve many-particle quantum problems in low spatial dimensions (especially 1D).

Understanding when these methods work well requires us to understand *entanglement,* which also leads to another useful definition of the MBL state.

Quantum entanglement

Sometimes a pure quantum state of a bipartite system *AB* is also a pure state of each subsystem separately:

Example: S_z=1 state of two s=1/2 spins, A and B

$$|\Psi_{AB}\rangle = |\uparrow_A\rangle \otimes |\uparrow_B\rangle$$

a "product" state

Sometimes a pure quantum state of a bipartite system *AB* is **not** a pure state of each subsystem separately:

Example: singlet state of two s=1/2 spins

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A\rangle \otimes |\downarrow_B\rangle - |\downarrow_A\rangle \otimes |\uparrow_B\rangle\right)$$

an "entangled" state

"Maximal knowledge of the whole does not imply maximal knowledge of the parts"

Entanglement entropy

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\rangle \otimes |\downarrow_B\rangle - |\downarrow_A\rangle \otimes |\uparrow_B\rangle)$$

an “entangled” state

In an entangled state, the state of subsystem A or B is not a pure quantum state, but rather a **density matrix**

For the singlet

$$\rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \rho_B$$

A classical uncertainty or **entropy** has been created by the operation of looking at only part of the system.

Entanglement entropy

Definition: the entanglement entropy of a pure state, **with respect to a given partition into A and B,** is the von Neumann entropy of the partial density matrices

$$\langle\phi_1|\rho_A|\phi_2\rangle = \sum_j (\langle\phi_1| \times \langle\psi_j|)|\psi\rangle\langle\psi|(|\phi_2\rangle \times |\psi_j\rangle)$$

$$S(\rho) = -\mathrm{Tr}\rho_A \log_2 \rho_A = -\mathrm{Tr}\rho_B \log_2 \rho_B$$

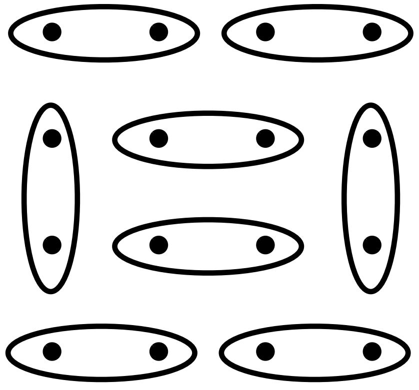
The singlet generates one bit of classical entropy when the two spins are separated

Note that the partial density matrix for subsystem A gives the results of *all* experiments limited to A

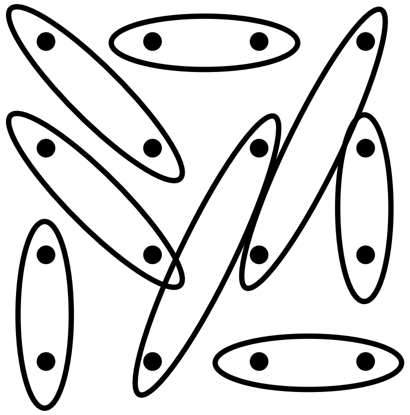
How much entanglement entropy occurs in ground states of local Hamiltonians?

To get some intuition for how entanglement behaves in statistical physics, consider "valence bond states" of $s=1/2$ systems:

Rule: every spin forms a singlet with some other spin



Short-ranged VBS
(= "dimer covering")



Long-ranged VBS

In these states, entanglement entropy S just counts singlets: S = 1 bit for each singlet crossing the *AB* boundary. (But real states are usually a bit more complicated.)

How much entanglement entropy occurs in ground states of local Hamiltonians?

Consider partitions of a *d*-dimensional infinite system *AB* into a subregion *A* of linear size *L* and an infinite subregion *B*.

How should entanglement entropy scale with *L*?

If we can ignore entanglement between points farther apart than some length scale ξ, then entanglement entropy should be determined by a shell of thickness ~ ξ around the *AB* boundary:

$$S \sim L^{d-1}\xi \Rightarrow S \sim L^{d-1} \text{as } L \to \infty \text{ with system parameters fixed}$$

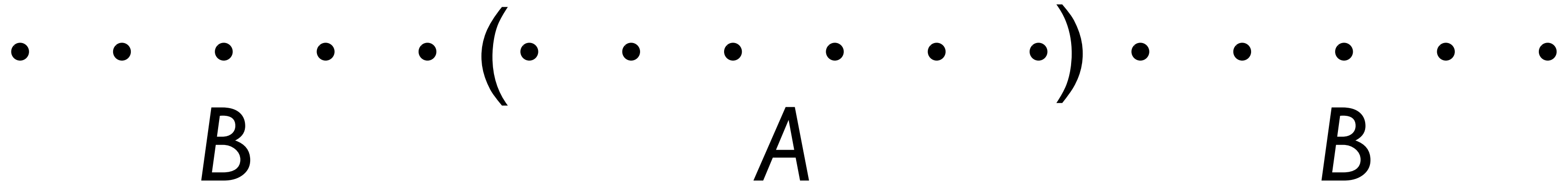
the "area law"

If there is no notion of locality, any site in *A* is as likely to be entangled with a site in *B* as with another site in *A*, and $S \sim L^d$

How much entanglement entropy occurs in ground states of local Hamiltonians?

We start with “pure” (translation-invariant), local Hamiltonians in one dimension.

Consider a partition for which *A* is a contiguous set of *N* spins inside an infinite chain:

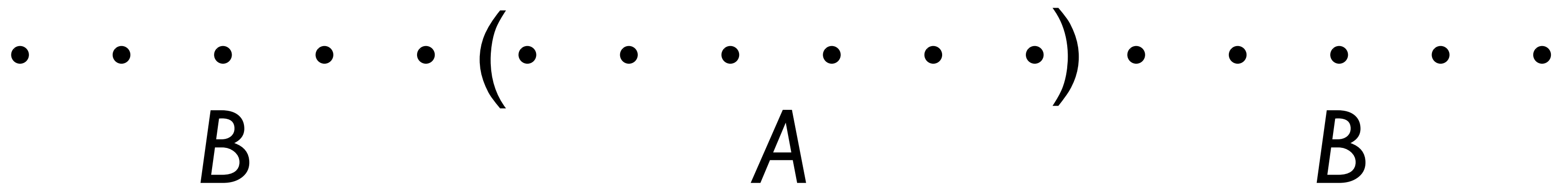


Away from critical points (i.e., when correlations are short-ranged), entanglement is localized in the vicinity of the boundary and the “area law” is satisfied:

$$\lim_{N\to\infty} S = C < \infty$$

But what about quantum critical states? Is there qualitatively more entanglement?

How much entanglement entropy occurs in critical states of local Hamiltonians?



Example of a quantum critical ground state: (c=1) Heisenberg AF

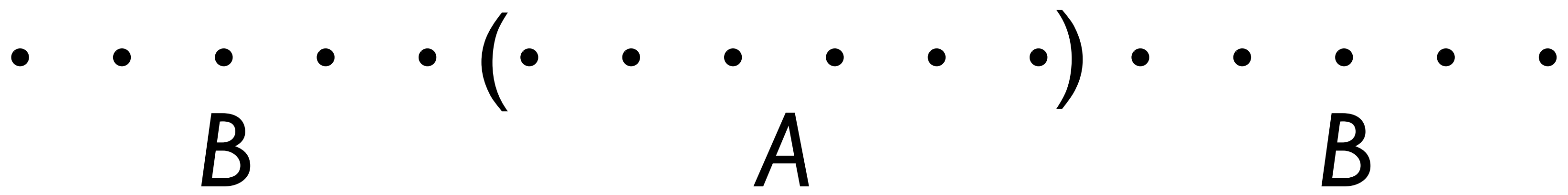
$$H = J \sum_i \mathbf{s}_i \cdot \mathbf{s}_j , \qquad J > 0$$

At criticality, the entanglement of a connected subset of N spins, with the remaining spins, is (note: violates area law)

$$\lim_{N \to \infty} S \sim \frac{c}{3} \log N \to \infty$$

At clean and conformally invariant quantum critical points in d=1, there is logarithmically divergent entanglement with a coefficient related to the "central charge" of associated CFT. (Holzhey, Wilczek et al. 94, Vidal 03, Calabrese and Cardy 04).

Uses of entanglement entropy in d=1



For the subset of 1D quantum critical points that are described by 2D conformal field theories:

The appearance of the central charge in the ground-state entanglement is consistent with its appearance in other quantities related to entropy, such as the free energy at finite temperature

$$f = \frac{F}{L} = f_0 - \frac{\pi}{6} c (kT)^2 \hbar v$$

The central charge is an important quantity, but only defined for a subset of quantum critical points.

Entanglement entropy can be defined at *any* quantum critical point. Does it still show similar behavior, with a universal coefficient? *Yes!*

Studying quantum correlations with classical algorithms: applied entanglement entropy

Basic (hazy) concept: "Entanglement entropy determines how much classical information is required to describe a quantum state."

Example:
how many classical real numbers are required to describe a *product* (not entangled) state of N spins?

simple product $|\psi\rangle = A_{s_1} A_{s_2} A_{s_3} A_{s_4} |s_1 s_2 s_3 s_4\rangle$

Answer: ~ N (versus exponentially many for a general state)

How do we efficiently manipulate/represent moderately entangled states?

Applied entanglement entropy

The remarkable success of the density-matrix renormalization group algorithm in one dimension (White, 1992; Ostlund and Rommer, 1995) can be understood as follows:

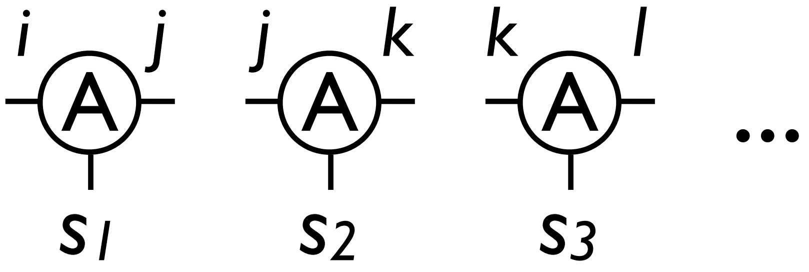
DMRG constructs "matrix product states" that retain local entanglement but throw away long-ranged entanglement.

Example states for four spins:

simple product $$|\psi\rangle = A_{s_1} A_{s_2} A_{s_3} A_{s_4} |s_1 s_2 s_3 s_4\rangle$$

matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Graphical tensor network representation:



“Infinite system” methods

Note that we can impose translation invariance simply by requiring constant matrices *A*.

In other words, for quantities in a translation-invariant system, *we just calculate A, rather than a large finite system.*
(Idea 1 of renaissance; see Vidal ’07, for example)

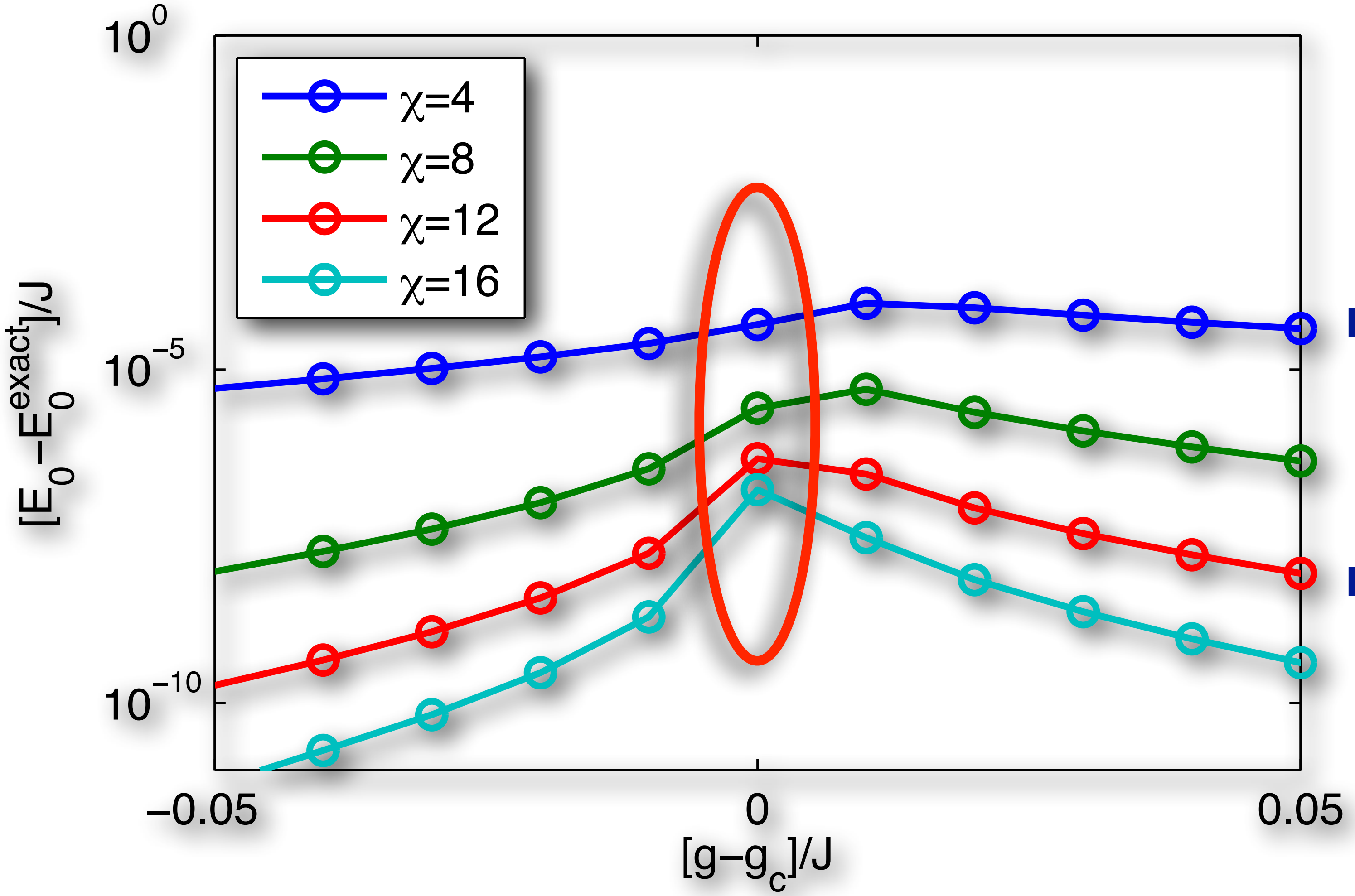
matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

So where is the approximation?
A finite matrix *A* can only capture a finite amount of entanglement.

In the early DMRG days, it was often thought:
1. To study an infinite system, we should study a large finite one.
2. Gapless/critical systems are hard. (Gapped uniform systems converge…)
3. Dynamical properties are hard
4. Finite temperature is hard
But none of these is strictly correct.

- find the ground state of a system by using **imaginary time evolution** (almost unitary for small time steps)
- parallel updates for **infinite/translational invariant** systems: **iTEBD** [Vidal '07]
- **example**, transverse Ising model:

$$H = \sum_i \left(J\sigma_i^z \sigma_{i+1}^z + g\sigma_i^x\right)$$



➡ convergence of wave function is worst at the **critical point**

➡ **conformal invariance**

Criticality: finite-entanglement scaling

All numerical methods have difficulty with quantum critical points. In DMRG-type approaches, this can be understood from the divergence of entanglement entropy at such points: the entanglement in a matrix product state is limited by dim *A*.

matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Quantitatively, it is found that dim *A* plays a role similar to imposing a finite system size:

$$L_{\mathrm{eff}} \propto \chi^{\kappa}, \quad \chi = \dim\ A$$

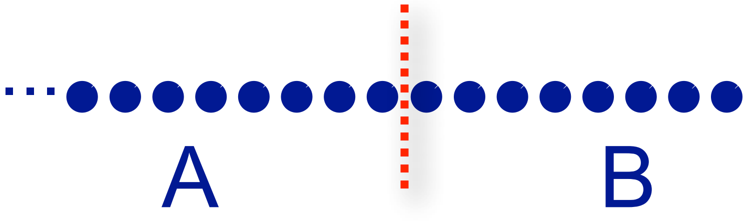
(Tagliacozzo et al., PRB 2008).

Finite matrix dimension effectively moves the system away from the critical point.

What determines this "finite-entanglement scaling"?
Is it like "finite-size scaling" of CFT's (cf. Blöte, Cardy, & Nightingale)

A way to picture the entanglement of a state

- **Schmidt decomposition** of the state (**SVD**):
- $$\begin{aligned} |\psi\rangle &= \sum_{i=1}^{N_A}\sum_{j=1}^{N_B} C_{ij}|i\rangle_A|j\rangle_B \\ &= \sum_{\alpha=1}^{\min(N_A,N_B)} \lambda_\alpha|\phi_\alpha\rangle_A|\phi_\alpha\rangle_B \end{aligned}$$



with $\lambda_\alpha \geq 0$ and $\sum_\alpha \lambda_\alpha^2 = 1$

- a natural measure of the entanglement is the **entropy:**

$$S_A = S_B = S = -\sum_\alpha \lambda_\alpha^2 \log(\lambda_\alpha^2)$$

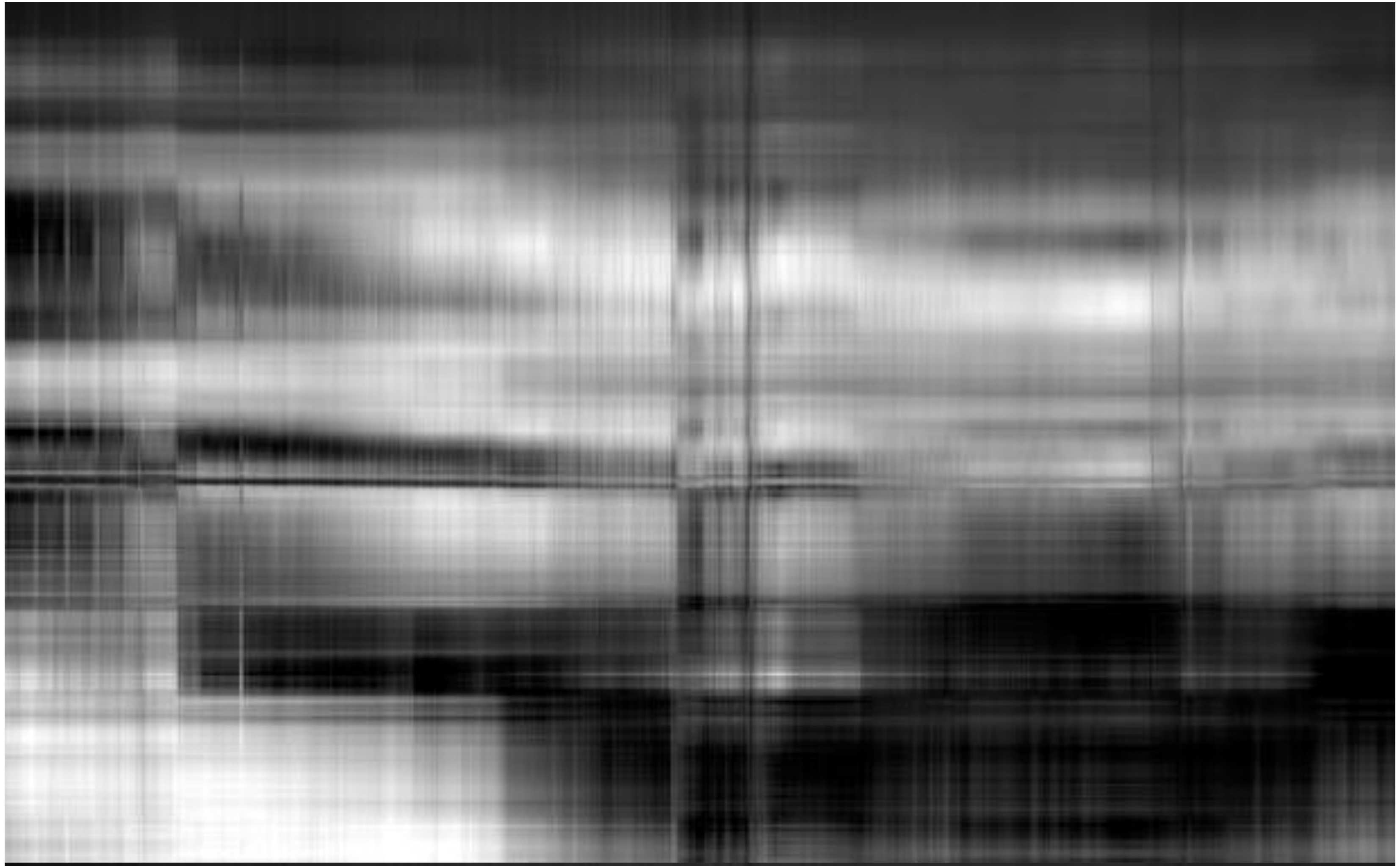
Efficient representation of quantum states?

- Hilbert-space dimension of many-body problems increases **exponentially** with number of sites
  example: spin 1/2 system on “classical” computers
  (store **one state** in double precision)

- need an efficient way to “**compress**” quantum states so that the matrices studied remain finite-dimensional

  ➡slightly entangled 1D systems: **Matrix Product States**

  ➡DMRG, TEBD, ...



$\chi = 4$

$\chi = 16$

$\chi = 64$

$\chi = 256$

- (Li-Haldane) “entanglement spectrum” [Calabrese et al ‘08]

$$n(\lambda) = I_0\left(2\sqrt{-b^2 - 2b\log\lambda}\right)$$

of $\hat{\lambda}$‘s greater than λ

with $b = \frac{S}{2} = \frac{c}{12}\log\xi = -2\log\lambda_{\text{max}}$

continuum of Schmidt values $|\psi\rangle = \sum_{\alpha=1}^{\infty}\lambda_\alpha|\phi_\alpha\rangle_A|\phi_\alpha\rangle_B$

- Want to explain how at a critical point, finite matrix size χ effectively moves the system away from criticality, leading to universal relations like

$$L_{\text{eff}} \propto \chi^{\kappa}, \quad \chi = \dim\ A$$

- **A heuristic argument** for the asymptotic case (using a continuum of Schmidt values and $\chi \to \infty$)

➡**universal finite-entanglement scaling relations**

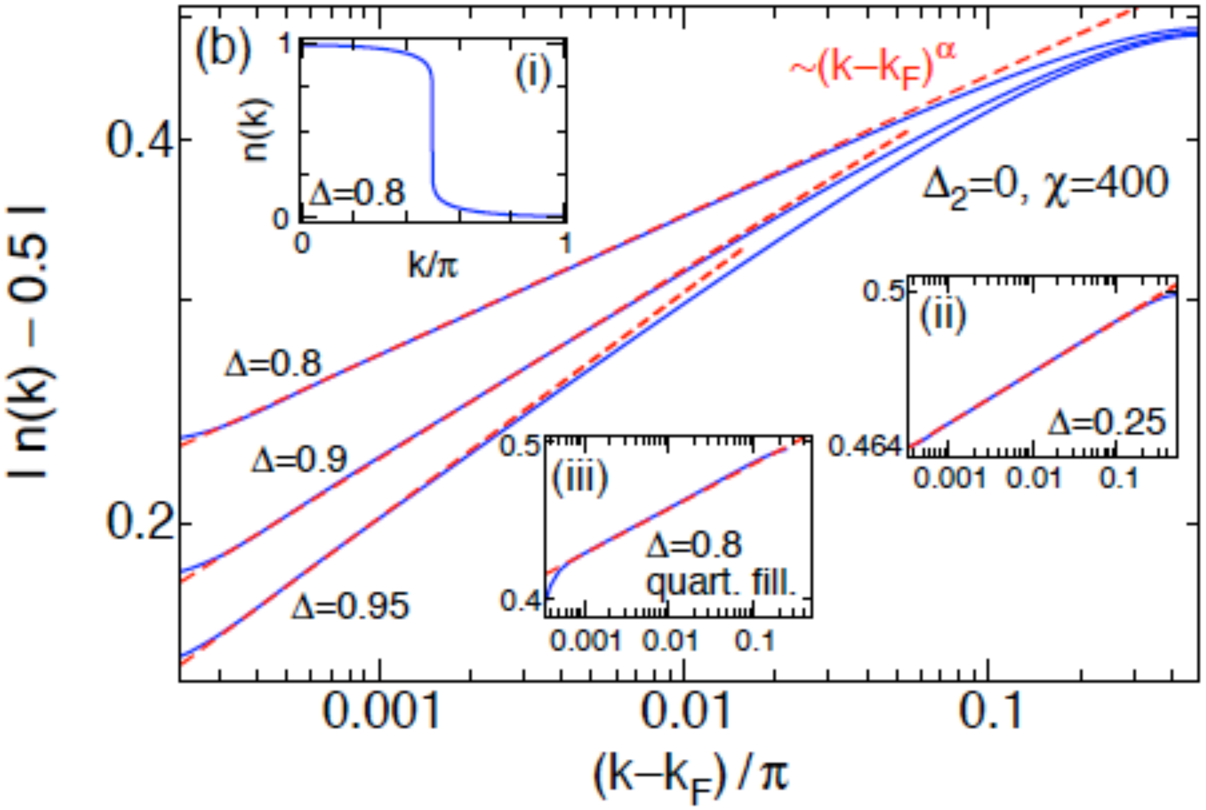
$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}}+1\right)} \Rightarrow S = \frac{1}{\sqrt{\frac{12}{c}}+1} \log \chi$$

F. Pollmann, S. Mukerjee, A. Turner, and J.E. Moore, PRL 2009
Some checks for various critical theories are in that paper, and the recent work
B. Pirvu, G. Vidal, F. Verstraete, L. Tagliacozzo, arXiv:1204.3934

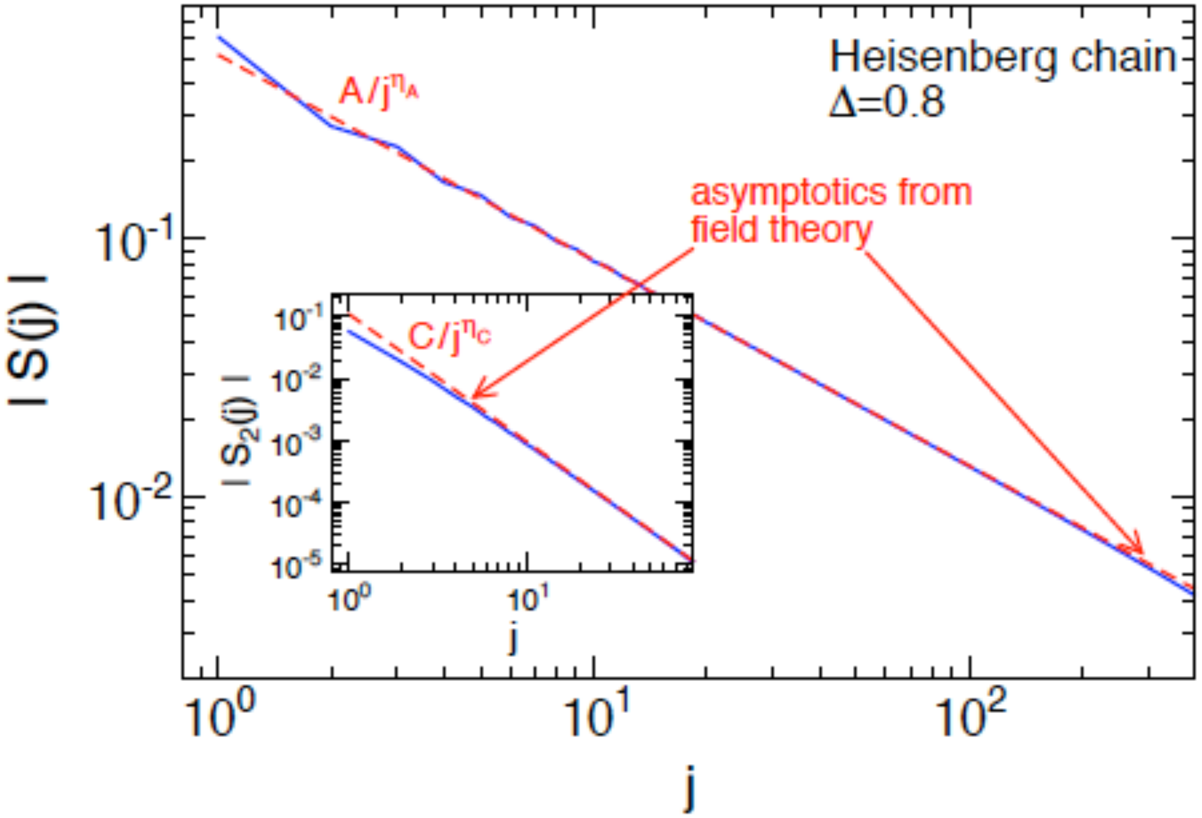
So critical points are worse than gapped points, but in a controlled way.
What does this mean in practice?

Remark: Entanglement spectra are qualitatively different for random critical spin chains than for pure ones, though entanglement entropies similar (M. Fagotti, P. Calabrese, JEM).

Tests of Luttinger liquid behavior in the XXZ model



Momentum distribution n(k)



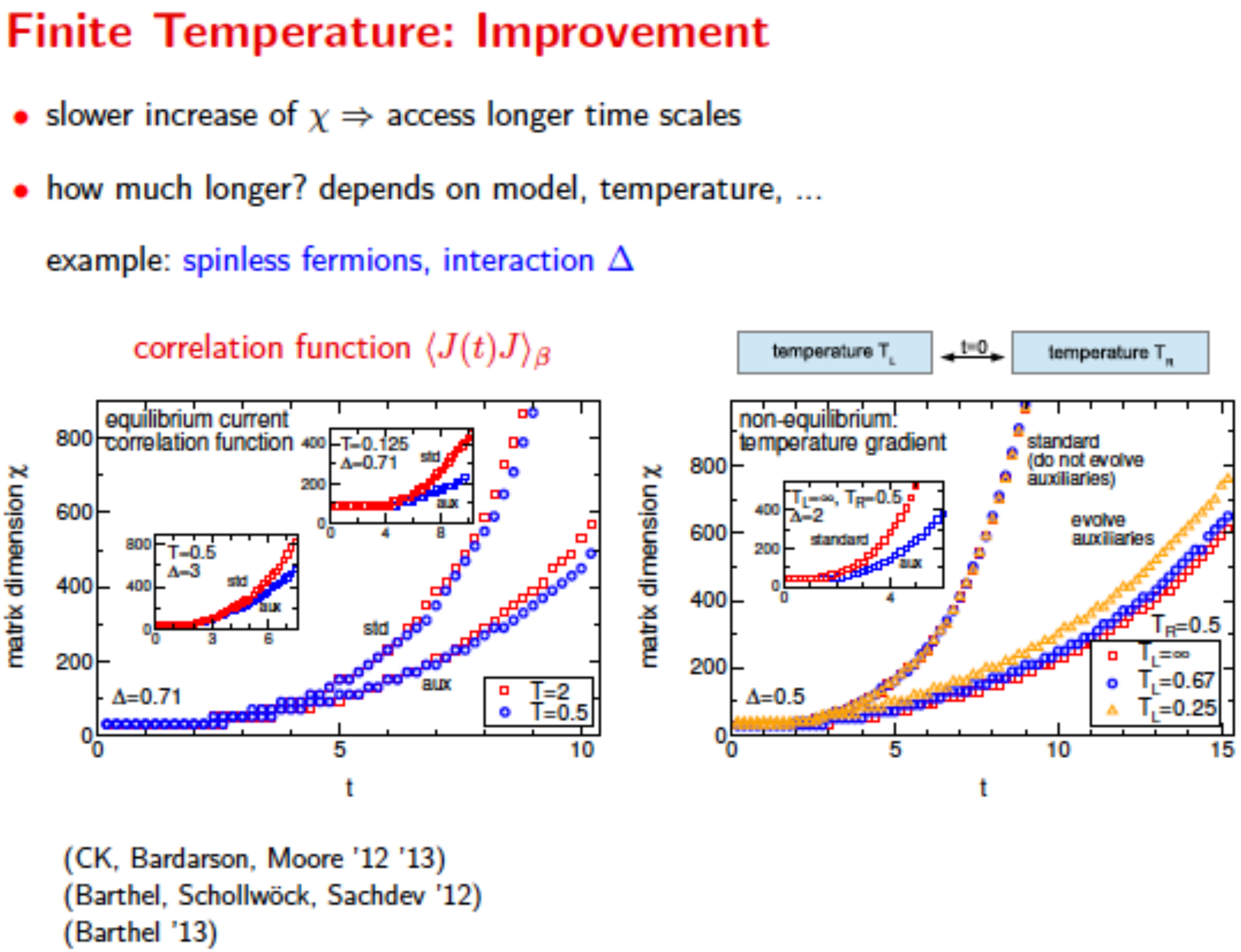
Check of leading staggered and uniform correlators against Lukyanov and Terras

(C. Karrasch and JEM, PRB)

Current ground-state applications moving to 2D: FQHE, spin liquids, …

Next: try to solve an open problem of dynamical properties at finite temperature.

One methodology slide (C. Karrasch)
Time-dependent DMRG at nonzero T



Finite Temperature: Improvement

- slower increase of $\chi \Rightarrow$ access longer time scales
- how much longer? depends on model, temperature, ...

example: spinless fermions, interaction Δ

correlation function $\langle J(t)J\rangle_\beta$

(CK, Bardarson, Moore '12 '13)
(Barthel, Schollwöck, Sachdev '12)
(Barthel '13)

References and questions to keep in mind

I. How is the MBL state different from ordinary Anderson localization (in entanglement, Bardarson, Pollmann, and JEM, PRL (2012); in random matrix ensemble, Serbyn and JEM, PRB (2016))?

II. How do these differences show up in observable dynamics ("revivals", Parameswaran, Vasseur, JEM, PRB (2014))?

III. How is MBL-type integrability different from Yang-Baxter integrability?

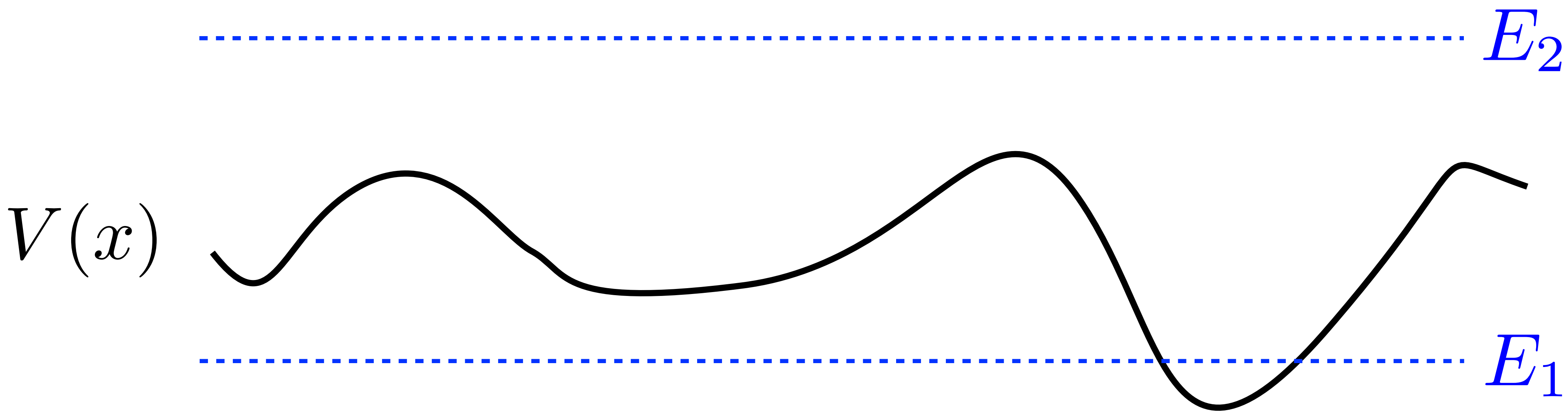
Probably won't get to:
What MBL-like behavior exists in translation-invariant systems? Can look for compact density response to a weak, arbitrarily broad perturbation…

Things that won't be discussed:
higher dimensions; conventional or topological order; response to a *local* quench; theories of thermalizing transition

Intro to disordered electronic systems

Consider a quantum particle, described by the Schrödinger equation, moving in a random potential.

Intuitively, we might expect:
at low energy, eigenstates are trapped (“localized”) in potential minima
at high energy, eigenstates are scattering states

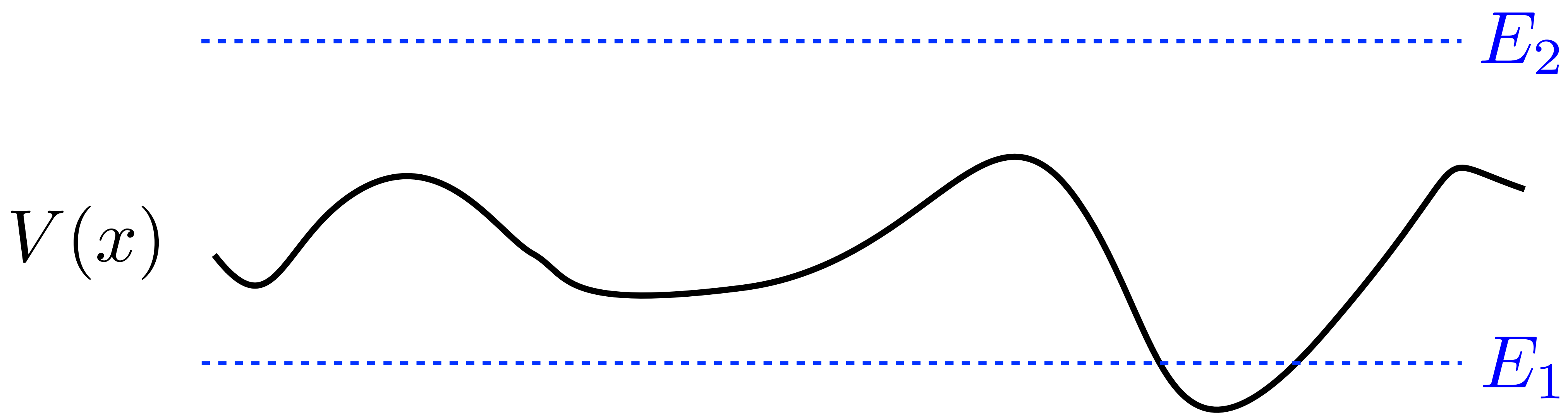


In 3D, this intuition is basically correct, and there is a specific energy (the “mobility edge”) that separates localized from disordered states.

Argument for mobility edge: (Mott) coexistence of localized and extended states at same energy is unstable, as a small perturbation will mix and give only extended states.

Intro to disordered electronic systems

This intuition breaks down in one or two dimensions: all electronic states are localized up to arbitrarily high energies, although the localization length increases with *E*.



Why is 2D special? Consider the stability of scattering states. We can model the scattering state as a random walk.

A random walk above 2D revisits any point only a finite number of times on average, so a weak potential fluctuation cannot be amplified infinitely. In 2D or below, a point (say the starting point) is visited an infinite number of times, and a "weak" potential can become strong.

Intro to disordered electronic systems

For non-interacting systems, we understand essentially completely the effects of disorder, at least away from transitions.
For the simplest symmetries (orthogonal and unitary ensembles), disorder is localizing for essentially all states in 1D and 2D.

The combination of interactions and disorder in closed systems (“many-body localization”) is not well understood even in 1D.
are the only two possibilities diffusive and localized? can there be subdiffusive scaling? (e.g., “glassy”: $r \sim \log t$)

CM experimental systems typically have “dephasing” from interactions with phonons, which ultimately leads to a finite diffusion constant.

Systems of atoms in an ultra cold lattice do not have phonons, so may be better.

Intro to disordered electronic systems

How do we see localization experimentally?

Localization in the sense described here requires *interference* (constructive interference of self-intersecting trajectories).

Hence it is a quantum property and disappears if the electrons lose their phase coherence by interacting with a their environment (e.g., a "bath" of phonons).

If that happens on a phase-breaking time scale τ_ϕ

then this acts as a cutoff on the effects of localization, e.g., on the reduction of conductivity.

Treating localization perturbatively ("weak localization theory") has been very powerful. Interaction effects can be incorporated (Altshuler-Aronov, Finkelstein, others) in this framework.

But in isolated systems (e.g., ultracold atomic systems), or possibly in femtosecond experiments on electrons, the system can be phase-coherent.

Including the bath also sidesteps some basic questions.

Intro to disordered electronic systems

So one-particle localization is very sensitive to dimensionality.

It is also sensitive to *symmetries*. For example, if we broke time-reversal symmetry with a magnetic field, then in 2D extended states survive at isolated energies.

If we assume that disorder breaks all symmetries except for two discrete symmetries T (time reversal) and C (chiral/charge conjugation), and that each of these can square to +1 or -1 if present, then there are 10 symmetry classes.

Why 10?

Just considering T gives 3 "Wigner-Dyson" classes: orthogonal ($T^2 = +1$), symplectic ($T^2 = -1$), and unitary (T broken).

Adding C gives 9 classes (3 times 3). There is also the possibility of having CT symmetry without either C or T separately, hence 10 "Altland-Zirnbauer" classes.

How do we see localization experimentally?
Why is it important for some basic physics questions?
Is there more to the story than symmetry and dimensionality?

Periodic table of insulators

Schnyder et al., Kitaev: 10-fold way classification, periodic in dimension
3 Wigner-Dyson cases + particle-hole symmetry in superconductors = 10
Better to think of as 2+8: see Freed and G. Moore, "Twisted Equivariant Matter"

There can be insulator-metal transitions, like the Anderson transition at the mobility edge, and also insulator-insulator transitions, like the quantum Hall plateau transition.

Numerics is not easy at these transitions but arguably easier than analytical theory.

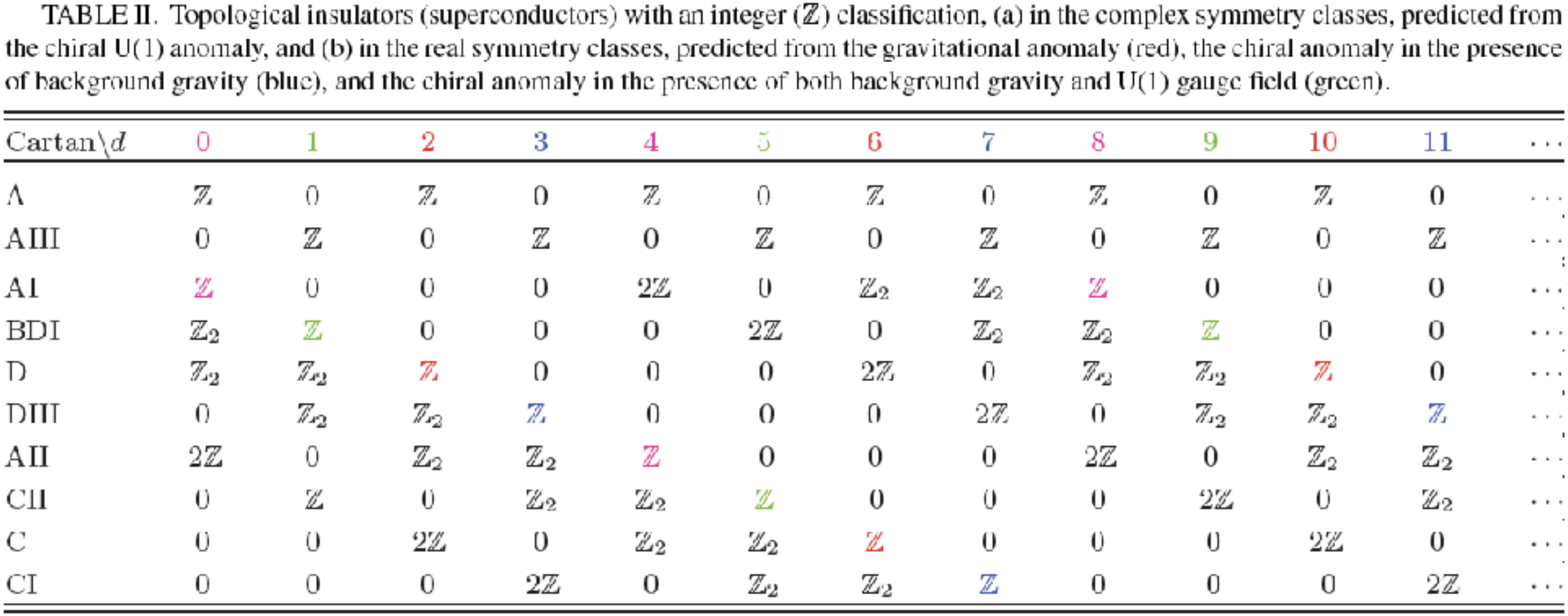


TABLE II. Topological insulators (superconductors) with an integer ($\mathbb{Z}$) classification, (a) in the complex symmetry classes, predicted from the chiral U(1) anomaly, and (b) in the real symmetry classes, predicted from the gravitational anomaly (red), the chiral anomaly in the presence of background gravity (blue), and the chiral anomaly in the presence of both background gravity and U(1) gauge field (green).

Cartan\\d	0	1	2	3	4	5	6	7	8	9	10	11	⋯
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	⋯
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	⋯
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	⋯
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	⋯
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	⋯
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	⋯
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	⋯
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	⋯
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	⋯
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	⋯

MBL can be motivated by the basic question

Does an isolated quantum system with interactions and disorder show localization?

which is related to the equally basic question

When do isolated quantum systems thermalize?

The connection is that localization is the most plausible physical way to avoid thermalization: localized particles cannot move around and equilibrate. In a delocalized system, we expect that a test particle sees other particles as a thermal "bath".

Will focus on 1D. Besides symmetry and dimensionality, what else controls localization in the interacting case? What are the new properties of the localized phase? Which are interaction-specific?

What about MBL versus ergodic states?

A thermalizing state should have volume-law entanglement of eigenstates according to the eigenstate thermalization hypothesis (ETH).

ETH=local measurements on an eigenstate of a thermalizing system are consistent with a thermal ensemble.

A picture of the MBL state is that it is similar to the ground state of a localized system and has an *area law* for entanglement.
(Bauer and Nayak, …)

So far we have three things we can look for to diagnose an MBL transition: vanishing of the conductivity, or absence of thermalization, or the change in the entanglement properties of eigenstates.

Note that the first two are slightly different: we might have a subdiffusive but thermalizing phase, for example.
See Bar Lev et al '14, Hulin et al '90, Agarwal et al '14, Potter et al '14, Vosk et al '14

Strong MBL and conservation laws

As far as I know, only one model has a rigorously established MBL phase (Imbrie 2014): an Ising model with random couplings and longitudinal & transverse fields

$$H = \sum h_i S_i^z + \sum \gamma_i S_i^x + \sum_i J_i S_i^z S_{i+1}^z$$

What is established is that there is an infinite set of local conserved quantities for a finite range of parameters. Let's call this "Strong MBL".

Immediately implies non-thermalization.

Surprising: a stable range of "integrability", unlike Yang-Baxter case.

Mostly people have studied interacting (Dirac) fermion models instead.

Recent "no-go" work suggests that strong MBL is quite difficult to achieve: specific to 1D; only short-ranged interactions; no SU(2) or other non-Abelian symmetry; … but can get very long time scales for thermalization.

Many-body localization at infinite temperature

$$H = J_{xx}\sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

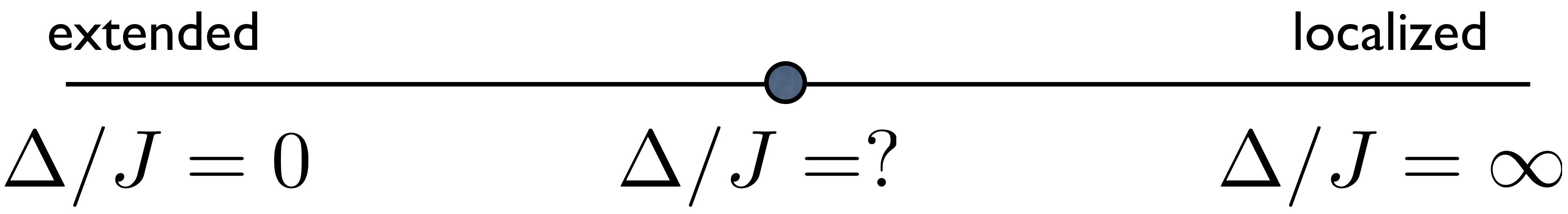
Clean XXZ chain + random z-directed Zeeman field

$$h_i \in (-\Delta, \Delta)$$

Claim: look at "infinite-temperature" dynamics but with no dephasing; evolve an arbitrary initial state by the Schrödinger equation

Heisenberg phase diagram:

(Oganesyan-Huse spin chain version of BAA)



extended

localized

$\Delta/J = 0$

$\Delta/J = ?$

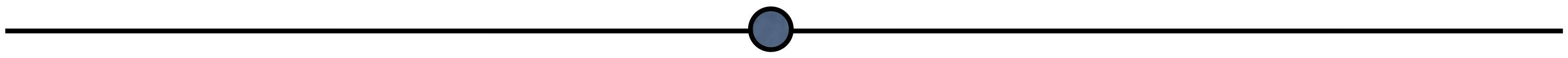
$\Delta/J = \infty$

or is there an intermediate "ergodic non-metal"?

Many-body localization at infinite temperature

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Heisenberg phase diagram:



extended

localized

$\Delta/J = 0$ $\Delta/J = ?$ $\Delta/J = \infty$

Transition(s) should be detectable in:

level statistics: (Wigner-Dyson vs. Poisson) Oganesyan & Huse, 2008
dynamical correlation functions
correlation distributions Pal & Huse, 2010; Reichman et al. 2010
entanglement growth/thermalization (JHB,FP,JEM 2012)
entanglement variance (recent work of Alet et al., Bardarson et al., …)

This spin chain problem is a numerically easier reformulation of many-body localization in continuum Fermi systems at nonzero T (Basko, Aleiner, Altshuler 2007)

Hoped to be generic for 1D local interactions, disorder, U(1) symmetry.

Many-body localization at infinite temperature

$$H = J_{xx}\sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

level statistics: (Wigner-Dyson vs. Poisson) Oganesyan & Huse, 2008

The idea is that diffusive and integrable systems have different level statistics, which is a simple property of the eigenvalues alone.

An MBL system is like an *integrable* system, which normally means a translation-invariant system with a complete set of conservation laws (return to this point in a moment).

The key difference (and let's look for it numerically) is that the *integrability of an MBL system is stable to disorder, while conventional integrability is not, nor even to translation-invariant perturbations that break the Yang-Baxter equation (factorization of scattering).*

Integrability in MBL

$$H = J_{xx}\sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Another way to define the MBL phase and explain its lack of thermalization is in terms of a complete set of conserved quantities that are genuinely local (Serbyn, Papic, Abanin; Imbrie)

(i.e., local as in the non-interacting case, not translation-invariant sums of local objects)

We expect to see Poisson statistics in the MBL phase simply because nearby states are likely to be localized in different parts of the sample, and hence not repel each other. Will come back to this.

Many-body localization at infinite temperature

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Numerical experiment: start with an arbitrary product state (local Sz eigenstate) and evolve under *H*. Can view as a "global quench".

"Extended phase": expect S grows linearly with t (Calabrese and Cardy)

"One-particle localized phase": (Jz = 0) eigenstates are Slater determinants of localized one-particle states; S saturates to a finite value.

What happens if we add interactions to the localized phase?

Note: this is efficiently simulable because for early times the system has small entanglement (Prelovsek et al., 2007)

Jens Bardarson, Frank Pollmann, and JEM, PRL **109**, 017202 (2012).

Many-body localization at infinite temperature

$$H = J_{xx}\sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

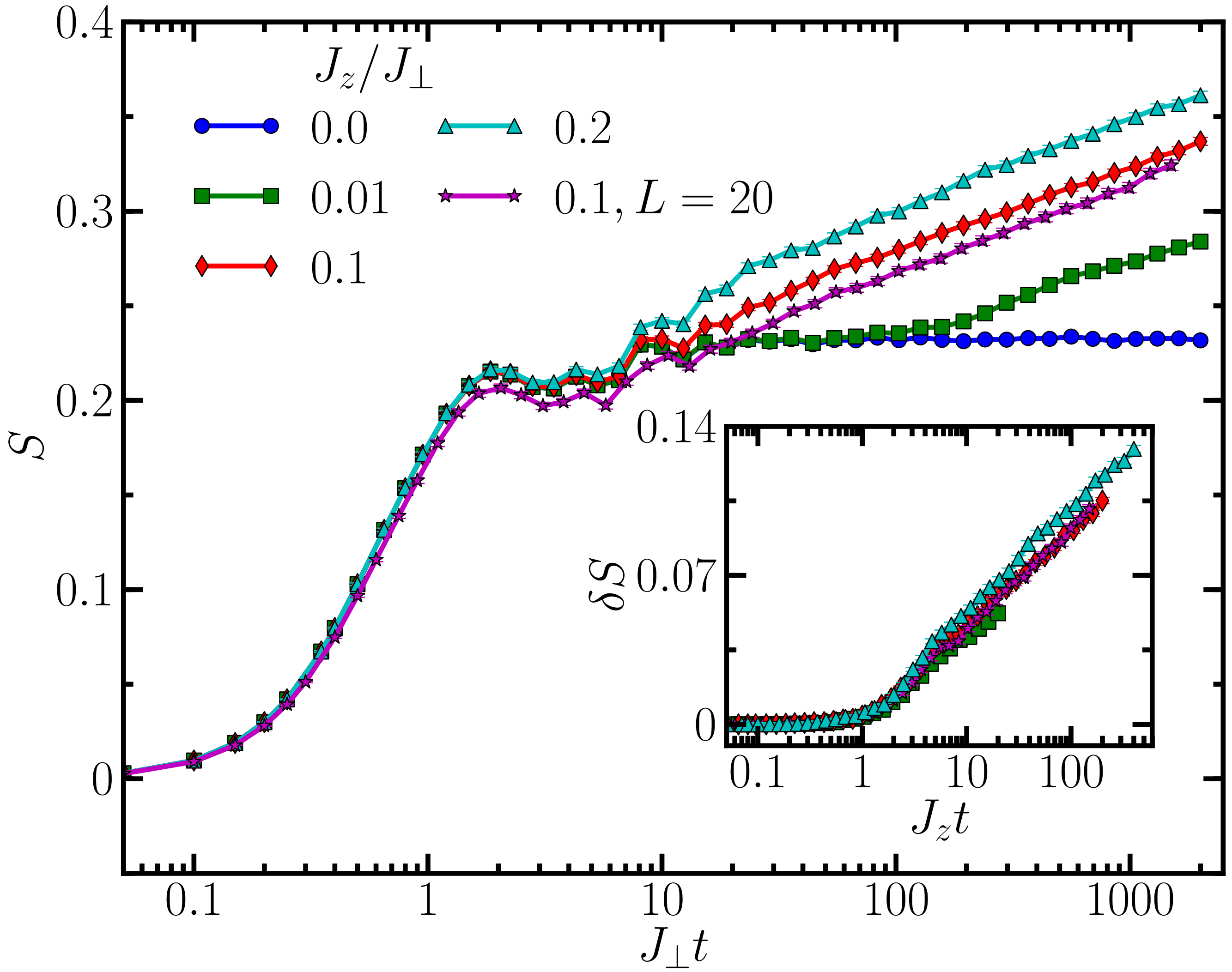
Numerical experiment: start with an arbitrary z-product state (local Sz eigenstate) and evolve under *H*. Can view as a "global quench".

Half-chain entanglement saturates with no interactions.

Interactions increase entanglement growth (consistent with previous work: De Chiara et al., Prelovsek et al.).

Surprise:
Interactions are a singular perturbation.

Even a very weak interaction leads eventually to a *slow* but *unbounded* increase of entanglement.



Many-body localization at infinite temperature

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Numerical experiment: start with an arbitrary z-product state (local Sz eigenstate) and evolve under *H*. Can view as a "global quench".

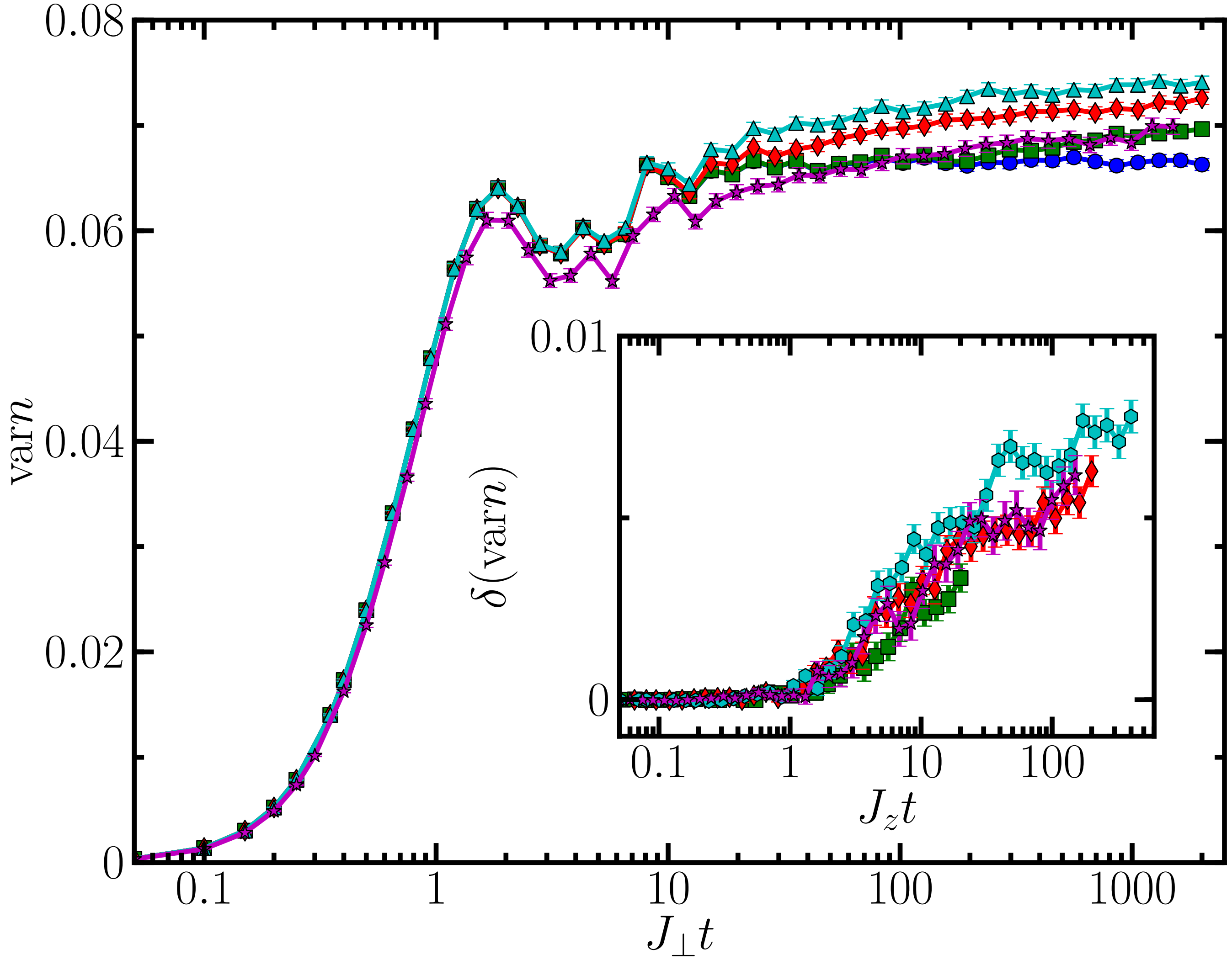
What about transport of the U(1) quantity?

Effect of interactions is less obviously singular--it could be that conductivity is zero.

We cannot rule out that the only physics with interactions is extended and that there is eventually thermalization.

But there is a long, possibly infinite, time range over which dynamics is very slow.

(Slower log log dynamics at low energy in random singlet phase--Igloi et al. PRB 2012)



Question: Is entanglement “physical”?

Yes, but hard to measure (although see Greiner et al. 2015); are other properties sensitive to this logarithmically slow dynamics?

Eigenstates versus dynamics of observables

One way to view the MBL phase: all eigenstates are basically similar, because a slight change in the potential will change which eigenstate is the ground state.

For example, all (or almost all) eigenstates are area-law (cf. Bauer-Nayak).

Different from the diffusive case, where the ground state is special (area law versus volume law, for example).

The arbitrariness in the MBL phase suggests that it may be difficult to prepare a single excited eigenstate; more generally, it is nontrivial to connect dynamics of observables (e.g., after a quench) to the properties of eigenstates.

Testing “dephasing without delocalization”

Favored scenario: (Huse-Oganesyan, Papic-Serbyn,-Abanin, Vosk-Altman, …)

The entanglement increase can be understood in terms of independent pairs with interaction energy scale

$$J_{\mathrm{eff}} = J_0 \exp(-L/\xi_0)$$

which under the (short-time) assumption that pairs contribute independently to entanglement gives

$$S \sim \xi_0 \log(J_0 t)$$

An experimentally practical way to test this log: Romain Vasseur, Siddharth Parameswaran, and JEM, PRB 2015

“Revivals”: how often, in a single realization of disorder, does a single spin’s expectation return to its original value?

This is basically a probe of how many frequencies are involved in the spin’s dynamics. That increases dramatically between Anderson localization and MBL.

The dephasing picture has to break down as we approach the transition to a delocalized phase.

Model: XXZ chain plus "probe spin" at edge

$$
\begin{aligned}
H &= H_{\mathrm{XXZ}}[\{\sigma_i\}] + \frac{\lambda}{2}\left(S^+\sigma_1^- + S^-\sigma_1^+\right) \\
H_{\mathrm{XXZ}} &= \frac{1}{4}\sum_{i=1}^{L-1} J_\perp\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+\right) + J_z\sigma_i^z\sigma_{i+1}^z + \frac{h_i}{2}\sigma_i^z
\end{aligned}
$$

Question: "revivals"

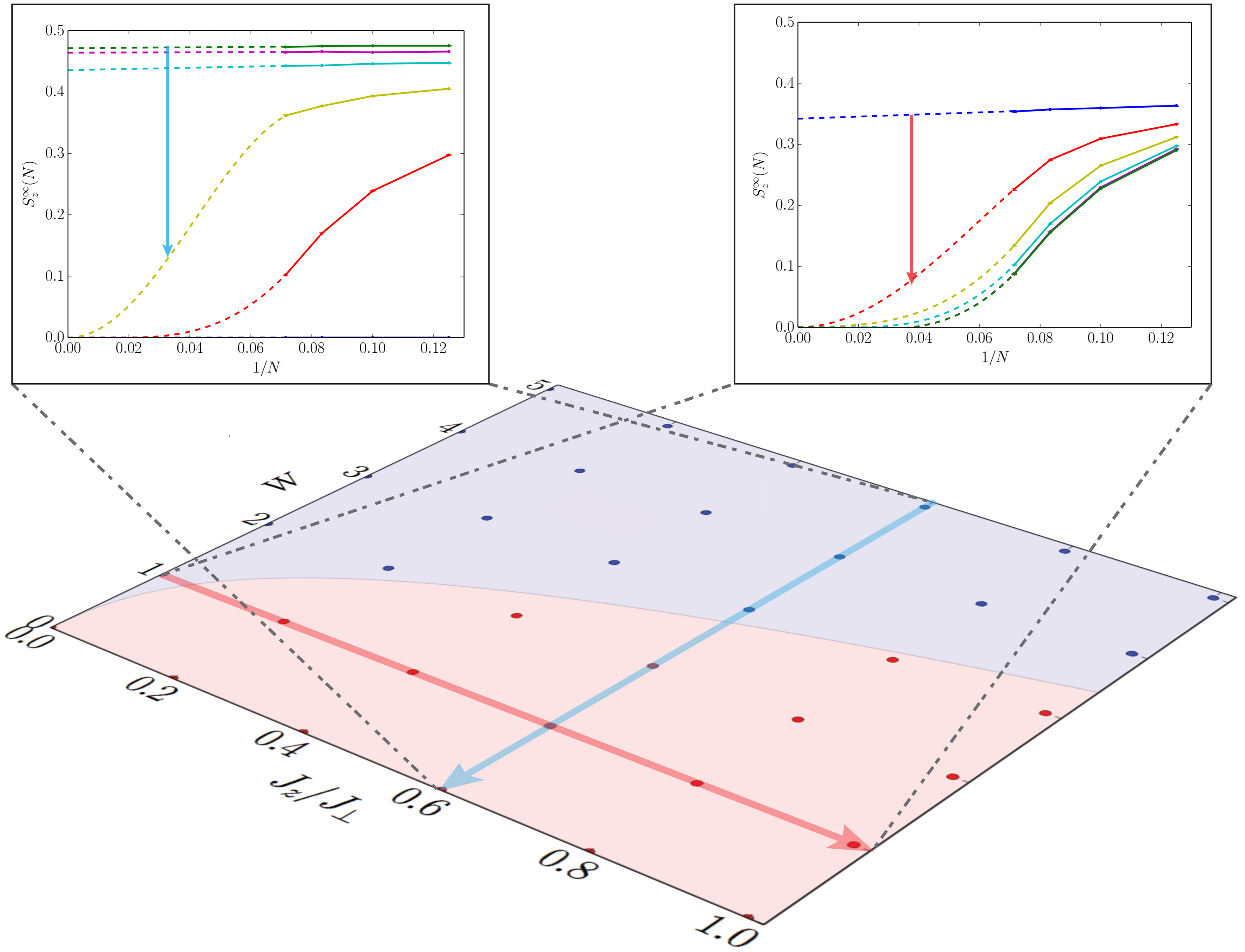
If probe spin is initially polarized, how frequently does its polarization return to nearly the initial value?

Qualitative motivation:
already in a classical system, Poincare recurrence time is a measurement of phase space volume.
Larger phase space to explore = lower rate of revivals.

Numerical experiment

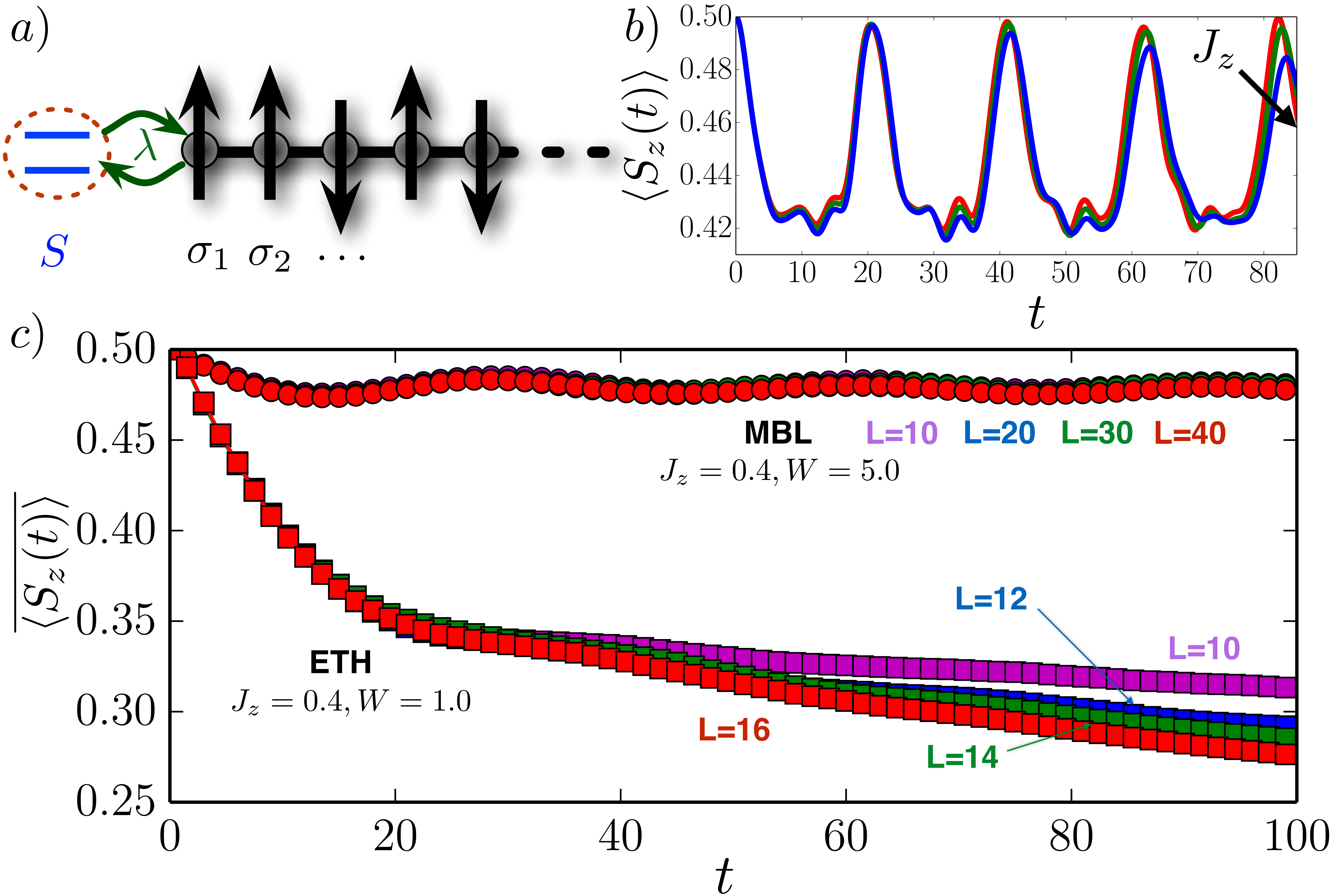
Initial state is probe spin up and random initial state of chain (with and without constraint of total Sz = 0).

Evolve in time and record a “revival” whenever average probe spin is within (1-epsilon) of initial value.



1. Estimate phase diagram via residual magnetization

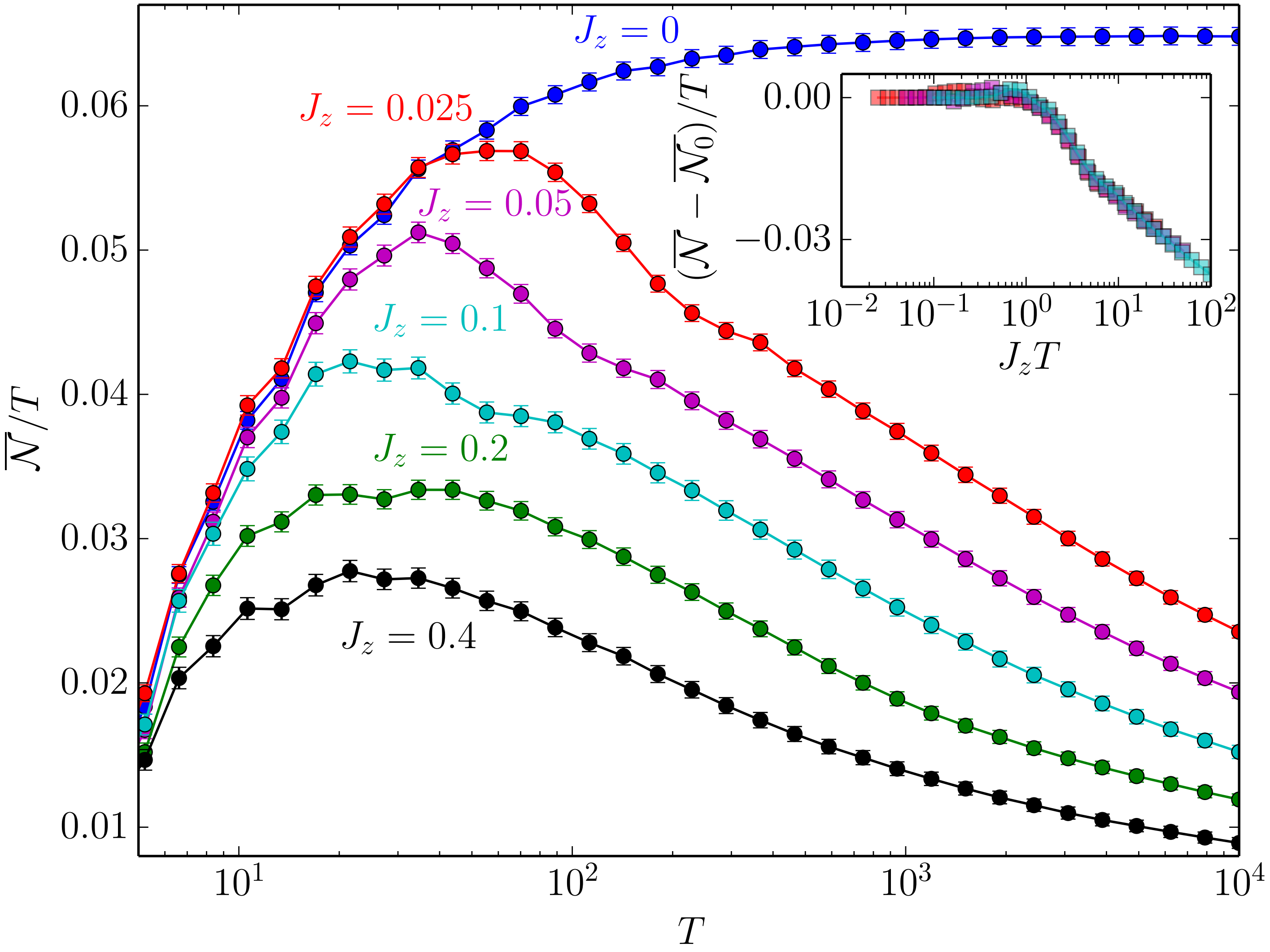
$$\overline{S^z_\infty} \equiv \overline{\sum_\alpha \langle\alpha|S^z|\alpha\rangle \, |\langle\Psi_0|\alpha\rangle|^2},$$



a)
λ
S
σ_1 σ_2 $\cdots$
b)
$\langle S_z(t) \rangle$
0.50
0.48
0.46
0.44
0.42
J_z
0 10 20 30 40 50 60 70 80
t
c)
$\overline{\langle S_z(t) \rangle}$
0.50
0.45
0.40
0.35
0.30
0.25
MBL
$J_z = 0.4, W = 5.0$
L=10 L=20 L=30 L=40
ETH
$J_z = 0.4, W = 1.0$
L=12
L=10
L=16
L=14
0 20 40 60 80 100
t

Numerical experiment

2. How does the revival rate show the effects of interactions?



Quantum Revivals. Disorder-averaged revival rate $\overline{\mathcal{N}(T)}/T$ as function of total time, T. Upon adding interactions of strength J_z, revivals are suppressed beyond $T^* \sim J_z^{-1}$. (Inset) The same data collapses onto a universal curve when plotted against J_zT.

Scaling collapse

We can do better than just saying that interactions = fewer revivals. Actually the same phenomenology that explains entanglement growth appears here as well: the difference in revival rates is

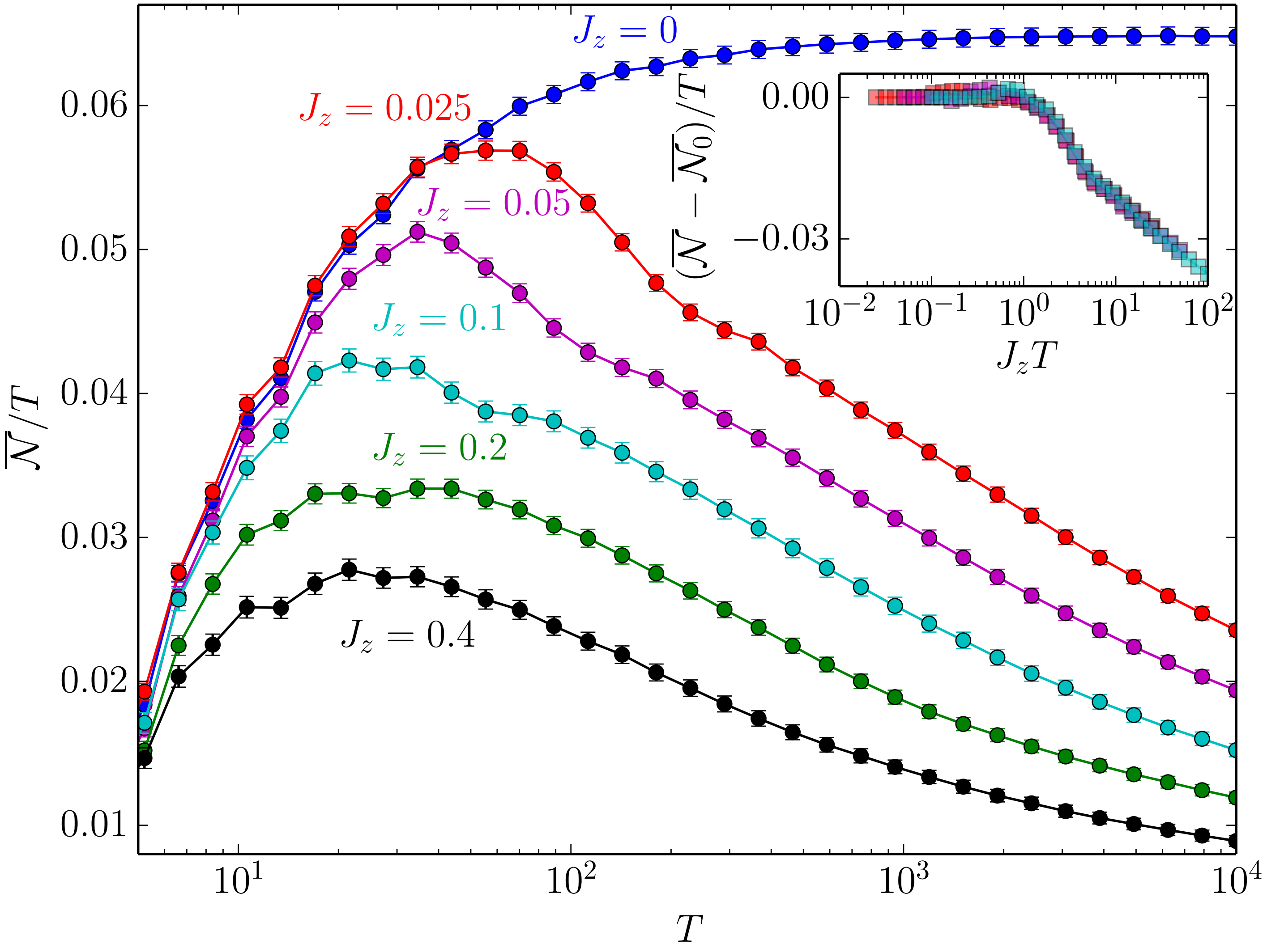
$$\frac{\overline{\mathcal{N}-\mathcal{N}_0}}{T} \approx \nu(N+\alpha\log J_z t)-\nu(N),$$

where \nu(N) is the revival rate when N different frequencies matter (expect an exponential dependence, but details turn out to be irrelevant).

The numerics show that the revival rate indeed shows a collapse with logarithmic time over most of the MBL phase (presumably not all of it)…

Numerical experiment

2. How does the revival rate show the effects of interactions?



Quantum Revivals. Disorder-averaged revival rate $\overline{\mathcal{N}(T)}/T$ as function of total time, T. Upon adding interactions of strength J_z, revivals are suppressed beyond $T^* \sim J_z^{-1}$. (Inset) The same data collapses onto a universal curve when plotted against J_zT.

Result: a simple picture

The “real-space Fermi liquid” form

$$H = \sum_i \epsilon_i n_i + \sum_{i,j} U_{ij} n_i n_j + \ldots$$

controls not just entanglement growth but more “physical” observables over a wide range of the MBL phase.

The resulting logarithmic time evolution (assuming U falls off exponentially) is likely to be a generic property of dynamics of observables in the MBL phase. This log scaling may be the most important observable difference between MBL and Anderson phases.

Point: two simple guesses (revival rate saturates as in Anderson case, or as 1/exp(xi)) are wrong.

It would be nice to understand (a) what is the long-time state of a block in the MBL phase starting from some physical preparation process (typically volume law but not ETH); (b) how *H* becomes more complicated (3-body, etc.) close to the transition.

Many-body localization at infinite temperature

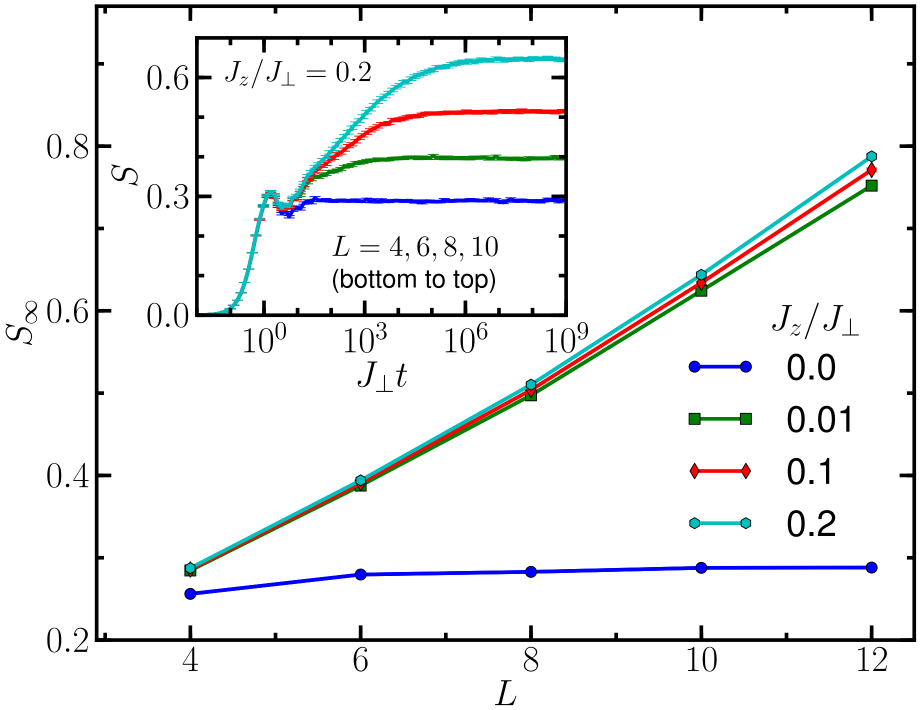
$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

What does entanglement entropy growth mean?

The entanglement entropy comes from the reduced density matrix, which governs any local experiment.

So any measurement of entropy in a subsystem will show that the interacting system is "more thermalized" than the Anderson one.

However, studies of the saturation of small blocks suggest that the full thermal entropy is not reached: O(L) but small.



Long-time behavior: some version of GGE?

A difference from normal GGE in Yang-Baxter integrability:

In something like XXZ or Lieb-Liniger, the conserved quantities are sums of local operators and hence *extensive*. Microcanonical and Gibbs ensembles are equivalent because fluctuations in an extensive operator like energy are relatively small.

This does not hold for the local quantities in MBL; it would be nice to have a useful means to calculate the long-time evolution of a generic initial state (which is related to the question of operators beyond two-spin).

Nature of phase transition

Believed to be “purely dynamical”, which we know already from the non-interacting case to be difficult.
(e.g., the Anderson and quantum Hall plateau transition)

Some theories in 1D based on real-space renormalization group
(Vosk-Altman-Huse, Vasseur-Parameswaran-Potter)

So far, not yet clear agreement between theory and numerics.

What happens near the MBL transition?

One big piece of news is that there could be two transitions:

as disorder decreases, in 1D we first get a thermalizing but subdiffusive phase, before (eventually?) transitioning to diffusion.

The resulting logarithmic time evolution (assuming U falls off exponentially) is likely to be a generic property of dynamics of observables in the MBL phase. This log scaling may be the most important observable difference between MBL and Anderson phases.

Point: two simple guesses (revival rate saturates as in Anderson case, or as 1/exp(xi)) are wrong.

It would be nice to understand (a) what is the long-time state of a block in the MBL phase starting from some physical preparation process (typically volume law but not ETH); (b) how *H* becomes more complicated (3-body, etc.) close to the transition.

What happens near the MBL transition?

Even the vicinity of the thermalizing transition is complicated.

M. Serbyn and JEM, 2015: analyze MBL transition with tools from Anderson transitions (generalized level statistics and multifractality)

For eigenvalues, use Dyson-type plasma model (an interaction between eigenvalues *V* and an overall confining potential *U*)

$$P(\{s_i\}) = \frac{e^{-\beta H}}{Z}, \; H = \sum_i W(s_i) + \sum_{i<j} U(s_i - s_j),$$

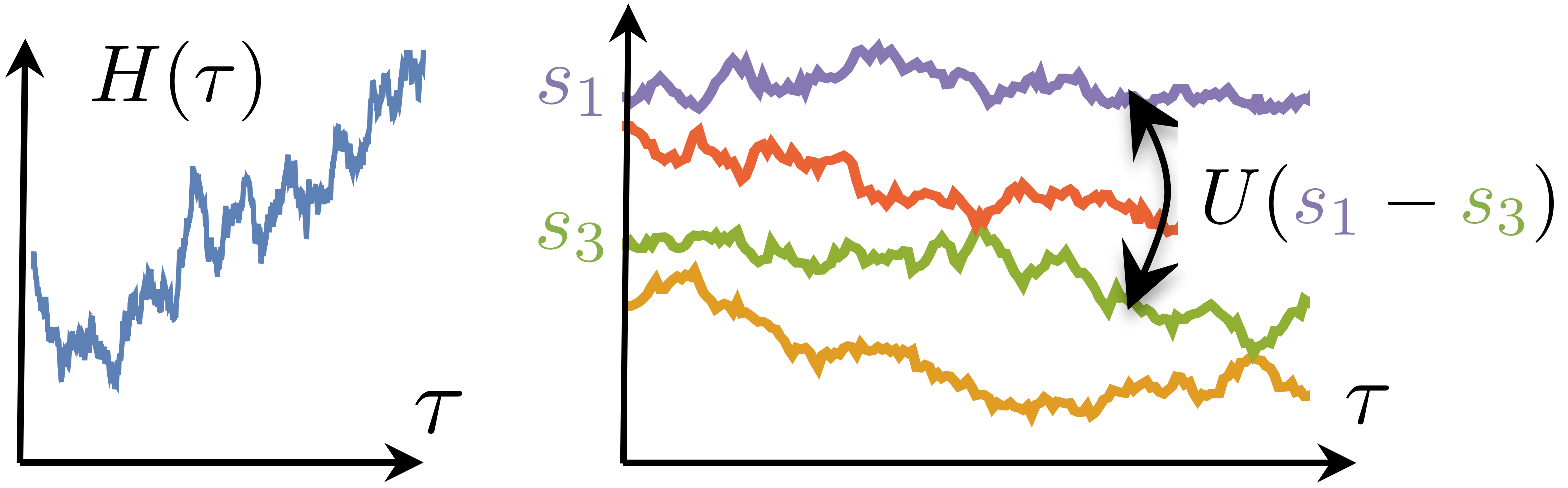
$\beta = 1$ for orthogonal matrix ensemble which will be of primary interest. The confining potential $W(s) = s^2/2$ is parabolic, and interaction is $U(s_i - s_j) = \ln|s_i - s_j|$.

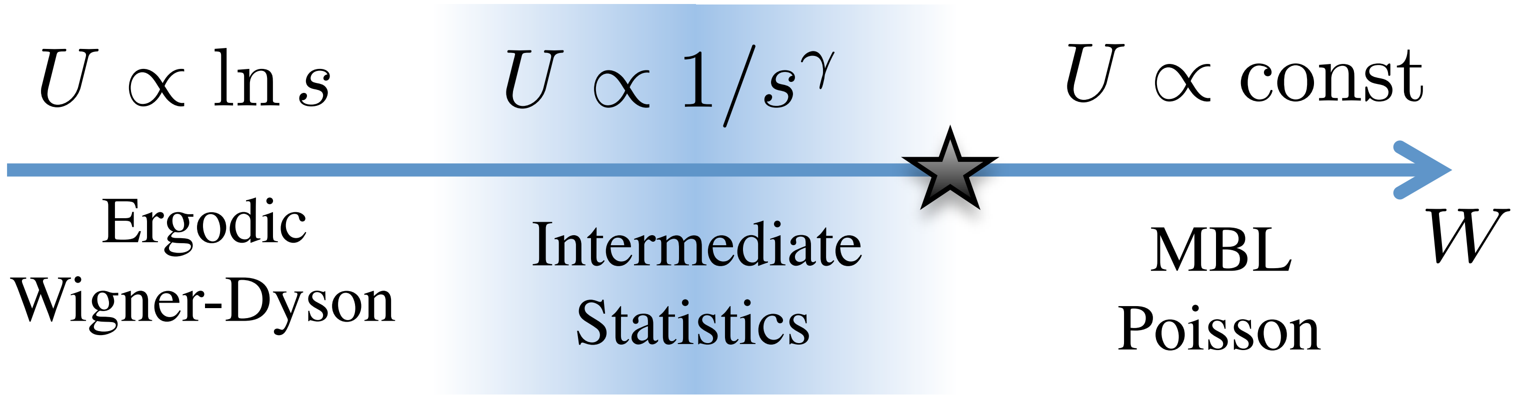
Challenges: statistics

We need to pick a way to fit intermediate level statistics:

There are ~100 proposals to generalize/interpolate between Poisson and Wigner-Dyson statistics, most of which have little microscopic justification.

We find a good fit by changing the interaction in the plasma model.
Dynamical "random walk" picture:





$U \propto \ln s$

Ergodic
Wigner-Dyson

$U \propto 1/s^\gamma$

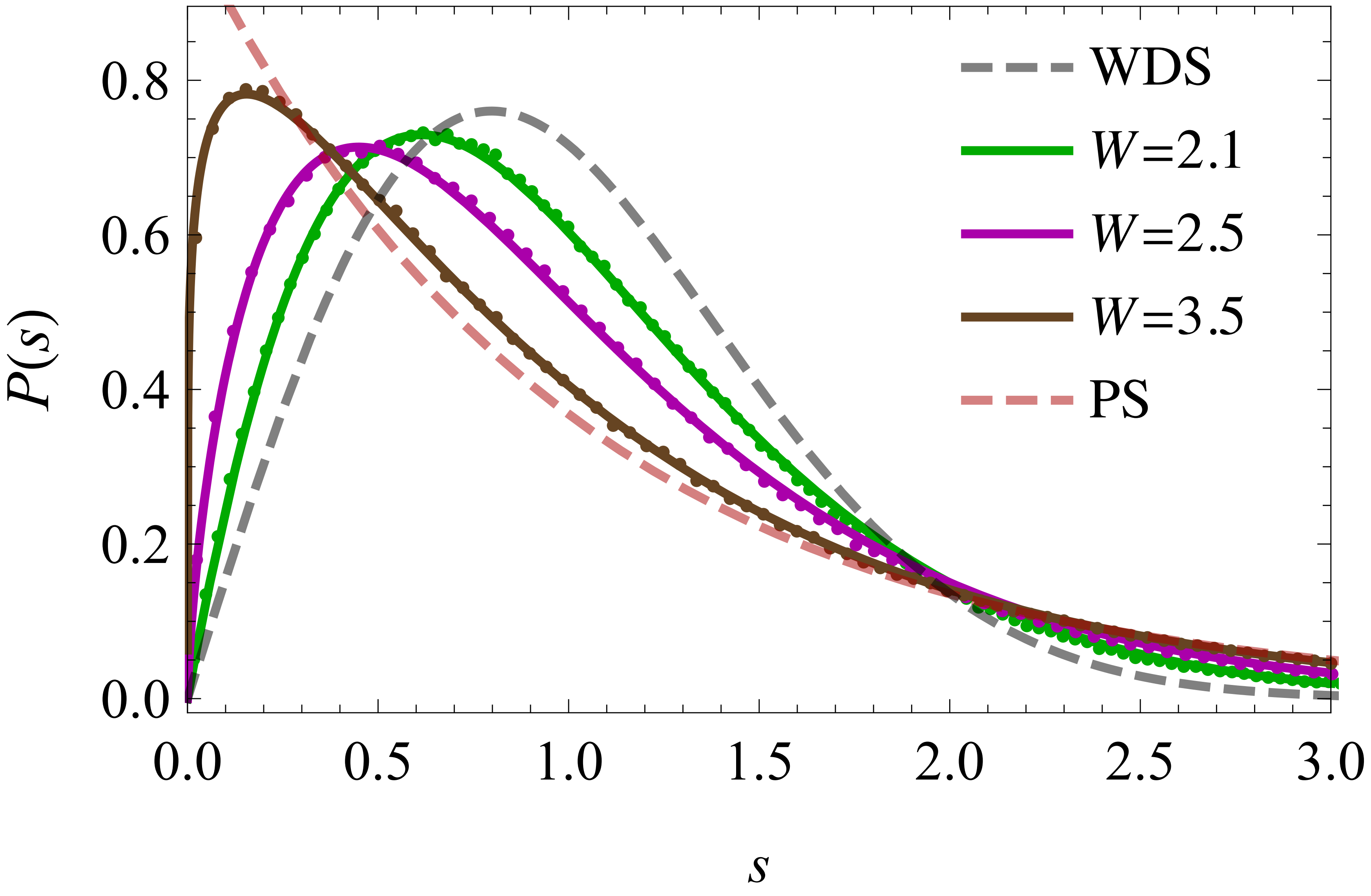
Intermediate
Statistics

$U \propto \text{const}$

MBL
Poisson

W

Challenges: statistics



We propose the following form of the level spacing distribution and spectral rigidity to interpolate between WDS and PS,

$$P(s;\beta,\gamma_P) = C_1 x^{\beta} \exp\left(-C_2 x^{2-\gamma_P}\right), \quad \mathrm{var}\, N = \chi N^{\gamma_{\mathrm{var}}}, \tag{1}$$

where the parameter $1 \geq \gamma_P, \gamma_{\mathrm{var}} \geq 0$ controls the tails of the statistics and level rigidity, and $1 \geq \beta \geq 0$ determines the level repulsion.

Challenges: multifractality of wavefunctions

One reason why even Anderson transitions are hard: “multifractality” means that there is a continuum of critical exponents.

$$\langle\langle|\psi(0)\psi(r)|^n\rangle\rangle \sim \frac{1}{r^{2\Delta(n)}}$$

Very different from a simple thermal phase transition (e.g., Ising has only *two* fundamental exponents).

Our strategy to test the model: compute several properties from both eigenvalues and eigenvectors, and see if they “collapse” (many results are explained by few parameters)

What happens near the MBL transition?

For eigenfunctions, follow Chalker RW approach, specialized to a spin chain:

$$\langle V_{nn}V_{mm}\rangle = v^2 d_{nm} = \frac{1}{\delta}\langle n|S_i^z|n\rangle\langle m|S_i^z|m\rangle, \tag{1}$$

$$\langle V_{nm}V_{mn}\rangle = v^2 c_{nm} = \frac{1}{\delta}|\langle m|S_i^z|n\rangle|^2, \tag{2}$$

where we normalized matrix elements by δ, the many-body level spacing, so that s_n represent unfolded energy spectrum. The correlator (**??**) sets the spectrum of a random noise, while c_{nm} determines the interaction between levels in the ensemble.

To make analytic progress we use a mean-field like approximation [**?**], assuming that d_{nm} and c_{nm} can be replaced by their ensemble averages,

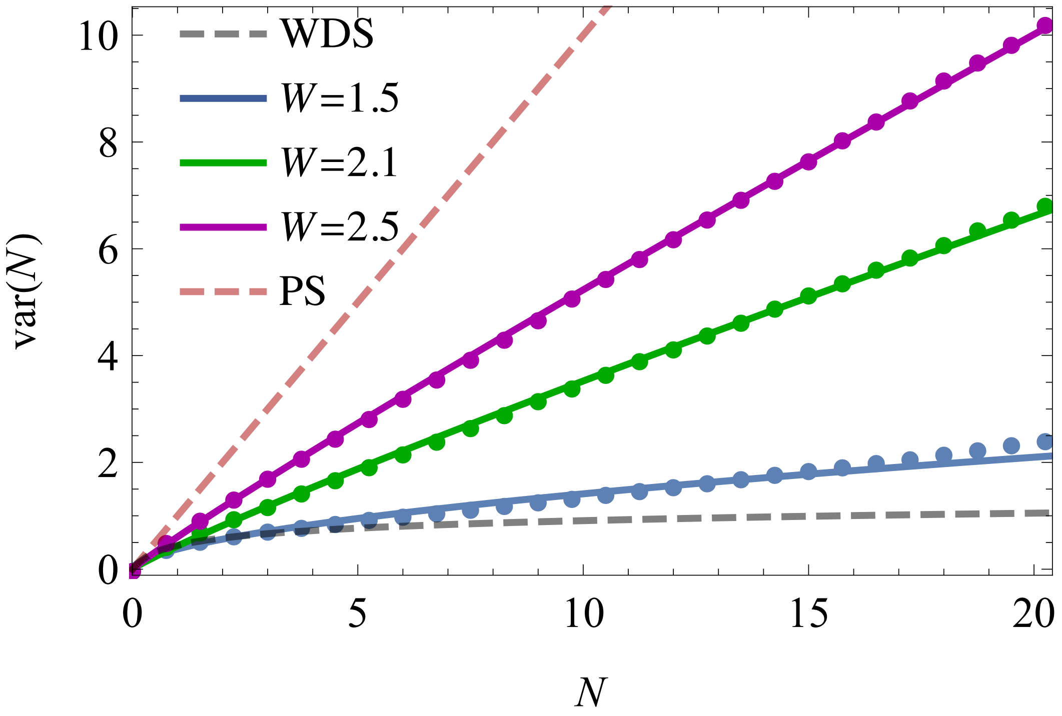
$$c(\omega) = \langle c_{nm}\delta(s_n - s_m - \omega)\rangle, \tag{3}$$

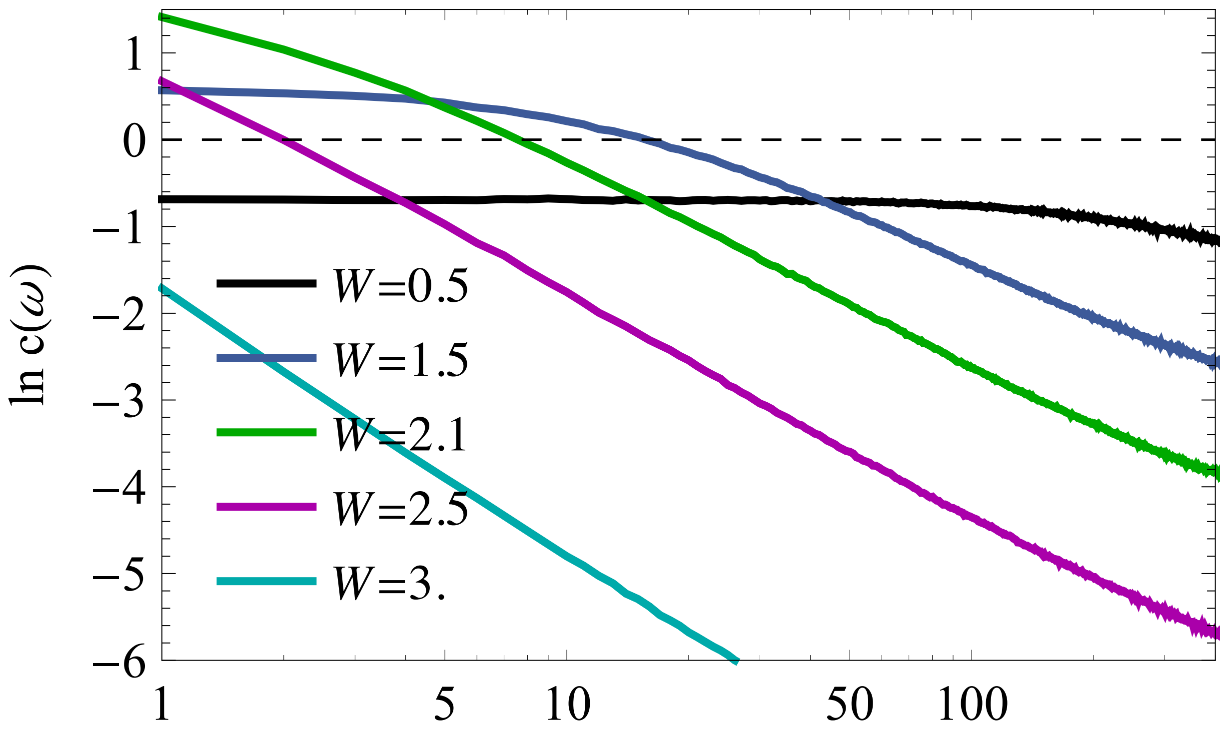
(and similar expression for d_{nm}) which now depend only on the energy difference between eigenstates.

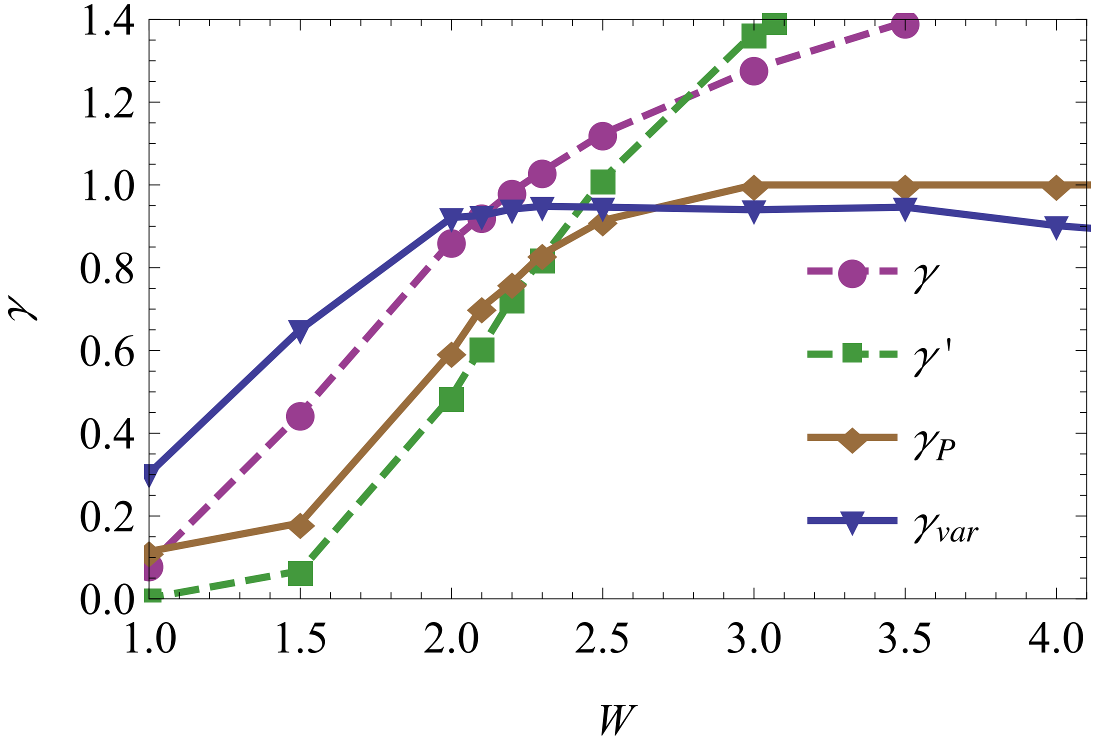
For the single-particle Anderson localization, the c_{nm} and d_{nm} necessarily coincide [**?**], as they both are given by wave functions overlaps, $c_{nm} = d_{nm} \propto \int dx|\psi_n(\tau,x)|^2|\psi_m(\tau,x)|^2$. **Not true for MBL!**

Collapse of different properties

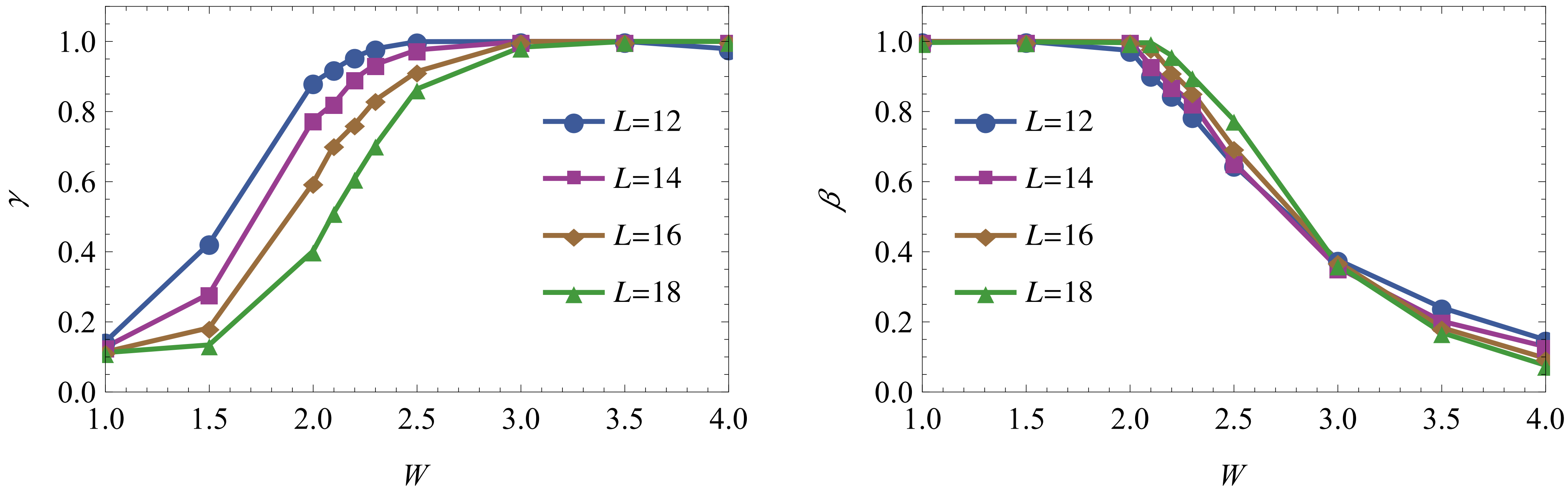
For additional information on multifractality and possible extrapolation to thermodynamic limit, see Kravtsov talk.







What happens near the MBL transition?



Within the picture of Brownian motion [**?**, **?**], the level statistics is controlled by the effective interaction between energy levels, see Fig. 1. In particular, deep in the metal phase, the WD statistics emerges from the partition function of a one-dimensional Coulomb gas, where particles interact with a logarithmic potential $U(s) = \log|s|$. At a first stage, upon approaching the MBL transition, the effective interaction starts to decay as a power-law: $U(s_i - s_j) = |s_i - s_j|^{-\gamma}$ when $|s_1 - s_2| \geq N_{\text{erg}}$. The power-law interaction changes tails of the level statistics, so it can be approximately described by the plasma model, and is intermediate between PS and WDS case. At the second stage, when exponent γ becomes bigger than one, the interaction becomes effectively short-ranged, and level spacing distribution tends to the semi-Poisson distribution [**?**]. In this regime it is the range of the interaction which changes with disorder/system size. As soon as the range of interactions reaches zero, we arrive at Poisson statistics.

Part II: no disorder

Remark: what are MBL-like features in translation-invariant systems? (M. Mueller et al., Yao et al., Papic et al.)

Point: can make artificial models that show MBL with translation invariance.

For example, we could interpret a random potential model as resulting from infinitely heavy particles that create a potential for light particles.

Are these stable? For example, if the heavy particles have large finite mass, does MBL still exist?

In our example, it does not, but very long time scales are needed to see it. (Of course, this isn't a general proof.)

Similarly unstable are "conventional" integrable systems. How is their long-time dynamics modified by the existence of infinitely many conservation laws?

Numerics

Most numerics on MBL so far were done with "exact diagonalization" (ED): find all eigenvalues, or a subset, of the Hamiltonian matrix.

ED is great for small systems as it gives essentially complete information and its implementation and convergence are well understood.

It doesn't scale very well: cost for all eigenvalues goes as the cube of the matrix dimension, so beyond 20 spin-half sites becomes expensive.

Good news: there has been enormous progress 1992-present in DMRG/"matrix product state" methods to solve many-particle quantum problems in low spatial dimensions (especially 1D).

Understanding when these methods work well requires us to understand *entanglement,* which also leads to another useful definition of the MBL state.

Studying quantum correlations with classical algorithms: applied entanglement entropy

Basic (hazy) concept: "Entanglement entropy determines how much classical information is required to describe a quantum state."

Example:
how many classical real numbers are required to describe a *product* (not entangled) state of N spins?

simple product $$|\psi\rangle = A_{s_1} A_{s_2} A_{s_3} A_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Answer: ~ N (versus exponentially many for a general state)

How do we efficiently manipulate/represent moderately entangled states?

Applied entanglement entropy

The remarkable success of the density-matrix renormalization group algorithm in one dimension (White, 1992; Ostlund and Rommer, 1995) can be understood as follows:

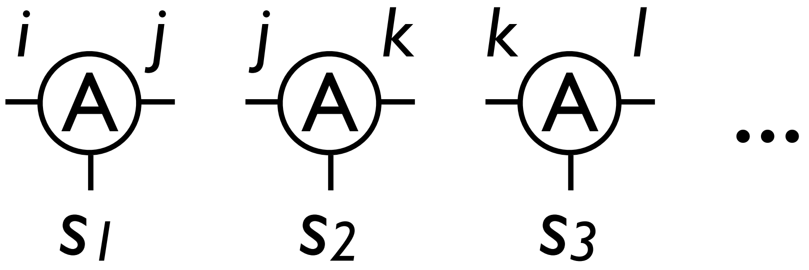
DMRG constructs "matrix product states" that retain local entanglement but throw away long-ranged entanglement.

Example states for four spins:

simple product $$|\psi\rangle = A_{s_1} A_{s_2} A_{s_3} A_{s_4} |s_1 s_2 s_3 s_4\rangle$$

matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Graphical tensor network representation:



“Infinite system” methods

Note that we can impose translation invariance simply by requiring constant matrices *A*.

In other words, for quantities in a translation-invariant system, *we just calculate A, rather than a large finite system.*
(Idea 1 of renaissance; see Vidal ’07, for example)

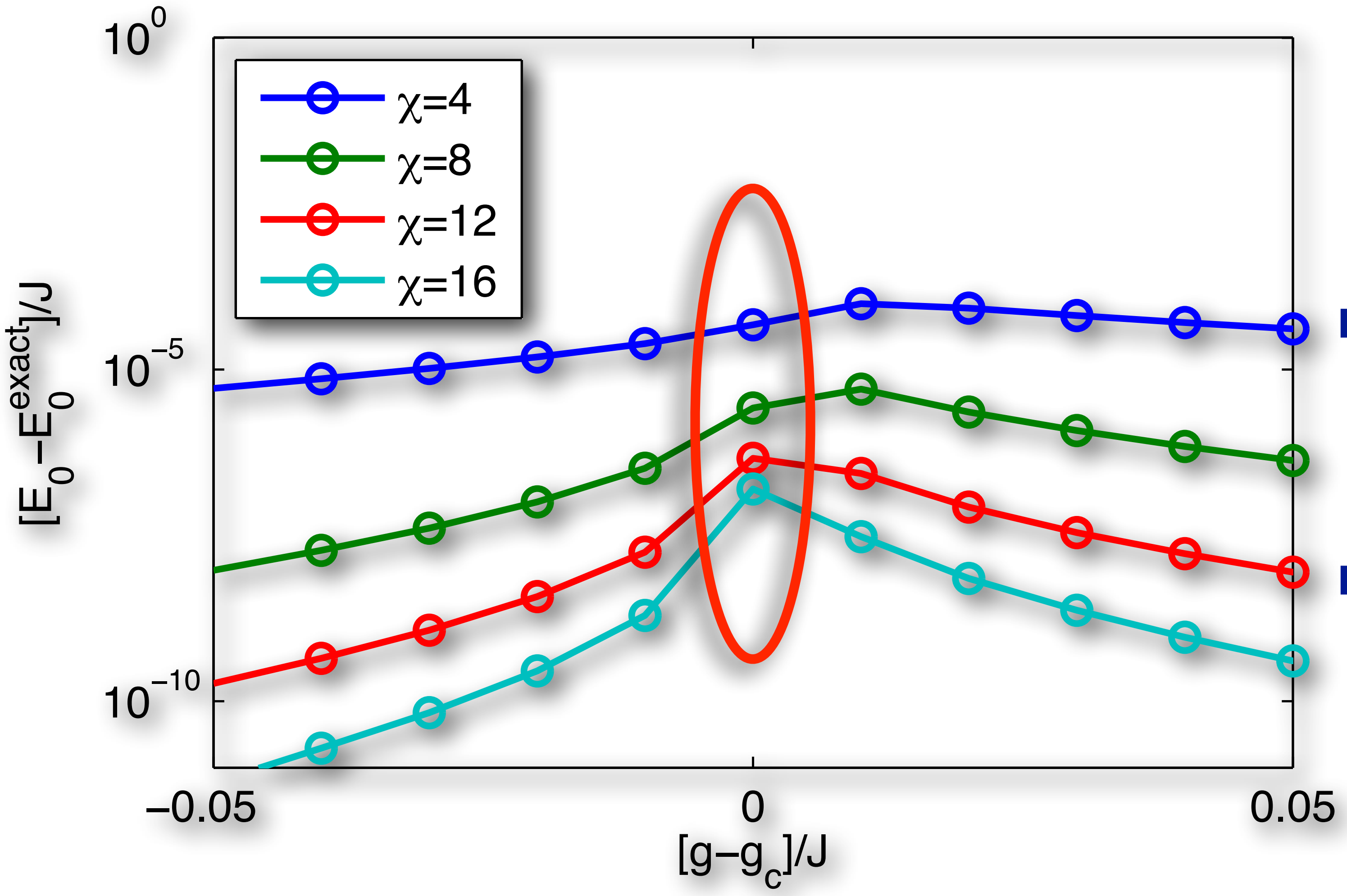
matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

So where is the approximation?
A finite matrix *A* can only capture a finite amount of entanglement.

In the early DMRG days, it was often thought:
1. To study an infinite system, we should study a large finite one.
2. Gapless/critical systems are hard. (Gapped uniform systems converge…)
3. Dynamical properties are hard
4. Finite temperature is hard
But none of these is strictly correct.

- find the ground state of a system by using **imaginary time evolution** (almost unitary for small time steps)
- parallel updates for **infinite/translational invariant** systems: **iTEBD** [Vidal '07]
- **example**, transverse Ising model:

$$H = \sum_i \left(J\sigma_i^z \sigma_{i+1}^z + g\sigma_i^x \right)$$



➡convergence of wave function is worst at the **critical point**

➡**conformal invariance**

Criticality: finite-entanglement scaling

All numerical methods have difficulty with quantum critical points. In DMRG-type approaches, this can be understood from the divergence of entanglement entropy at such points: the entanglement in a matrix product state is limited by dim *A*.

matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Quantitatively, it is found that dim *A* plays a role similar to imposing a finite system size:

$$L_{\text{eff}} \propto \chi^{\kappa}, \quad \chi = \dim\ A$$

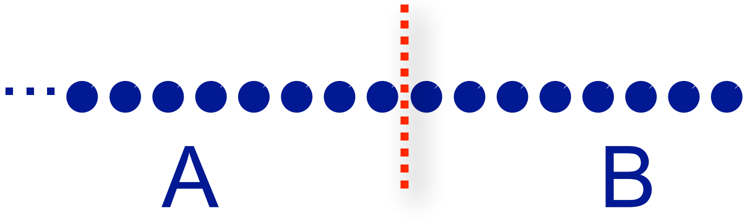
(Tagliacozzo et al., PRB 2008).

Finite matrix dimension effectively moves the system away from the critical point.

What determines this “finite-entanglement scaling”?
Is it like “finite-size scaling” of CFT’s (cf. Blöte, Cardy, & Nightingale)

A way to picture the entanglement of a state

- **Schmidt decomposition** of the state (**SVD**):
- $$\begin{aligned} |\psi\rangle &= \sum_{i=1}^{N_A}\sum_{j=1}^{N_B} C_{ij}|i\rangle_A|j\rangle_B \\ &= \sum_{\alpha=1}^{\min(N_A,N_B)} \lambda_\alpha|\phi_\alpha\rangle_A|\phi_\alpha\rangle_B \end{aligned}$$



with $\lambda_\alpha \geq 0$ and $\sum_\alpha \lambda_\alpha^2 = 1$

- a natural measure of the entanglement is the **entropy:**

$$S_A = S_B = S = -\sum_\alpha \lambda_\alpha^2 \log(\lambda_\alpha^2)$$

Efficient representation of quantum states?

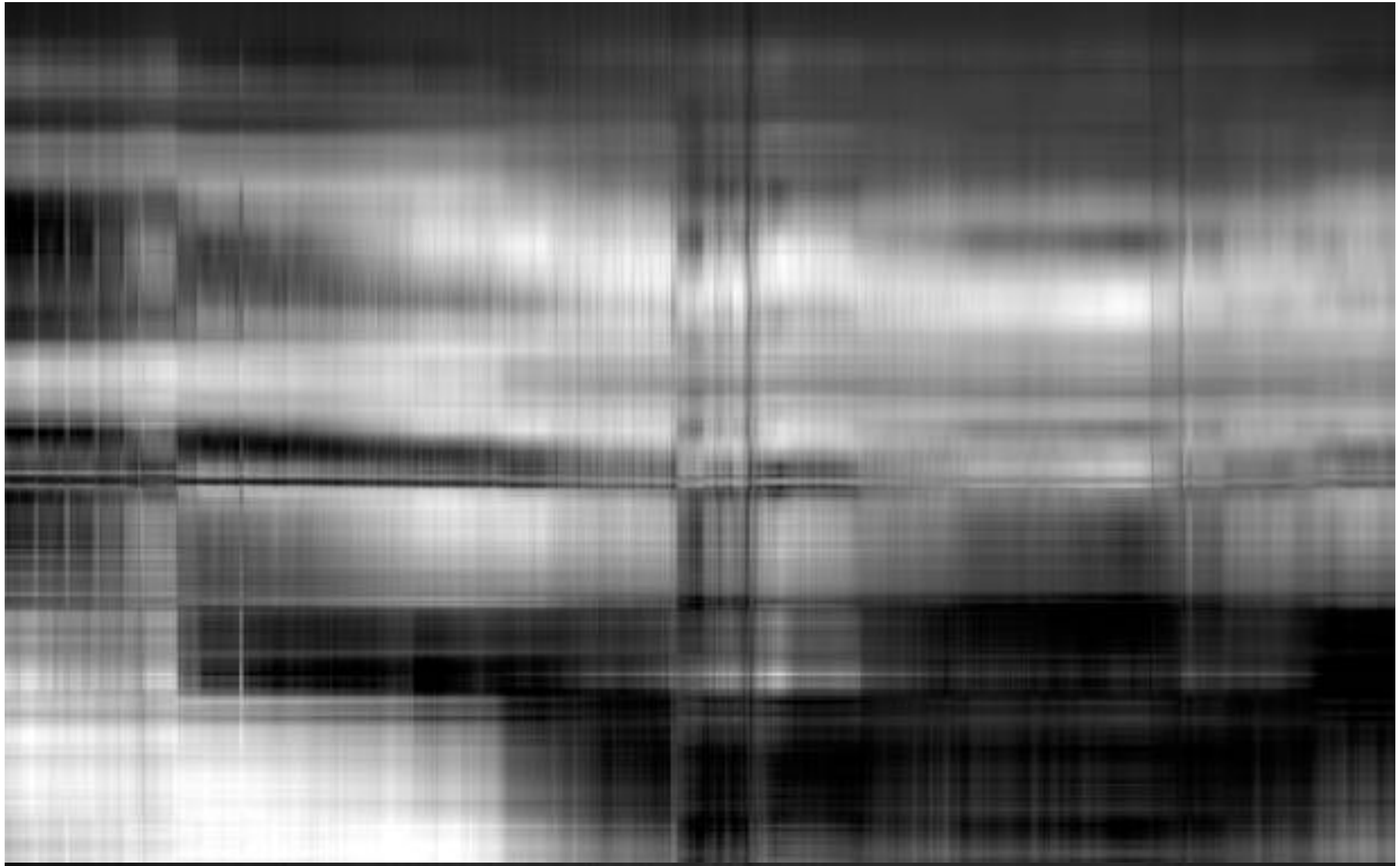
- Hilbert-space dimension of many-body problems increases **exponentially** with number of sites
  example: spin 1/2 system on “classical” computers
  (store **one state** in double precision)

- need an efficient way to “**compress**” quantum states so that the matrices studied remain finite-dimensional

  ➡slightly entangled 1D systems: **Matrix Product States**

  ➡DMRG, TEBD, ...

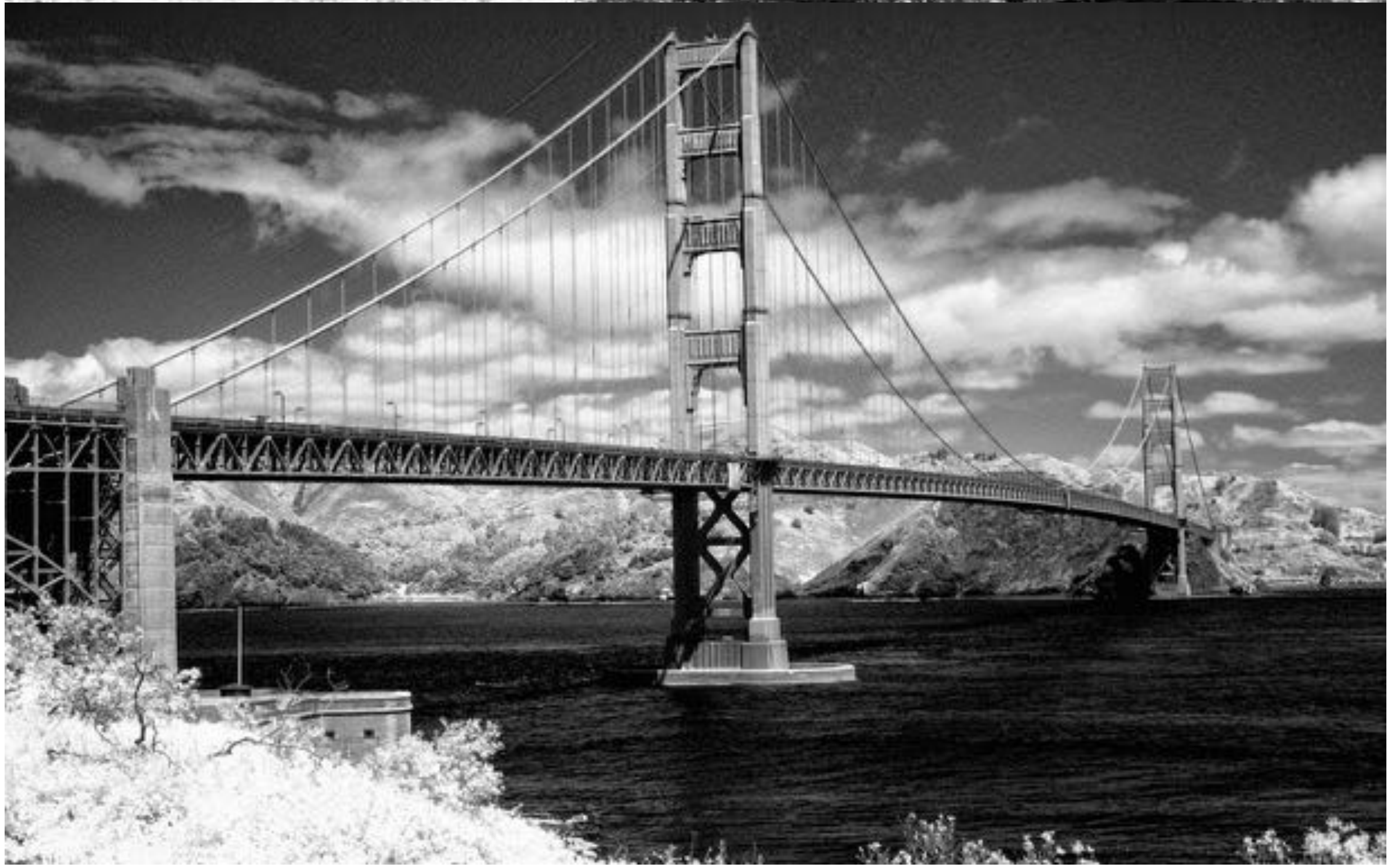
$\chi = 4$



$\chi = 16$

$\chi = 64$

$\chi = 256$



- (Li-Haldane) “entanglement spectrum” [Calabrese et al ‘08]

$$n(\lambda) = I_0\left(2\sqrt{-b^2 - 2b\log\lambda}\right)$$

of $\hat{\lambda}$ ‘s greater than λ

with $b = \frac{S}{2} = \frac{c}{12}\log\xi = -2\log\lambda_{\mathrm{max}}$

continuum of Schmidt values $|\psi\rangle = \sum_{\alpha=1}^{\infty}\lambda_\alpha|\phi_\alpha\rangle_A|\phi_\alpha\rangle_B$

- Want to explain how at a critical point, finite matrix size χ effectively moves the system away from criticality, leading to universal relations like

$$L_{\mathrm{eff}} \propto \chi^{\kappa}, \quad \chi = \dim\, A$$

- **A heuristic argument** for the asymptotic case (using a continuum of Schmidt values and $\chi \to \infty$)

➡**universal finite-entanglement scaling relations**

$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}}+1\right)} \Rightarrow S = \frac{1}{\sqrt{\frac{12}{c}}+1} \log \chi$$

F. Pollmann, S. Mukerjee, A. Turner, and J.E. Moore, PRL 2009
Some checks for various critical theories are in that paper, and the recent work
B. Pirvu, G. Vidal, F. Verstraete, L. Tagliacozzo, arXiv:1204.3934

So critical points are worse than gapped points, but in a controlled way.
What does this mean in practice?

Remark: Entanglement spectra are qualitatively different for random critical spin chains than for pure ones, though entanglement entropies similar (M. Fagotti, P. Calabrese, JEM).

Part II: translation-invariant systems with Yang-Baxter integrability

1. Outline: start with a model problem: two-reservoir quench in 1D

Simple cases: free bosons; CFTs
Review of ballistic linear response (Drude weight)

In less simple integrable models: exact results for some quantities even arbitrarily far from equilibrium; can compare to DMRG simulations for XXZ
(Vasseur, Karrasch, JEM PRL 2015)
Background to hydrodynamical/kinetic theory approaches for soliton gases, Lieb-Liniger, XXZ.

2. Test of kinetic theory predictions in more general cases: expansions from smooth initial conditions.

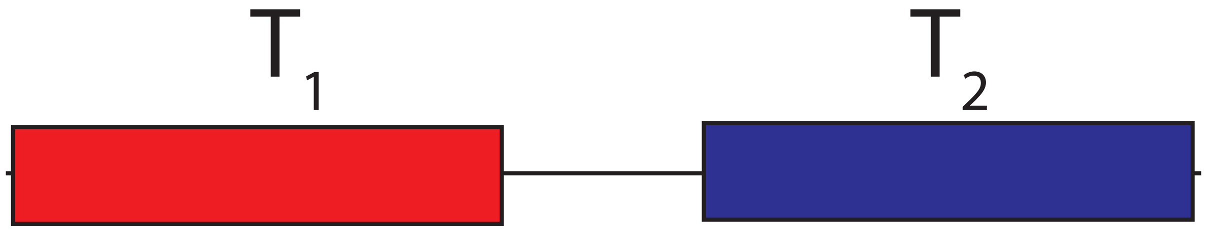
When is hydrodynamics (i.e., Bethe-Boltzmann equation) valid? Can compare to *microscopic simulations at nonzero T,* when hydrodynamics should be generic. (At T=0, coarse-graining length diverges at least in CFTs)
What does hydrodynamics miss?

3. (if time permits) Hall viscosity in d=2 hydrodynamics

Non-equilibrium energy transport in XXZ

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

1. Create two different temperatures in two disconnected, infinite 1D "leads".
2. Connect them by a finite region (e.g., one bond).
3. Evolve in time for as long as possible.



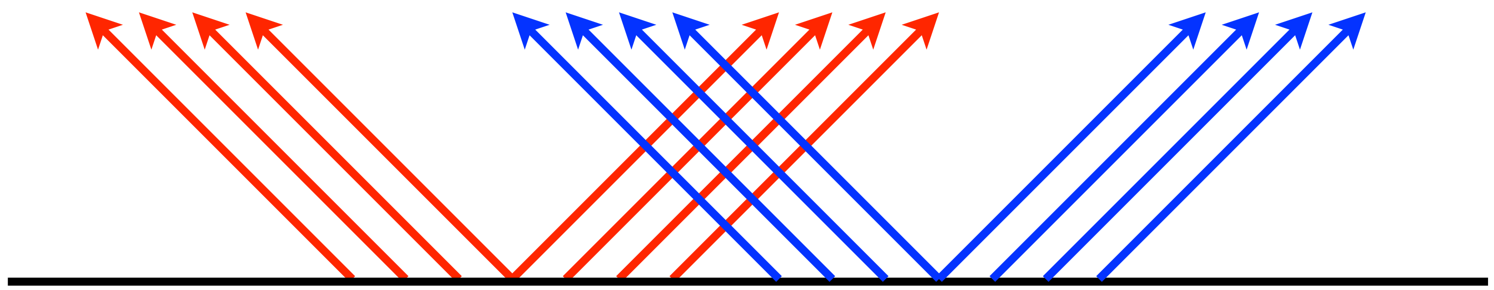
Is a steady-state heat current reached?

Is non-equilibrium (finite bias) thermal transport determined by linear-response thermal conductance?

We observe two different outcomes, depending on integrability of the leads and whether the connected system is homogeneous.

Stefan-Boltzmann picture

Idea: the right lead is prepared at one temperature and the left lead at a different temperature.



In a ballistic system like a CFT, there is no local temperature at x=0 at later times; rather the right-movers are at a different temperature than the left-movers. The thermal current is the difference between total radiation from left and right.
(Sotiriadis-Cardy, Bernard-Doyon)

Warmup: free bosons

We compute the right-moving energy current from a lead at temperature $T+dt$ and subtracting the left-moving energy current from a lead at T. Assume one-dimensional free bosons as in the Schwab et al. experiment mentioned above. Using k for momentum, we have that the total energy current (units of energy per time) is

$$J_E = J_E^R - J_E^L = \int_0^{\pi/a} \frac{dk}{2\pi}\, \left[f_{T+dt}(\hbar\omega_k) - f_T(\hbar\omega_k)\right] \hbar\omega_k v_k. \tag{1}$$

Here $v_k = d\omega_k/dk$ and $f_T(E)$ is the Bose factor $(e^{E/k_BT} - 1)^{-1}$. So

$$J_E = (dt) \int_0^{\omega_{max}} \frac{d\omega}{2\pi k_B T^2}\, \frac{e^{\hbar\omega/k_BT}}{(e^{\hbar\omega/k_BT} - 1)^2} \hbar^2\omega^2. \tag{2}$$

Here ω_{max} is the highest phonon frequency. If we assume that the temperature is small compared to this, so that $x = \hbar\omega/k_BT$ runs from 0 to infinity, then we obtain (note that we need to multiply by $(k_BT/\hbar)^3$)

$$J_E = \frac{{k_B}^2 T}{2\pi\hbar}(dt) \int_0^{\infty} dx\, \frac{x^2 e^x}{(e^x - 1)^2}. \tag{3}$$

The dimensionless integral gives $\pi^2/3$, so

$$G_0 = \frac{J_E}{dt} = \frac{\pi^2 {k_B}^2 T}{3h}. \tag{4}$$

An interesting fact about the thermal conductance G_0 is that it is the same for bosons or fermions (or indeed anyons), unlike charge transport. The Schwab et al. experiment observed one thermal conductance quantum G_0 for each low-temperature phonon mode.

Dissipationless transport

When is there a nonzero Drude weight *D*?

$$\sigma(\omega) = D\delta(\omega) + \ldots$$

Two easy examples:

I. Superconductors (transport by condensate)

II. Part of the current is conserved: Mazur lower bound

$$D = \frac{1}{2LT} \lim_{t\to\infty} \langle J(t)J(0)\rangle \geq \frac{1}{2LT} \sum_k \frac{\langle J Q_k\rangle^2}{\langle {Q_k}^2\rangle}$$

Dissipationless transport

When is there a nonzero Drude weight *D*?

$$\sigma(\omega) = D\delta(\omega) + \dots$$

Example of Mazur bound: suppose momentum is conserved, and current is proportional to momentum (e.g., if only one kind of particle).

Technical note: the Drude weight is not thermodynamic:

$$D = \frac{1}{L}\left[\langle\Theta\rangle - \hbar\sum_{n,m}(1-\delta_{\epsilon_n,\epsilon_m})\frac{e^{-\beta\epsilon_n} - e^{-\beta\epsilon_m}}{Z(\epsilon_m - \epsilon_n)}|\langle n|J|m\rangle|^2\right], \quad D_m = \frac{1}{L}\left[\langle\Theta\rangle - \hbar\sum_{n,m}\frac{e^{-\beta\epsilon_n} - e^{-\beta\epsilon_m}}{Z(\epsilon_m - \epsilon_n)}|\langle n|J|m\rangle|^2\right]$$

where D_m is "Meissner stiffness" (response to flux). Always $D \geq D_m$. (Mukerjee and Shastry, PRB 2007). Here

$$\Theta = \lim_{k\to 0}\frac{1}{k}[\rho(k), J(-k)]$$

$$\sigma(\omega) = D\delta(\omega) + \ldots$$

$$D = \frac{1}{2LT} \lim_{t\to\infty} \langle J(t)J(0)\rangle \geq \frac{1}{2LT} \sum_k \frac{\langle JQ_k\rangle^2}{\langle {Q_k}^2\rangle}$$

What about "integrable" models with an infinite number of conserved local quantities, none of which gives a lower bound?

Actually this happens quite often in 1D--simplest case is spinless interacting fermions (XXZ model in zero magnetic field).

$$H = \sum_i \left[J_{xx}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \Delta S_i^z S_{i+1}^z + hS_i^z\right]$$

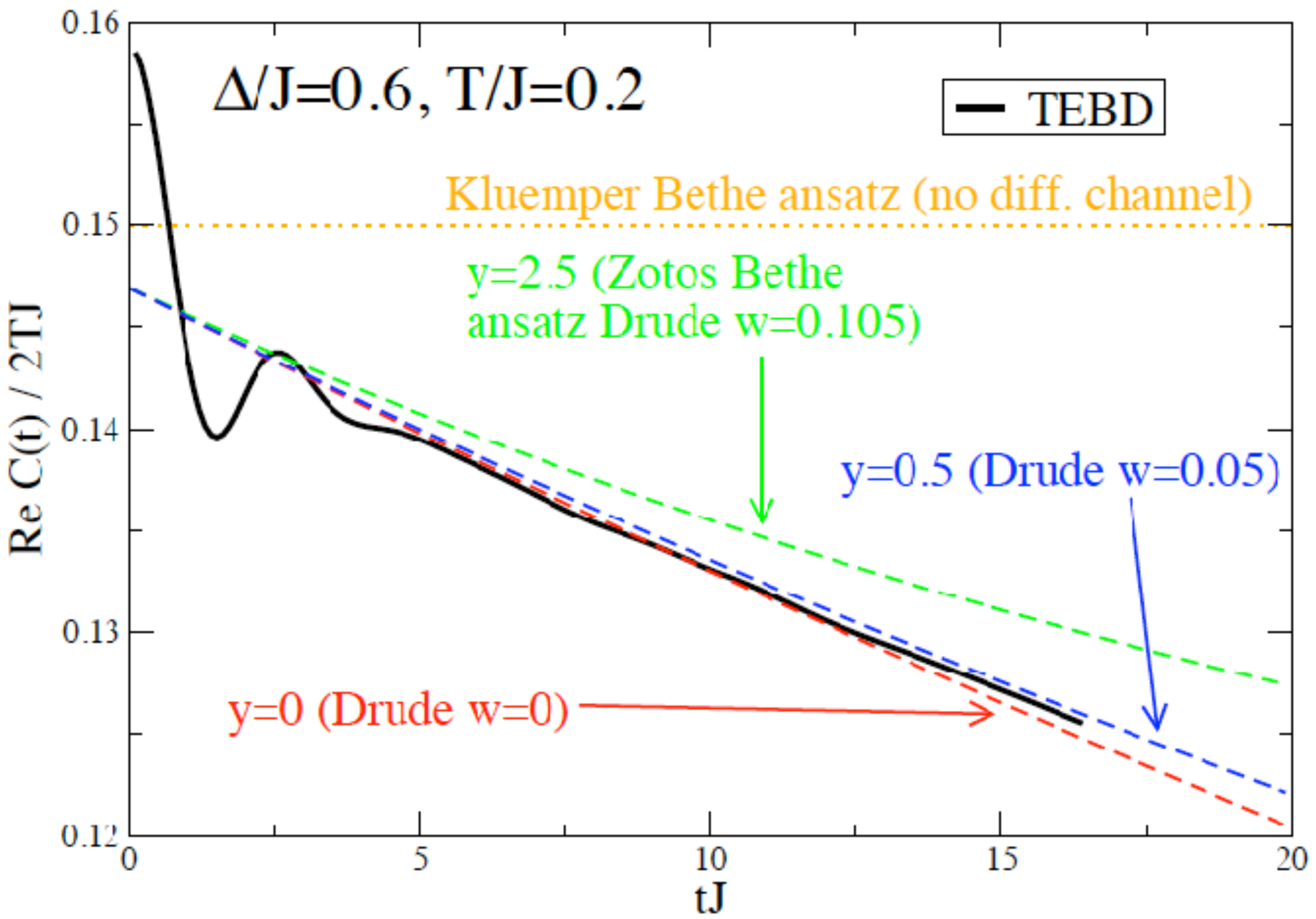
The Drude weight is easy to calculate and nonzero at T=0.
20+ years of efforts to calculate it (or even prove that it is nonzero) at T>0, h=0, by either analytical or numerical methods.

(cf. Sirker, Pereira, Affleck, PRB 2011)
(Thermal Drude weight is easier, for reason said later: found by Klumper and Sakai)

$$\sigma(\omega) = D\delta(\omega) + \ldots$$

$$D = \frac{1}{2LT} \lim_{t \to \infty} \langle J(t) J(0) \rangle \geq \frac{1}{2LT} \sum_k \frac{\langle J Q_k \rangle^2}{\langle {Q_k}^2 \rangle}$$

data from Sirker, Pereira, Affleck, PRB 2011



Drude weight progress, from 2011

$$\sigma(\omega) = D\delta(\omega) + \dots$$

$$D = \frac{1}{2LT} \lim_{t\to\infty} \langle J(t)J(0)\rangle \geq \frac{1}{2LT} \sum_k \frac{\langle JQ_k\rangle^2}{\langle {Q_k}^2\rangle}$$

Prosen: there is an iterative process to construct a nonlocal quantity that gives a lower bound that depends non-analytically on anisotropy, with cusps at Δ=cos(π/n). (PRL 2011)
(subsequent work generalizing this result: Ilievski-Prosen, …)

Karrasch-Bardarson-JEM: The Drude weight can be calculated numerically for all but the lowest temperatures at positive Δ, and essentially all temperatures at negative Δ.

The lower bound appears to saturate the full value at the cusps.

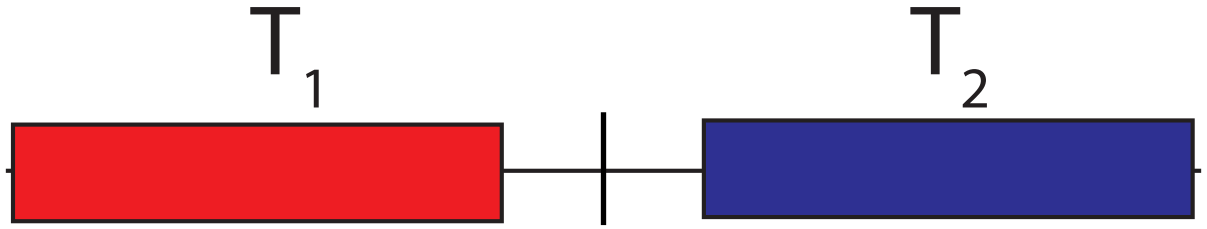
“Non-equilibrium expansions”

Almost everything that follows will be specific to 1D systems, where we have special analytical and numerical tools.

A. “Point current”: A natural question about electrons is to compute the (charge or energy) current through a *point*.

Many beautiful works, especially in quantum impurity models. Older works on translation-invariant case: Sotiriadis-Cardy, Bernard-Doyon, Karrasch-Ilan-JEM

Solved (not too easily) by new hydrodynamical methods mentioned earlier



B. “Expansion”: with atoms, it is more natural to *image* the full distribution of atoms (or conceivably energy). Two nice features:

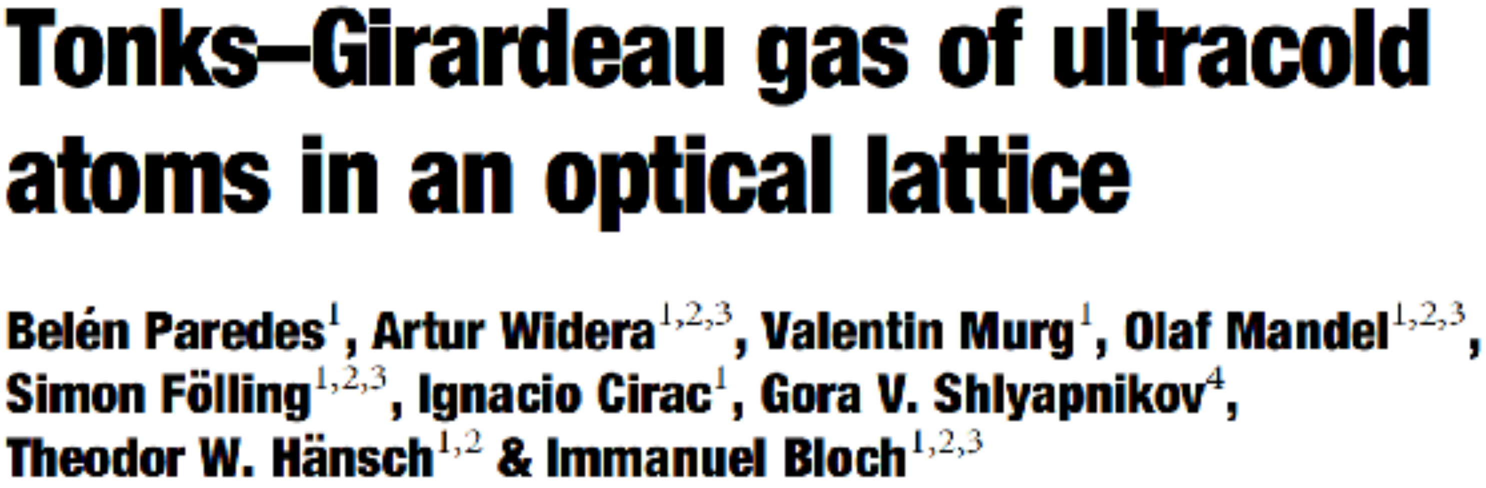
1. The most natural model of 1D interacting atoms, the Lieb-Liniger model, is integrable.

2. For charge current in Lieb-Liniger, or energy current in the XXZ model, there is conservation of the spatially integrated current, which turns out to have remarkable consequences: expansion is controlled by a form of non-equilibrium thermodynamics.

“Non-equilibrium expansions”

Lieb-Liniger model = Bose gas with delta-function interactions

Thermodynamics (Yang and Yang) interpolates from free bosons to free fermions as interaction strength increases.



Tonks–Girardeau gas of ultracold atoms in an optical lattice

Belén Paredes[1], Artur Widera[1,2,3], Valentin Murg[1], Olaf Mandel[1,2,3], Simon Fölling[1,2,3], Ignacio Cirac[1], Gora V. Shlyapnikov[4], Theodor W. Hänsch[1,2] & Immanuel Bloch[1,2,3]

One experimental example:
Nature, 2004

Originally, it was only possible to measure momentum-space distributions; now several groups have achieved imaging of individual sites of an optical lattice (Greiner, Chin, Bloch, …).

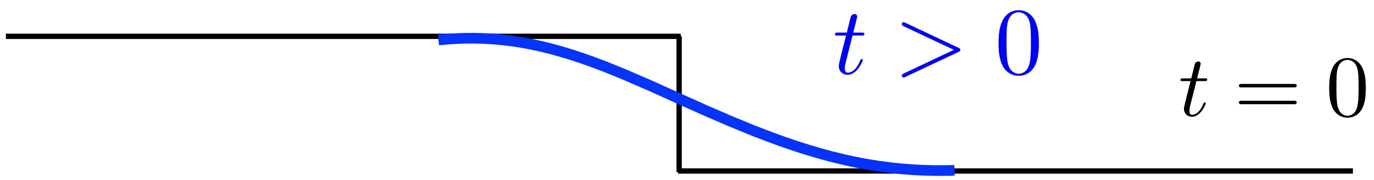
How to quantify an expansion?

There is a great deal of theoretical work, especially on the Lieb-Liniger case (~100 papers; Stringari, Caux-Konik, Gangardt,…). Three time scales:

Short time: initial transient, which we ignore

Intermediate time: (becomes infinite if reservoirs are infinite)

$\rho(x)$ or $\rho_E(x)$

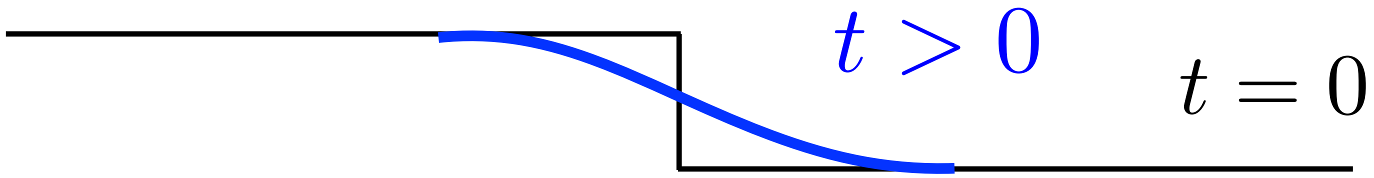


In a ballistic (nonzero Drude weight system), the first moment increases quadratically in time.

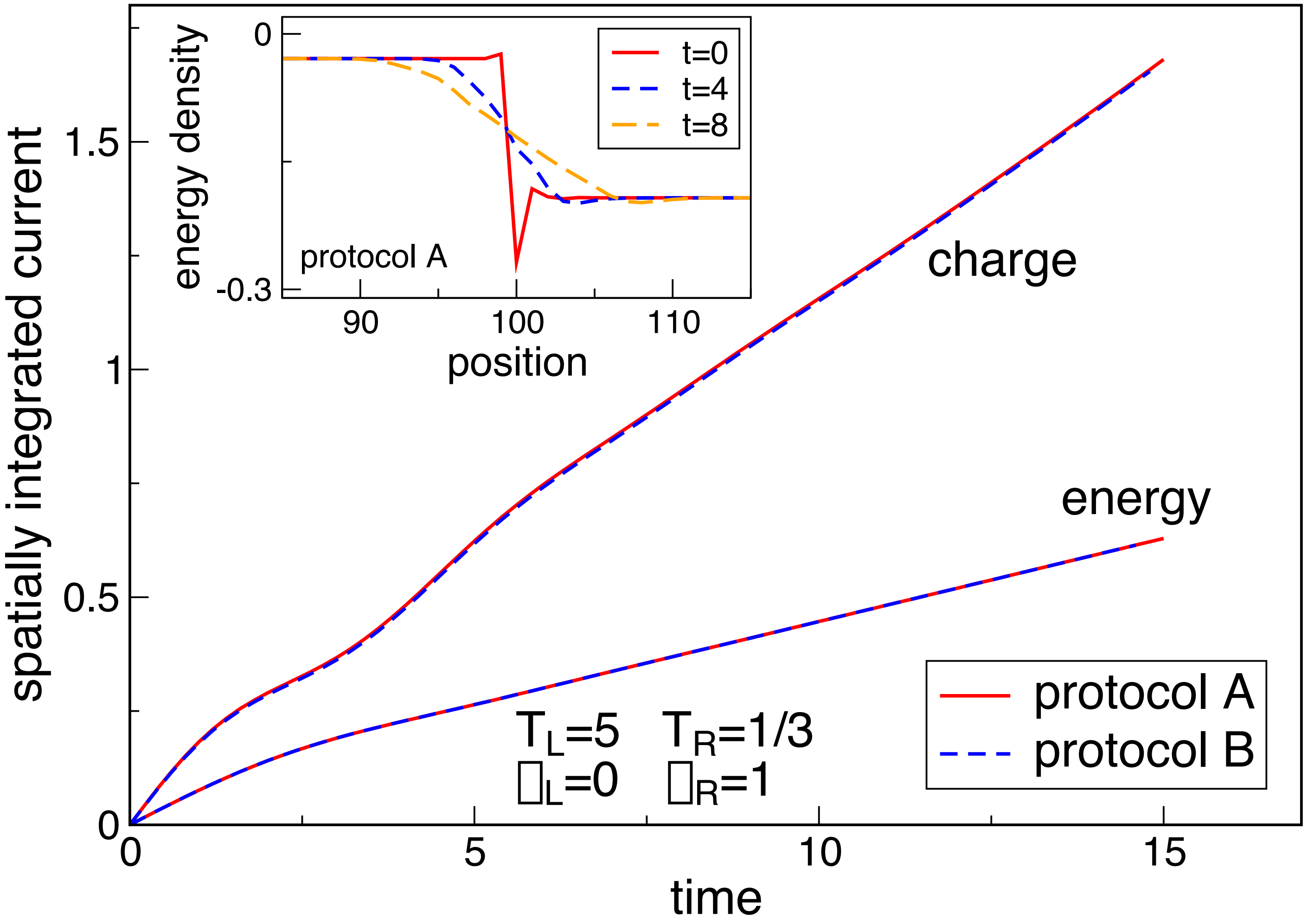
Long time: long-time expansion into vacuum can be analyzed relatively simply in BA because asymptotic density is zero.

How to quantify an expansion?

$\rho(x)$ or $\rho_E(x)$



In a ballistic (nonzero Drude weight system), the first moment increases quadratically in time.



How to quantify an expansion?

At $t = 0$, prepare two leads at (μ_1, T_1) and (μ_2, T_2). The initial state on the boundary between the two leads does not matter after some initial transient. We can quantify the expansion by the time dependence of the first moment of particle density (or similarly for energy)

$$M_1 = \int_{-\infty}^{\infty} n(x) x \, dx. \tag{1}$$

The continuity equation relates density and current:

$$\partial_t n + \partial_x j = 0. \tag{2}$$

Now

$$\partial_t M_1 = \int_{-\infty}^{\infty} x \partial_t n \, dx = -\int_{-\infty}^{\infty} x \partial_x j \, dx = \int_{-\infty}^{\infty} j \, dx, \tag{3}$$

where in the integration by parts we have assumed $j(x)$ vanishes rapidly at $x = \pm\infty$. We will make considerable use of the fact that in many problems of interest

$$\left[\int_{-\infty}^{\infty} j \, dx, H \right] = 0. \tag{4}$$

Now go back to basic ideas of equilibration…

Linear and non-linear response: point current

For the final *H* a homogeneous integrable model, there is numerically a "generalized Stefan-Boltzmann law" to high accuracy (to be defined in a moment), which led us to the idea that this picture can be made exact for expansions.

For final *H* homogeneous and non-integrable, we do not observe a steady state. We believe that the temperature gradient is decreasing and Fourier's law is setting in, but cannot access very long times.

For final H inhomogeneous, there can be a steady state if the leads are integrable and *J* is a function of both temperatures jointly.

We can see the onset of the nontrivial power-laws in tunneling between Luttinger liquids as temperature is lowered.

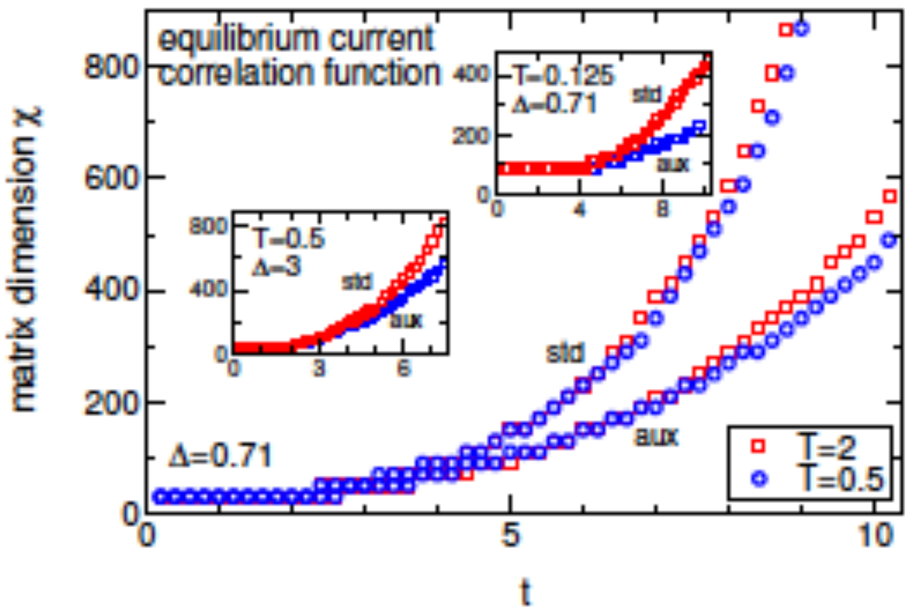
One methodology slide (C. Karrasch)
Time-dependent DMRG at nonzero T

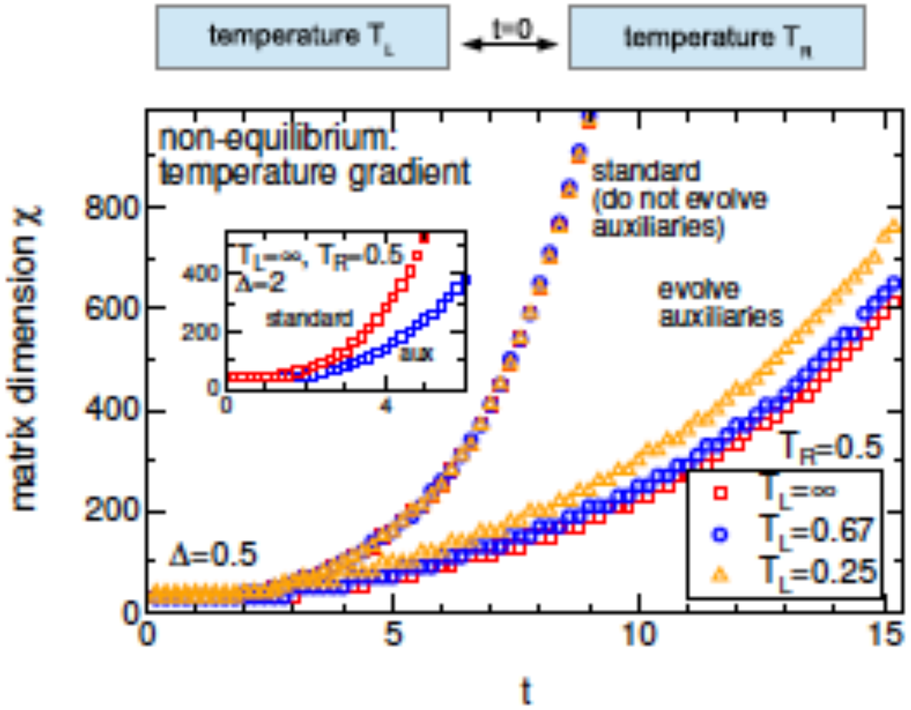
Finite Temperature: Improvement

- slower increase of $\chi \Rightarrow$ access longer time scales
- how much longer? depends on model, temperature, ...

  example: spinless fermions, interaction Δ

correlation function $\langle J(t)J\rangle_\beta$





(CK, Bardarson, Moore '12 '13)
(Barthel, Schollwöck, Sachdev '12)
(Barthel '13)

Alternative hydrodynamics: from more conservation laws

Energy transport in XXZ is special: because integrated energy current commutes with the Hamiltonian, we expect an additional continuity equation

$$\partial_t \rho_E + \partial_x j_E = 0, \quad \partial_t j_E + \partial_x Q_4 = 0$$

This is familiar from Lorentz-invariant models (cf. Bernard-Doyon): the energy current is itself a conserved density, by symmetry of the stress-energy tensor.

In other words, energy transport in XXZ is like that in a Lorentz-invariant model, and expansion occurs according to a expansion potential or generalized pressure Q4. (“cyclic law”)

The “q-boson” model is a lattice example of similar physics for *charge* current rather than energy current.

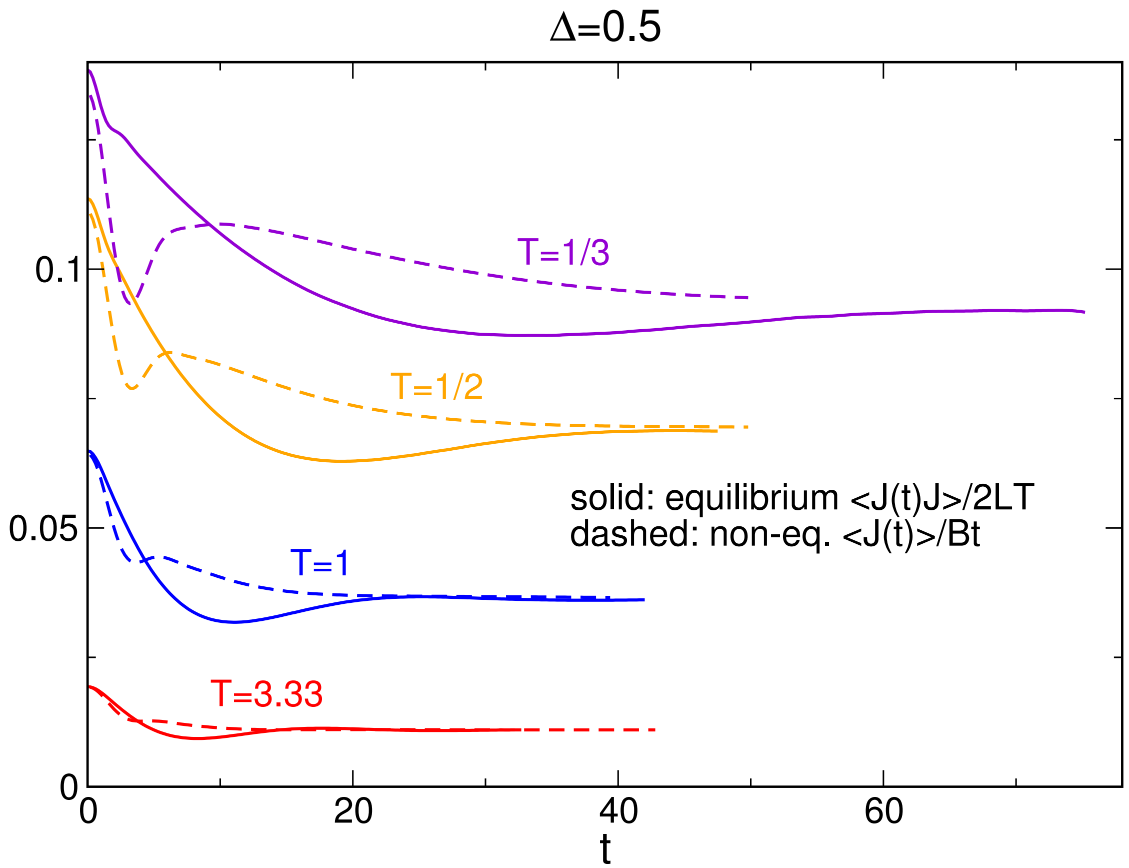
Linear response: Drude weight

The cyclic law means that linear-response is enough to predict non-equilibrium. The increase of the moment at linear-response can be related to the Drude weight: focusing for the moment on energy current and a purely thermal gradient, we find

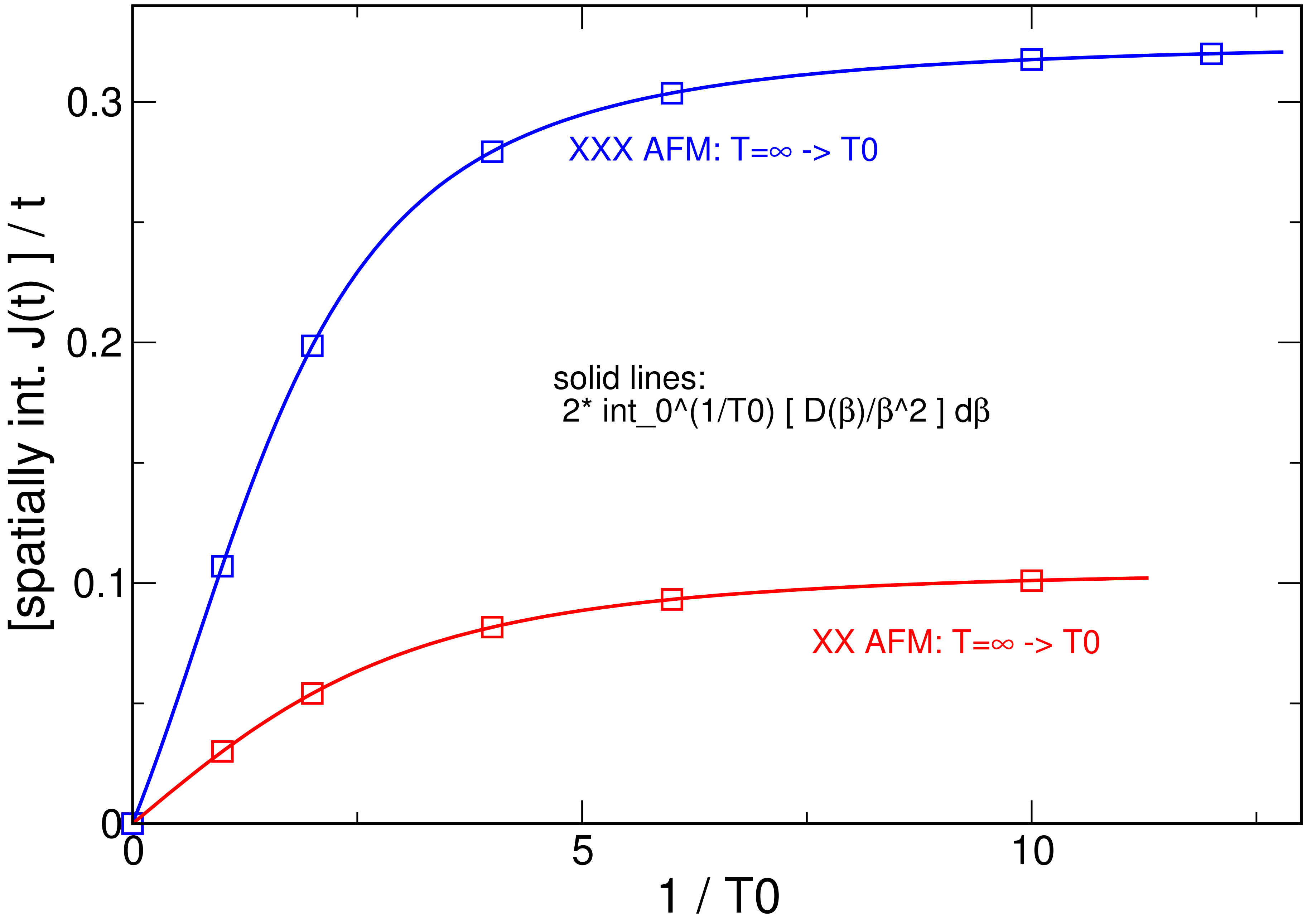
$$\partial_t^2 M_1^{\mathrm{th}} = \partial_t \int_{-\infty}^{\infty} j\,dx = G^{\mathrm{th}} \times (\Delta T). \tag{1}$$

This can be checked numerically by comparing the rate of expansion to the thermal Drude weight of the XXZ model computed by Klümper and Sakai.

Actually this connection exists in LR even without current conservation: example is *charge* current in XXZ model



Combine cyclicity with Drude weight: Exact far-from-equilibrium energy expansion in XXZ



Comparison is rate of increase of energy current versus temperature integral of Drude weight

Recent progress

The above is a very specialized trick to get some exact results for one model. Can one develop a more general approach for hydrodynamics in integrable models?

Yes: recent work on (almost certainly) asymptotically exact solutions for this initial condition by

Castro-Alvaredo/Doyon/Yoshimura, PRX 2016 (Lieb-Liniger)
Bertini/Collura/De Nardis/Fagotti, PRL 2016 (XXZ)

1. Key steps of approach (in one language)

Physical picture of kinetic theory (Boltzmann equation): same spirit as El and Kamchatnov, PRL 2005

2. Does it pass XXZ numerical comparisons that previous similar ansatzes failed?

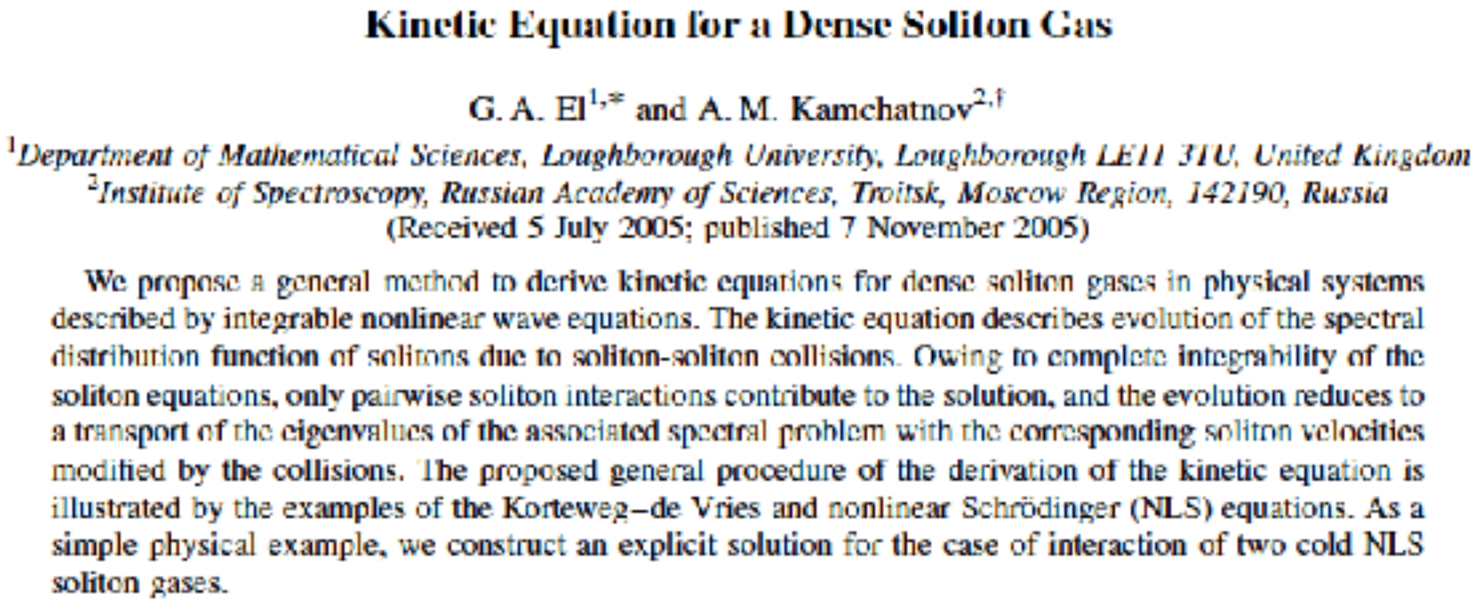
Our starting point: think of particles in an integrable model as streaming (with self-consistent velocity) but not colliding

“Bethe-Boltzmann equation”

$$\partial_t \rho(\lambda, x) + \partial_x \left(v(\{\rho\}\rho) \right) = 0.$$

No collision term since quasiparticles retain their identity; however, they modify each other’s velocities via phase shifts

This type of equation was written down in various older contexts: I think the most relevant for the models here is



Kinetic Equation for a Dense Soliton Gas

G. A. El[1,*] and A. M. Kamchatnov[2,†]

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(Received 5 July 2005; published 7 November 2005)

We propose a general method to derive kinetic equations for dense soliton gases in physical systems described by integrable nonlinear wave equations. The kinetic equation describes evolution of the spectral distribution function of solitons due to soliton-soliton collisions. Owing to complete integrability of the soliton equations, only pairwise soliton interactions contribute to the solution, and the evolution reduces to a transport of the eigenvalues of the associated spectral problem with the corresponding soliton velocities modified by the collisions. The proposed general procedure of the derivation of the kinetic equation is illustrated by the examples of the Korteweg–de Vries and nonlinear Schrödinger (NLS) equations. As a simple physical example, we construct an explicit solution for the case of interaction of two cold NLS soliton gases.

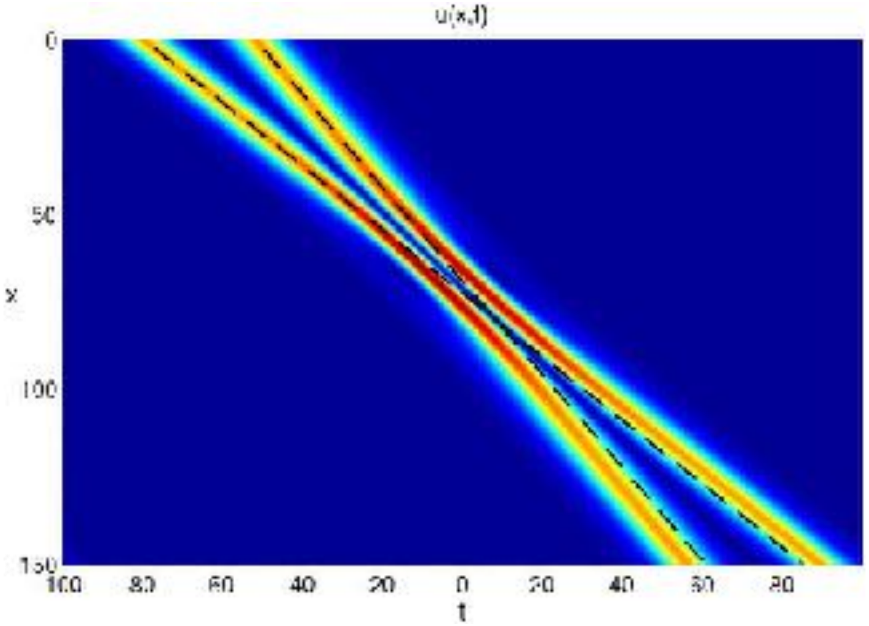
Why Boltzmann equation gets modified in (classical or quantum) integrable systems

Solitons/particles pass through each other even in dense system; no randomization of momentum and no collision term.

However, there is an interaction:

Classical

Solitons delay each other



so velocity depends on other solitons at spacetime point

Quantum

Phase shift from Bethe equations

but semiclassically an energy-dependent phase shift is also just a time delay (Wigner)

$$\tau = 2\hbar \frac{d\delta}{dE}$$

This derives the equation directly for NLS and KdV solitons as a kinetic theory;
An alternate route is via hydrodynamical equations (cf. Doyon talk)

Different integrable models just differ in the velocity form: three examples are

El and Kamchatnov for NLS solitons (dense generalization of Zakharov 1971):

$$s(\alpha,\gamma) = -4\alpha + \frac{1}{2\gamma}\int_{-\infty}^{\infty}\int_{0}^{\infty} \ln\left|\frac{\lambda-\bar{\mu}}{\lambda-\mu}\right|^{2} f(\xi,\eta) \times [s(\alpha,\gamma) - s(\xi,\eta)]\,d\xi\,d\eta.$$

Castro-Alvaredo/Doyon/Yoshimura for Lieb-Liniger

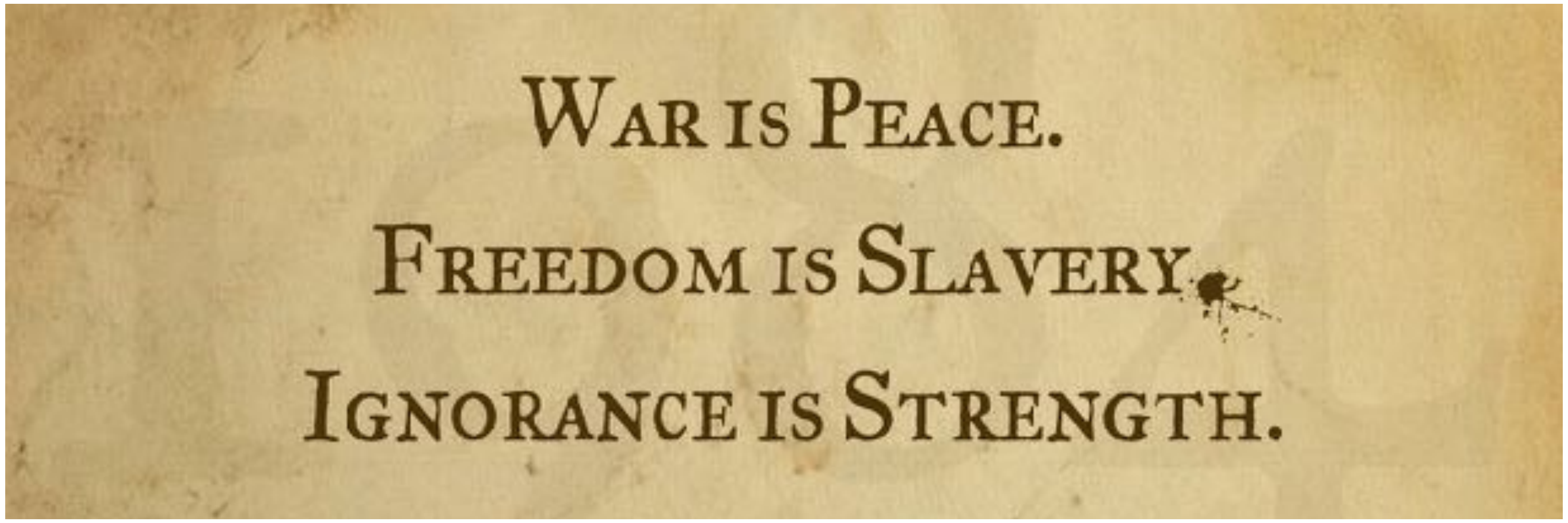
Bertini et al. for XXZ

arXiv:1605.09790 [pdf, other]
Transport in out-of-equilibrium XXZ chains: exact profiles of charges and currents
Bruno Bertini, Mario Collura, Jacopo De Nardis, Maurizio Fagotti

How do kinetic theory (Boltzmann equation) and
hydrodynamics (Euler equations) give the same description?

Equivalence is a little surprising, esp. in XXZ.

From *1984*, by Orwell:



War is Peace.
Freedom is Slavery.
Ignorance is Strength.

For integrable models:
(to at least as good an approximation as above)

TBA is GGE
Kinetic theory is hydrodynamics
One (functional) equation is an infinite hierarchy

Integrable hydrodynamics

Simplest case is Bose gas (Lieb-Liniger; Yang and Yang)
GGE = Generalized Gibbs Ensemble = include an infinite number of conservation laws:

$$\begin{aligned}
\int \rho(k,x,t)\,dk &= n(x,t) \\
\int k\rho(k,x,t) &= mv(x,t) \\
\int k^2\rho(k,x,t) &= 2m\epsilon(x,t) \\
&\vdots \\
\int k^n\rho(k,x,t)
\end{aligned}$$

Kinetic theory: might as well work with $\rho(k,x,t)$

instead of its moments.

GGE (conserved quantities) is equivalent to distribution function, rather than containing less information.

Somewhat surprising for XXZ, where the charges are quite complicated; somehow Takahashi's old TBA and Bertini et al. backflow leads to Drude weight, i.e., it "knows about" the deep quasilocal charges.

Summary so far

Normal fluid:
Initial state → Local equilibrium → Hydrodynamics

Integrable fluid:
Initial state → Local GGE → Boltzmann/hydrodynamics

So, for non-local-GGE initial conditions, still need to solve difficult "quench" problem, at least locally.

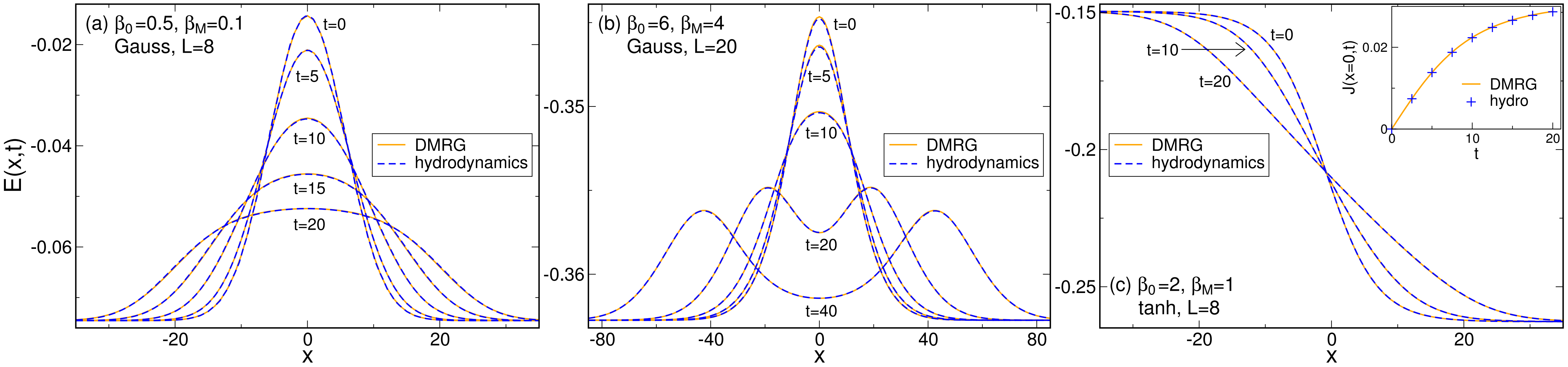
Two-reservoir problem already solved in 2016 papers: solution is function of one variable (x/t).

Let's look for full (x,t) solutions: are quantum dynamics really describable by these classical particle equations?

Mathematical properties of solutions ("semi-Hamiltonian structure"): Bulchandani, 2017, arXiv, as for NLS

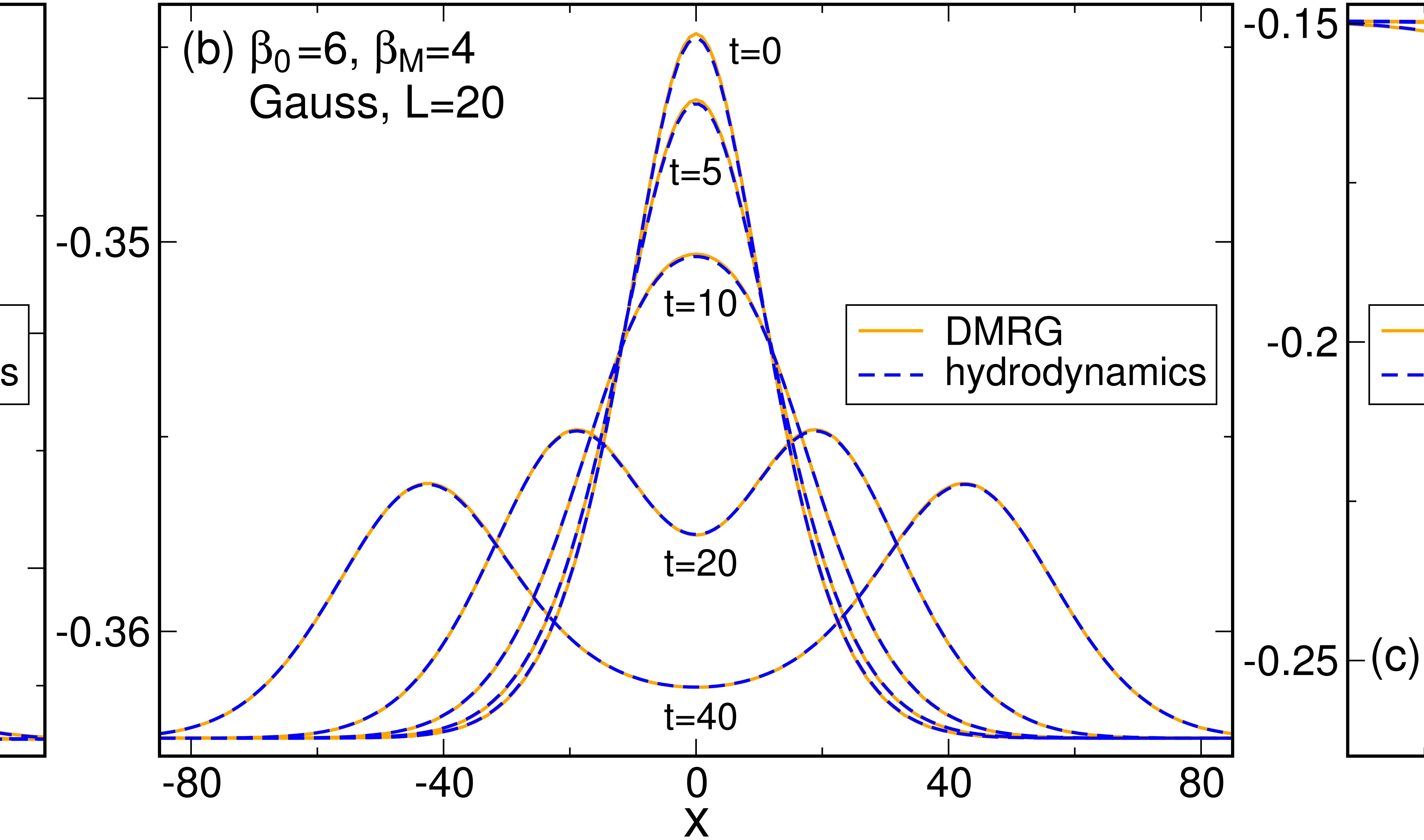
Take XXZ in zero magnetic field. Make a spatial variation of initial temperature. Watch the energy spread out in time.

Note: nonzero temperature is required for coarse-graining time to be finite, according to basic principle that systems can't relax faster than hbar/kT.
(Hence more physically generic than T=0 or Bethe-Bethe comparisons.)



These are comparisons for interacting spinless fermions (XXZ) between backwards Euler solution of Bethe-Boltzmann and microscopic DMRG simulations.
(figure from "Solvable quantum hydrodynamics", V. Bulchandani, R. Vasseur, C. Karrasch, and JEM, arXiv April 2017)

Zoom in!


(b) β_0=6, β_M=4
Gauss, L=20
t=0
t=5
t=10
t=20
t=40
DMRG
hydrodynamics
-0.35
-0.36
-80
-40
0
40
80
x
s
-0.15
-0.2
-0.25
(c)

Summary

1. The two-reservoir quench is now pretty well understood in various approaches.

2. For that and more general problems, it is useful to think about the Boltzmann equation for quasiparticles, whose only interaction is via delays (Bethe ansatz phase shifts), a.k.a. GHD.

3. This type of equation has a long history in classical integrable systems (El and Kamchatnov). Even for fairly small quantum systems, it can be remarkably successful in comparison to microscopics.

What is left out: for other initial states, need to solve initial GGE problem;
possible singularities;
corrections from integrability-breaking terms;
non-ballistic behavior (e.g., in gapped XXZ regions)
subleading terms (e.g., Schwarzian in Sotiriadis-Cardy)

Brief intro to quantum hydrodynamics above 1D: near-equilibrium

An example of recent progress on a long-standing question:

Are there intrinisic limits on how fast a system can relax to equilibrium?

Related to conductivity via the Kubo formula: how rapidly does the current-current correlation decay in time?

Also related to existence of “hydrodynamical” regimes of electron transport where quasiparticle scattering is not the right picture.

Some past formulations:

Mott-Ioffe-Regel: mean free path must be at least the lattice spacing

Sachdev: $\tau \geq \frac{\hbar}{k_B T}$ $\quad$ $\text{quantum } e^{-iHt/\hbar} \leftrightarrow e^{-H/(k_B T)} \text{ thermal}$

Kovtun-Son-Starinets: the viscosity is bounded below*

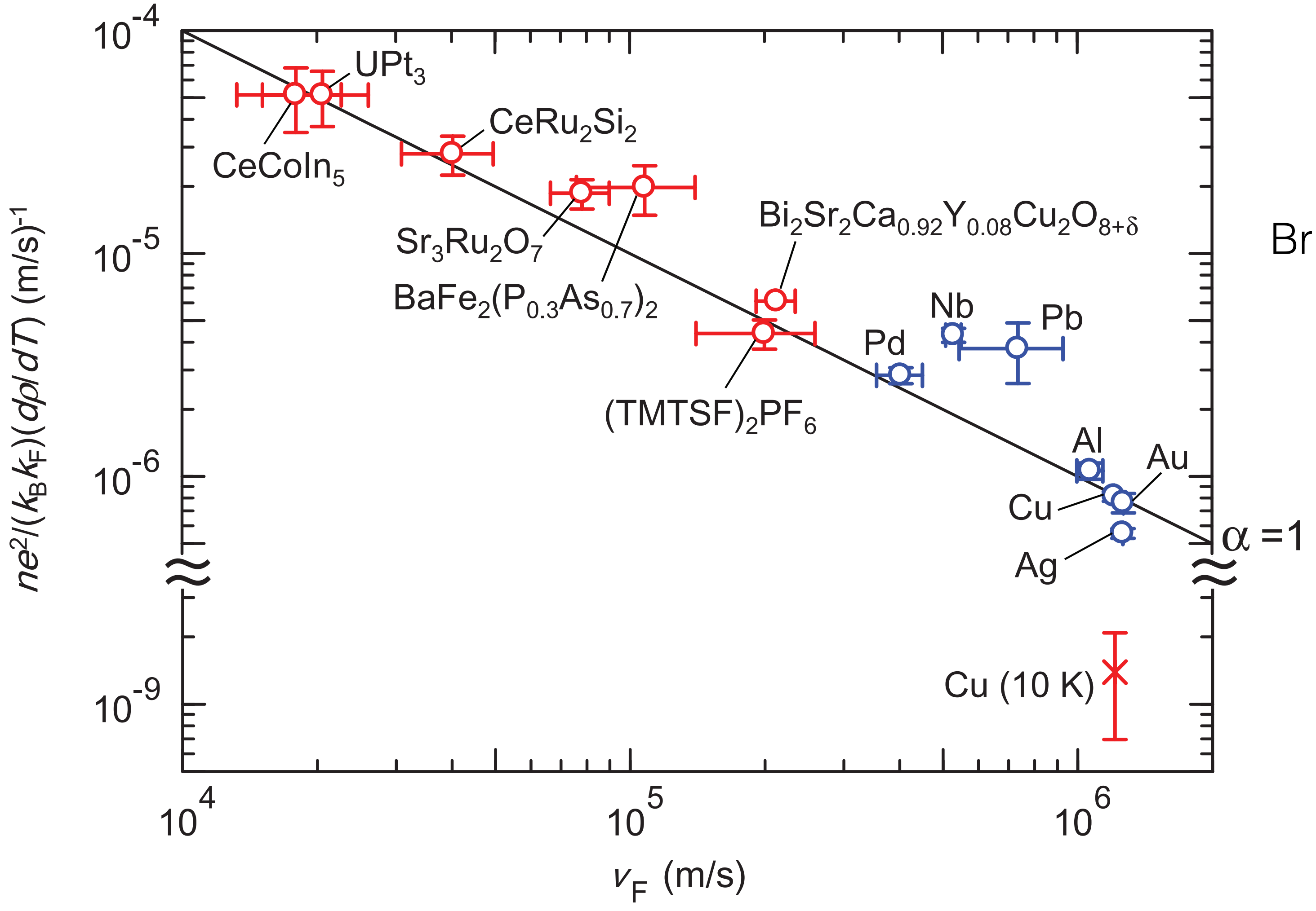
$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

Hartnoll: reinterpret viscosity bound as a lower limit on diffusion constant

Near-equilibrium

Why these bounds matter: nature seems to contain such behavior

$$\tau \geq \frac{\hbar}{k_B T}$$



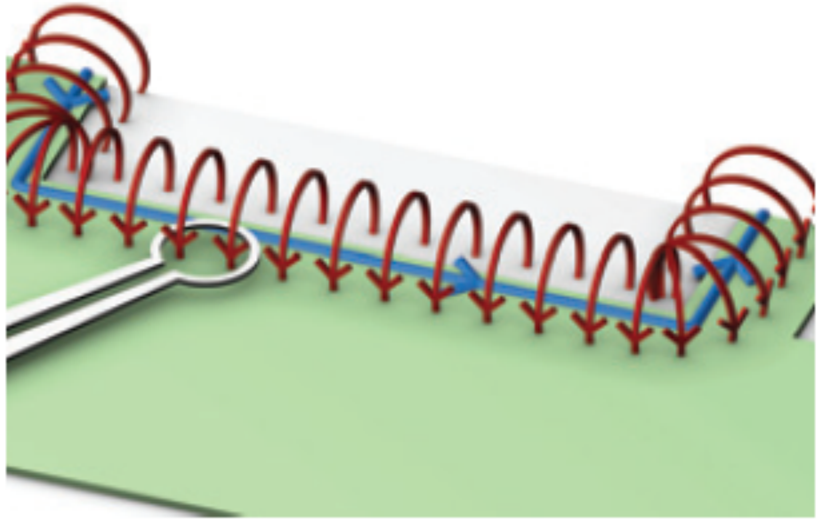
Bruin et al. (Mackenzie), Science 2013

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

is satisfied (I believe) in all experimental liquids, and within ~10 of saturation in helium and QGP, but there exist violations in exotic theories

Hydrodynamics of electrons

In materials that are very clean, momentum relaxation may take a relatively long time. It might be better to view electrons as a *fluid* rather than as independently scattering quasiparticles.



Nowack et al., Nat. Mat. 2013

Solid-state electrons where fluid properties measured

2DEGs (Molenkamp & others, 1990s)
Graphene (P. Kim; A. Geim)
Layered crystals (A. Mackenzie)
…

Near-equilibrium

New work, originally motivated by AdS but derivable without gravity dual:
(Kitaev, Maldacena-Shenker-Stanford, 2015):

The Lyapunov exponent for short-time onset of chaos is bounded

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar}$$

Other ways quantum mechanics modifies hydrodynamics:

“Hall viscosity” in topological states: (Avron; Read; Gurarie…) stress tensor is

$$T_{ij} = p\delta_{ij} - \lambda_{ijkl}\xi_{kl} - \eta_{ijkl}\dot{\xi}_{kl}$$

and T-breaking allows an odd contribution $\eta^{(A)}_{ijkl} = -\eta^{(A)}_{klij}$

What is Hall viscosity in T-breaking gapless systems?
Allowed by symmetry.

Hydrodynamics of electrons

What makes electron fluids different from classical fluids?

In 2D and 3D, can induce broken T by a magnetic field
and have a new kind of viscosity, "Hall viscosity"

Is significant, and could be observable, in simple *metals*:
T. Scaffidi, N. Nandi, B. Schmidt, AP Mackenzie, JEM, PRL 17

In the quantum Hall regime there are two contributions in the q^2 correction to Hall conductance

$$\sigma_{xy}(q) = \sigma_{xy}(0) + O(q^2)$$

that are comparable (Hoyos-Son): one from Hall viscosity and one from (inverse) internal compressibility. In a metal, the internal compressibility part is small and the Hall viscosity follows from a Boltzmann calculation.