Exact instanton summation in O(3) NLSM

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Outline

- Motivation
  - Trans-Series expansions in QM and QFT
  - $O(3)$-sigma model
  - Bukhvostov-Lipatov (BL) model and an exact instanton counting in $O(3)$-sigma model
- Some aspects of the BL model
  - Matsubara perturbation theory
  - Bethe Ansatz
  - NLIE for BL model
  - Conformal Perturbation Theory
- Quantum/Classical duality
  - Fateev model. Conformal Perturbation Theory.
  - Quantum BL model v.s. Classical integrable sinh-Gordon PDE
  - NLIE for $O(3)$ model

Trans-Series expansions in QM and QFT

- Perturbation theory $\longrightarrow$ divergent (asymptotic) series
- Non-perturbative effects $\longrightarrow$ Trans-Series expansions

$$F(g) \asymp \sum_{n\geq 0} c_n^{(0)} g^n + \sum_i \mathrm{e}^{-S_i/g} \sum_{n\geq 0} c_n^{(i)} g^n$$

  $\mathrm{e}^{-S_i/g}$- the multi-instanton corrections, or other type of non-trivial saddle points

- Resurgence (Écalle'80; Stokes'1850)
  Unification of perturbation theory and non-perturbative physics
- Recent developments
  - QM (Delabaere,Pham'99; Jentschura,Zinn-Justin'04, ...)
  - Matrix models (Mariño'14,...)
  - SUSY localizable field theories (Aniceto, Schiappa, Vonk'12,...)
  - Certain asymptotically free QFT (Dunne, Ünsal'13, ...)

$O(3)$ NLSM

- $$\mathcal{A}[\boldsymbol{n}] = \frac{1}{2f^2} \int \mathrm{d}^2x \, \left(\partial_\mu \boldsymbol{n}(x)\right)^2 , \qquad \boldsymbol{n} \in \mathbb{S}^2$$

  Euclidean Green functions: $\langle \mathcal{O}(\boldsymbol{n}) \rangle = \frac{1}{Z} \int \mathcal{D}\boldsymbol{n} \; \mathcal{O}(\boldsymbol{n}) \; \mathrm{e}^{-\mathcal{A}[\boldsymbol{n}]}$

- Dimensional transmutation (Polyakov'75)

$$\Lambda \, g_0^{-1} \mathrm{e}^{-\frac{1}{g_0}} \sim M \, , \qquad g_0 = \frac{f^2}{2\pi}$$

- Finite volume $x \in [0, R]$. Quasi-periodic BC

$$n_3(t, x{+}R) = n_3(t, x) \, , \quad n_\pm(t, x{+}R) = \mathrm{e}^{\pm 2\pi \mathrm{i} k} \, n_\pm(t, x) \;\; (n^\pm = n_1 \pm \mathrm{i} n_2)$$

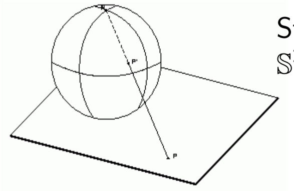
- Vacuum energy $E = E(r, k)$ $(r = MR)$

$$RE(r, k) \asymp a_0(k) + a_1(k) g(r) + a_2(k) \; g^2(r) + \ldots$$

$$g^{-1} \; \mathrm{e}^{-\frac{1}{g}} = r$$

Instantons in $O(3)$ NLSM

- Multi-instanton solutions (Belavin-Polyakov'75):



Stereographic projection of $\boldsymbol{n} \in \mathbb{S}^2$:

$$w = \frac{n_1 + \mathrm{i}\, n_2}{1 + n_3}$$

$$\partial_{\bar{z}} w = 0 \quad (z = x_0 + \mathrm{i} x_1) : \qquad w(z) = c\, \frac{(z - a_1) \cdots (z - a_q)}{(z - b_1) \cdots (z - b_q)}$$

Instanton contributions in Euclidean Green functions

- Fateev-Frolov-Schwarz'79:

$$w(z) = c\ \frac{(z-a_1)\cdots(z-a_q)}{(z-b_1)\cdots(z-b_q)}$$

$$Z_q[\mathcal{O}] = \frac{M^q}{(q!)^2}\int \mathcal{O}(a,b,c)\,\mathrm{e}^{-\epsilon_q(a,b)}\ \frac{\mathrm{d}^2 c}{\pi(1+|c|^2)^2}\prod_{j=1}^{q}\mathrm{d}^2 a_j \mathrm{d}^2 b_j$$

$$\epsilon_q(a,b) = -\sum_{i<j}^{q}\log|a_i - a_j|^2 - \sum_{i<j}^{q}\log|b_i - b_j|^2 + \sum_{i,j}^{q}\log|a_i - b_j|^2\ .$$

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- In the Gaussian approximation the partition function coincides with the partition function of the 2D neutral Coulomb system:

$$Z = \sum_q \frac{M^q}{(q!)^2} \int \mathrm{e}^{-\frac{\epsilon_q(a,b)}{T}} \prod_{j=1}^{q} \mathrm{d}^2 a_j \mathrm{d}^2 b_j$$

where $M \propto f^{-4}\ \mathrm{e}^{-\frac{2\pi}{f^2}}$ – fugacity and $T = 1$

"Small-instanton" divergency

- The divergency already appears in the one instanton sector

$$Z_1[\mathcal{O}] = M \int \mathcal{O}(a,b,c) \;\; \frac{\mathrm{d}^2 c}{\pi(1+|c|^2)^2} \; \frac{\mathrm{d}^2 a \mathrm{d}^2 b}{|a-b|^2}$$

  and comes from the singular contribution from the region of the instanton moduli space with $|a-b| \to 0$.
- The standard lattice description of the $O(3)$ sigma model has problems – for example, the lattice topological susceptibility does not obey naive scaling laws.
- Lüscher'82 has shown that this is because of the field configurations such as the winding of the $O(3)$-field around plaquettes of lattice size, giving rise to spurious contribution to quantities related to the zero point energy.

Free fermions

- The partition function of 2D neutral Coulomb system in the grand canonical ensemble:

$$Z = \sum_q \frac{M^q}{(q!)^2} \int \mathrm{e}^{-\frac{\epsilon_q(a,b)}{T}} \prod_{j=1}^{q} \mathrm{d}^2 a_j \mathrm{d}^2 b_j$$

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where $T = 1$.

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where $T = 1$.

- The complete instanton contribution in the partition function can be represented as the partition function of the free Dirac field

$$Z = \int \prod_x \mathrm{d}\bar{\psi}(x)\mathrm{d}\psi(x)\ \exp\left(-\int \mathrm{d}^2 x\, (\bar{\psi}\gamma_a\partial_a\psi + M\,\bar{\psi}\psi)\right)$$

Bukhvostov-Lipatov model

- Weak "i-a" interaction $\Rightarrow$ Bukhvostov-Lipatov (BL) model (1980)

$$\mathcal{L} = \sum_{\sigma=\pm} \bar{\psi}_\sigma \left(\mathrm{i}\gamma^\mu \partial_\mu - M \right) \psi_\sigma - g \left(\bar{\psi}_+ \gamma^\mu \psi_+ \right) \left(\bar{\psi}_- \gamma_\mu \psi_- \right)$$

- Renormalization counterterms

$$\mathcal{L}_{\mathrm{BL}} = \mathcal{L} - \sum_{\sigma=\pm} \left(\delta M \, \bar{\psi}_\sigma \psi_\sigma + \frac{g_1}{2} \left(\bar{\psi}_\sigma \gamma^\mu \psi_\sigma \right)^2 \right)$$

$$\frac{M}{M_0} = \left(\frac{M}{\Lambda_{\mathrm{UV}}} \right)^\nu \qquad (M_0 = M + \delta M)$$

Two parameter families (g, ν).

- Integrable BL model: $\nu = 0$, i.e., Finite mass renormalization for special g and g_1. Only vacuum energy diverges (small instanton divergency).

Bosonization

- Bosonic version of the BL-model

$$\tilde{\mathcal{L}}_{\mathrm{BL}} = \frac{1}{16\pi}\left((\partial\varphi_1)^2 + (\partial\varphi_2)^2 \right) + 4\mu \cos\left(\frac{\sqrt{a_1}}{2}\varphi_1\right) \cos\left(\frac{\sqrt{a_2}}{2}\varphi_2\right)$$

$$\nu = \frac{1}{2}\left(a_1 + a_2 - 2\right), \qquad \frac{g}{\pi} = \frac{a_2 - a_1}{2a_1 a_2}, \qquad \mu \propto M^{1-\nu} = M_0\, \Lambda_{\mathrm{UV}}^{-\nu}$$

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- Integrable case $\nu = 0$: $a_1 = 1-\delta, \qquad a_2 = 1+\delta$

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- For $\delta > 1$ the model

$$\tilde{\mathcal{L}}_{\rm BL} = \tfrac{1}{16\pi}\ \big((\partial\varphi_1)^2 + (\partial\varphi_2)^2 \big) + 4\mu\ \cosh\big(\tfrac{\sqrt{\delta-1}}{2}\varphi_1\big) \cos\big(\tfrac{\sqrt{\delta+1}}{2}\varphi_2\big)$$

is equivalent (Al. Zamolodchikov'95) to "sausage model" (Fateev, Onofri, Al. Zamolodchikov'93)

$$\mathcal{A}_{\rm sausage}[\boldsymbol{n}] = \frac{1}{2f^2}\int {\rm d}^2x\ \frac{\big(\partial_\mu \boldsymbol{n}(x)\big)^2}{1-\lambda^2 n_3^2/2f^4} \qquad \Big(\lambda = \frac{2\pi}{1-\delta}\Big)$$

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- **BL-model in the strong coupling regime $\delta \to \infty$ describes $O(3)$ sigma model. The instanton counting is exact!**

Perturbation theory

$$\mathcal{L} = \sum_{\sigma} \left[\bar{\psi}_{\sigma} \left(\mathrm{i}\gamma^{\mu}\partial_{\mu} - (M + \delta M) \right) \psi_{\sigma} - \frac{g_1}{2} \left(\bar{\psi}_{\sigma}\gamma^{\mu}\psi_{\sigma} \right)^2 \right] - g \left(\bar{\psi}_{+}\gamma^{\mu}\psi_{+} \right) \left(\bar{\psi}_{-}\gamma_{\mu}\psi_{-} \right)$$

Boundary conditions

$$\psi_{\pm}(x^0, x^1 + R) = -\mathrm{e}^{2\pi \mathrm{i} k_{\pm}}\, \psi_{\pm}(x^0, x^1)$$

Propagator (in x-space)

$$\boldsymbol{S}_{\sigma}(\mathbf{x}) = \left(M - \gamma^{a}\partial_{a} \right)\ G_{\sigma}(\mathbf{x})$$

$$G_{\sigma}(\mathbf{x}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^{n}\, \mathrm{e}^{2\pi \mathrm{i} n k_{\sigma}}\ K_0\left(|w - \mathrm{i}\, nr| \right), \qquad w = M\,(x^0 + \mathrm{i}\, x^1)$$

Vacuum energy as function of $r = MR$

$$\mathfrak{F}(r) = \frac{R}{\pi}\,(E_{\mathbf{k}} - R\,\mathcal{E})\,, \quad \mathfrak{F}(0) = -\frac{c_{\mathbf{k}}}{6}\,, \quad \mathfrak{F}(r)|_{r\to\infty} = 0,$$

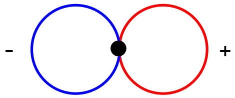
$$c_{\mathbf{k}} = 2 - 6(1-\delta)k_1^2 - 6(1+\delta)k_2^2\,, \quad k_{\pm} = (k_1 \pm k_2)/2$$

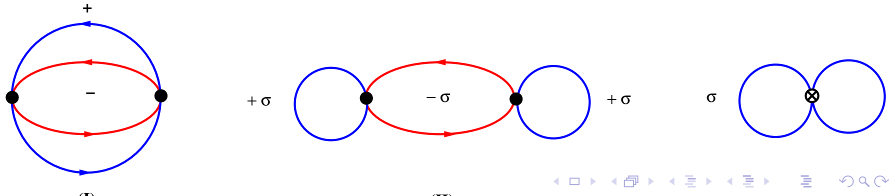
Perturbative expansion

$$\mathfrak{F} = \mathfrak{F}_0 + \mathfrak{F}_1\,\delta + \mathfrak{F}_2\,\delta^2 + \dots\,, \quad \mathfrak{F}_0 = \mathfrak{f}(r, k_+) + \mathfrak{f}(r, k_-)$$

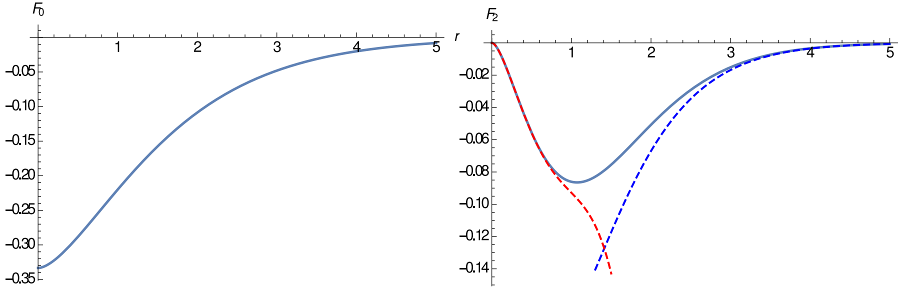
Free energy of the 1D relativistic Fermi gas:

$$\mathfrak{f}(r,k) = -\frac{r}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{d}\theta\,\cosh\theta\,\log\left[\left(1+\mathrm{e}^{2\pi\mathrm{i}k}\mathrm{e}^{-r\cosh\theta}\right)\left(1+\mathrm{e}^{-2\pi\mathrm{i}k}\mathrm{e}^{-r\cosh\theta}\right)\right]$$





Results of (Matsubara) perturbation theory (Bazhanov, Lukyanov, Runov'15)



Leading large r asymptotics $\mathfrak{F}_2 = O(r\mathrm{e}^{-2r})$

$$\mathfrak{F}(r) = -\frac{4}{\pi^2}\ \cos(\pi k_1)\ \cos(\pi k_2)\ r\,K_1(r) + o(\mathrm{e}^{-r})\ .$$

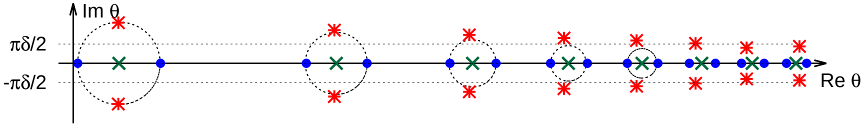
Bethe Ansatz (BL 1980)

$$-1 = e^{2\pi i(p_1-p_2)}\, e^{i\mathcal{M}R\sinh\theta_j} \prod_{\ell} \frac{\sinh\left(\theta_j - u_\ell - \frac{1}{2}\, i\pi\delta\right)}{\sinh\left(\theta_j - u_\ell + \frac{1}{2}\, i\pi\delta\right)}$$

$$-1 = e^{-4\pi i p_1} \prod_{\ell'} \frac{\sinh\left(u_\ell - u_{\ell'} + i\pi\delta\right)}{\sinh\left(u_\ell - u_{\ell'} - i\pi\delta\right)} \prod_{j} \frac{\sinh\left(u_\ell - \theta_j - \frac{1}{2}\, i\pi\delta\right)}{\sinh\left(u_\ell - \theta_j + \frac{1}{2}\, i\pi\delta\right)},$$

$j = 1, \ldots, 2N,\ \ell = 1, \ldots, N$

$$E_N = -\mathcal{M} \sum_j \cosh\theta_j\,.$$



$$\mathfrak{F}(r) = -\frac{c_{\mathbf{k}}}{6} + \lim_{\substack{N\to\infty \\ r-\text{fixed}}} \left(\frac{RE_N}{\pi} - \epsilon_2\, N^2 - \epsilon_0\, r^2\, \Big(\log(4N/r) + C \Big) \right) ,$$

$$c_{\mathbf{k}} = \sum_{i=1}^{2} \left(1 - 6a_i k_i^2\right) , \qquad \epsilon_2 = -(1+\delta) , \qquad \epsilon_0 = -\tfrac{1}{\pi^2}\, \cos^2\left(\tfrac{\pi\delta}{2}\right)$$

The scaling function $\mathfrak{F}(r, \mathbf{k})$ is expressed through the solution of a system of two Non-Linear Integral Equations (NLIE)

NLIE (Bazhanov, Lukyanov, Runov'15)

- $$\begin{aligned}\varepsilon_\sigma(\theta) &= r\,\sinh(\theta - \mathrm{i}\chi_\sigma) - 2\pi k_\sigma \\ &+ \sum_{\sigma'=\pm}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,G_{\sigma\sigma'}(\theta-\theta')\,\Im m\Big[\log\big(1+e^{-\mathrm{i}\varepsilon_{\sigma'}(\theta'-\mathrm{i}0)}\big)\Big]\end{aligned}$$

  $\sigma = \pm$, $(\chi_+, \chi_-) = (0,\, \pi a_1/2)$

- $G_{\pm\pm}(\theta) = G_{a_1}(\theta) + G_{a_2}(\theta)$, $\quad G_{\pm\mp}(\theta) = \hat{G}_{a_1}(\theta) - \hat{G}_{a_2}(\theta)$

  $$\begin{aligned}G_a(\theta) &= \int_{-\infty}^{\infty}\mathrm{d}\nu\,\frac{\mathrm{e}^{\mathrm{i}\nu\theta}\,\sinh(\frac{\pi\nu}{2}(1-a))}{2\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi\nu a}{2})} \\ \hat{G}_a(\theta) &= \int_{-\infty}^{\infty}\mathrm{d}\nu\,\frac{\mathrm{e}^{\mathrm{i}\nu\theta}\,\sinh(\frac{\pi\nu}{2})}{2\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi\nu a}{2})}\ .\end{aligned}$$

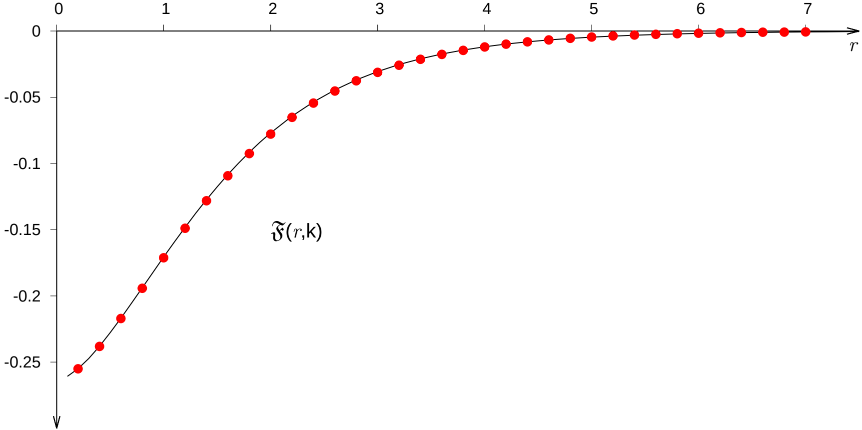
- $\mathfrak{F}(r,\mathbf{k}) = \pm\frac{r}{\pi}\,\Im m\Big[L_+(\pm\mathrm{i}) + \mathrm{e}^{\mp\frac{\mathrm{i}\pi}{2}a_1}\,L_-(\pm\mathrm{i})\Big]$

  $$L_\sigma(\nu) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{\pi}\,\mathrm{e}^{-\mathrm{i}\nu\theta}\,\log\big(1+e^{-\mathrm{i}\varepsilon_\sigma(\theta-\mathrm{i}0)}\big)$$

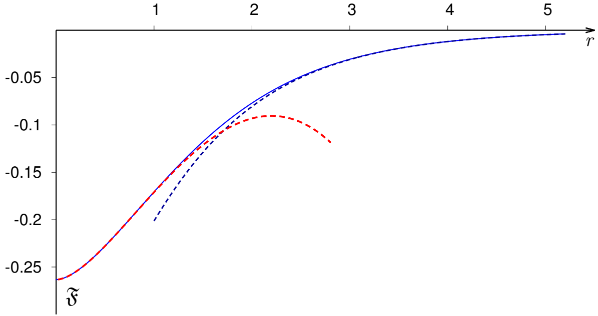
Numerical values obtained from

$$\mathfrak{F}(r) = -\frac{c_{\mathbf{k}}}{6} + \lim_{\substack{N\to\infty \\ r-\text{fixed}}} \left(\frac{RE}{\pi} - \epsilon_2\, N^2 - \epsilon_0\, r^2\, \left(\log(4N/r) + C \right) \right)$$

with $N = 500$ display good agreement (of at least three decimal places) with the more accurate results obtained from the NLIE



- NLIE consistent with the original coordinate BA
- The NLIE results show an excellent agreement with the predictions of renormalized perturbation theory.
- Another form of NLIE was proposed by Saleur'98 (singular for $\delta \to 0$)
- Even though our NLIE look totally different then the Saleur NLIE the resulting expressions for the scaling function are, in fact, exactly the same
- The numerical values for $\mathfrak{F}(r, \mathbf{k})$ show an excellent agreement with the Conformal Perturbation Theory



Conformal Perturbation Theory

- Use Bosonic version of the BL-model

$$\tilde{\mathcal{L}}_{\rm BL}=\tfrac{1}{16\pi}\,\big((\partial\varphi_1)^2+(\partial\varphi_2)^2\,\big)\quad+\quad 4\mu\ \cos\big(\tfrac{\sqrt{1-\delta}}{2}\varphi_1\big)\cos\big(\tfrac{\sqrt{1+\delta}}{2}\varphi_2\big)$$

$$2\pi\mu=M\cos(\pi\delta/2)$$

- In the bosonic formulation, the model is described by the Lagrangian with the potential term is periodic w.r.t. φ_i. Due to this periodicity, the space of states splits on the orthogonal subspaces $\mathcal{H}_{\mathbf{k}}$ characterized by two “quasimomentums” $\mathbf{k}=(k_1,k_2)$,

$$\varphi_i\to\varphi_i+\frac{4\pi}{\sqrt{a_i}}\quad:\qquad|\,\Psi_{\mathbf{k}}\,\rangle\mapsto \mathrm{e}^{2\pi \mathrm{i}k_i}\,|\,\Psi_{\mathbf{k}}\,\rangle\qquad\Big(k_\pm=\frac{1}{2}\,(k_1\pm k_2)\Big)$$

- The neutral (w.r.t. $U(1)\otimes U(1)$) sector of the theory is described by the Bose fields with periodic boundary conditions:

$$\varphi_i(x_1+R,x_2)=\varphi_i(x_1,x_2)\ .$$

- $$\mathfrak{F}(r)=-\tfrac{1}{6}\ c_k-16\,r^2\,\log(r)-\sum_{n=1}^{\infty}e_n(\delta)\,(2r)^{2n}\qquad(r=MR)$$

Fateev model. Symmetric regime.

$$\mathcal{L} = \frac{1}{16\pi}\sum_{i=1}^{3}(\partial\varphi_i)^2$$
$$+ \quad 2\mu\,\left(\mathrm{e}^{\mathrm{i}\alpha_3\varphi_3}\ \cos(\alpha_1\varphi_1+\alpha_2\varphi_2)+\mathrm{e}^{-\mathrm{i}\alpha_3\varphi_3}\ \cos(\alpha_1\varphi_1-\alpha_2\varphi_2)\right)$$

$$\alpha_i = \frac{1}{2}\sqrt{a_i}\ : \qquad a_1+a_2+a_3=2$$

- Unitary regime: $a_1 > 0, \quad a_2 > 0, \quad a_3 = 2 - a_1 - a_2 < 0$
- Symmetric regime: $a_i > 0$
  Stationary states can be chosen to be the Floquet states characterized by $\mathbf{k} = (k_1, k_2, k_3)$:

$$\varphi_i \mapsto \varphi_i + 2\pi/\alpha_i\ : \quad |\,\Psi_{\mathbf{k}}\,\rangle \mapsto \mathrm{e}^{2\pi\mathrm{i}k_i}\,|\,\Psi_{\mathbf{k}}\,\rangle\ , \qquad |\,\Psi_{\mathbf{k}}\,\rangle \in \mathcal{H}_{\mathbf{k}}$$

- k-vacuum energies:

$$E_{\mathbf{k}}|_{R\to 0} \to E_{\mathbf{k}}^{(0)} = -\frac{\pi c_{eff}}{6R}\ , \qquad c_{eff} = 3 - 6\sum_{i=1}^{3} a_i k_i^2$$

Conformal Perturbation Theory (Symmetric regime)

- Rescale the problem to the circle of circumference 2π

$$\mathcal{A} = \mathcal{A}_0 - \mu R \int \mathrm{d}x_2 \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi} \; V$$

$$\begin{aligned} V &= \mathrm{e}^{\mathrm{i}(+\alpha_1\varphi_1+\alpha_2\varphi_2+\alpha_3\varphi_3)} + \mathrm{e}^{\mathrm{i}(+\alpha_1\varphi_1-\alpha_2\varphi_2-\alpha_3\varphi_3)} \\ &+ \mathrm{e}^{\mathrm{i}(-\alpha_1\varphi_1-\alpha_2\varphi_2+\alpha_3\varphi_3)} + \mathrm{e}^{\mathrm{i}(-\alpha_1\varphi_1+\alpha_2\varphi_2-\alpha_3\varphi_3)} \end{aligned}$$

- $E_{\mathbf{k}} - E_{\mathbf{k}}^{(0)} - = -\frac{1}{L} \log \frac{\mathcal{Z}}{\mathcal{Z}_0} = -\frac{\mu R}{L} \int \mathrm{d}x_2 \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi} \langle V \rangle + \ldots$
-

$$\frac{R}{\pi} E_{\mathbf{k}}^{(0)} = -\frac{1}{2} + \sum_{i=1}^{3} a_i k_i^2 - \sum_{n=1}^{\infty} e_n \, (\mu R)^{4n}$$

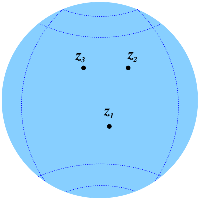
e_n are expressed in terms of *convergent* 2D Coulomb-type integrals.

Conformal Perturbation Theory (Symmetric regime)

- A real regular solution of the Liouville equation

$$\partial_z \partial_{\bar{z}} \eta_{\rm L} - {\rm e}^{2\eta_{\rm L}} = 0$$

on the Riemann sphere except at three points $z = z_i$



$$\eta_{\rm L} \to (a_i|k_i|-1)\,\log|z-z_i|+O(1)$$

$$\eta_{\rm L} \to -4\log|z| + O(1)\,, \quad z \to \infty$$

- Quadratic differential w.r.t. $PSL(2,C)$ on $S^2/\{z_1, z_2, z_3\}$:

$$\mathcal{P}(z) = \frac{(z_3-z_2)^{a_1}(z_1-z_3)^{a_2}(z_2-z_1)^{a_3}}{(z-z_1)^{2-a_1}(z-z_2)^{2-a_2}(z-z_3)^{2-a_3}}\,, \quad a_1+a_2+a_3 = 2$$

-

$$E_{\mathbf{k}} = -\frac{\pi}{6R}\sum_{i=1}^{3}\left(1-6a_i\,k_i^2\right) - \frac{\mu^4 R^3}{64}\int {\rm d}^2 z\,|\mathcal{P}(z)|^2\,{\rm e}^{-2\eta_{\rm L}} + O(R^5)$$

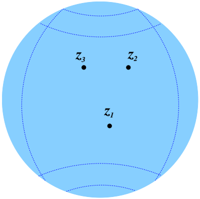
Conformal Perturbation Theory (Symmetric regime)

Lukyanov'13; Bazhanov, Lukyanov'13

- A real regular solution of the (modified) sinh-Gordon equation

$$\partial_z \partial_{\bar{z}} \eta - \mathrm{e}^{2\eta} + \rho^4 \, |\mathcal{P}(z)|^2 \, \mathrm{e}^{-2\eta} = 0 \qquad \left(\rho = \tfrac{1}{2} \, \mu R\right)$$

on the Riemann sphere except at three points $z = z_i$



$$\eta \to (a_i |k_i| - 1) \log |z - z_i| + O(1)$$

$$\eta \to -4 \log |z| + O(1) \, , \quad z \to \infty$$

$$\mathcal{P}(z) = \frac{(z_3 - z_2)^{a_1} (z_1 - z_3)^{a_2} (z_2 - z_1)^{a_3}}{(z - z_1)^{2 - a_1} (z - z_2)^{2 - a_2} (z - z_3)^{2 - a_3}} \, , \quad a_1 + a_2 + a_3 = 2$$

-

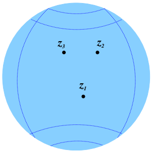
$$E_{\mathbf{k}} = -\frac{\pi}{6R} \sum_{i=1}^{3} \left(1 - 6 a_i \, k_i^2 \right) - \frac{\mu^4 R^3}{64} \int \mathrm{d}^2 z \, |\mathcal{P}(z)|^2 \, \mathrm{e}^{-2\eta}$$

Quantum/Classical correspondence

- Integrable Quantum Field Theory (IQFT)

$$\mathcal{L} = \frac{1}{16\pi}\sum_{i=1}^{3}(\partial\varphi_i)^2 + 2\mu\big(\mathrm{e}^{\mathrm{i}\alpha_3\varphi_3}\cos(\alpha_1\varphi_1+\alpha_2\varphi_2)+\mathrm{e}^{-\mathrm{i}\alpha_3\varphi_3}\cos(\alpha_1\varphi_1-\alpha_2\varphi_2)\big)$$

- Partial Differential Equation(s) (PDE)



$$\partial_z\partial_{\bar z}\eta-\mathrm{e}^{2\eta}+\rho^4\,\mathcal{P}(z)\bar{\mathcal{P}}(\bar z)\,\mathrm{e}^{-2\eta}=0$$

$$\rho^2=\tfrac{1}{2}\,\mu R$$

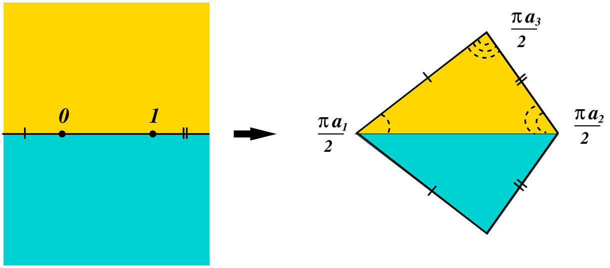
- The conformal map $w=\rho\int \mathrm{d}z\,\sqrt{\mathcal{P}(z)}$ transforms the PDE to the standard form of ShG equation

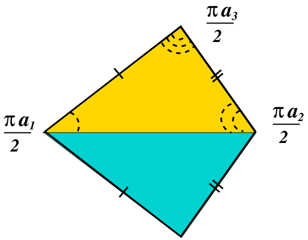
$$\partial_w\partial_{\bar w}\hat\eta-\mathrm{e}^{2\hat\eta}+\mathrm{e}^{-2\hat\eta}=0 \qquad \big(\hat\eta=\eta-\tfrac{1}{4}\,\log(\rho^4\mathcal{P}\bar{\mathcal{P}})\big)$$

- $PSL(2,C)$ transformation: $(z_1, z_2, z_3) \mapsto (0, 1, \infty)$

$$\mathcal{P}(z) = \frac{(z_3 - z_2)^{a_1}(z_1 - z_3)^{a_2}(z_2 - z_1)^{a_3}}{(z - z_1)^{2-a_1}(z - z_2)^{2-a_2}(z - z_3)^{2-a_3}}$$
$$\mapsto z^{a_1 - 2}(1 - z)^{a_2 - 2}$$

- $w = \rho \int \mathrm{d}z\, z^{\frac{a_1}{2} - 1}(1 - z)^{\frac{a_2}{2} - 1}$ – Schwartz-Christoffel mapping.
- For $a_1 > 0,\ a_2 > 0,\ a_3 = 2 - a_1 - a_2 > 0$:





$$\partial_w \partial_{\bar w} \hat\eta - \mathrm{e}^{2\hat\eta} + \mathrm{e}^{-2\hat\eta} = 0$$

$$\hat\eta \to 2\, l_i \ \log|w - w_i| + O(1)$$

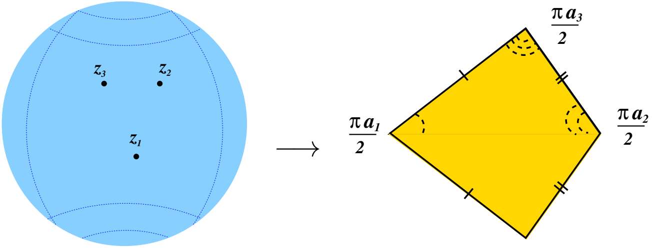
$$\mathfrak{F} = -\frac{8}{\pi} \int \mathrm{d}^2 w \ \sinh^2(\hat\eta) + \sum_i a_i l_i^2$$

$$\frac{R}{\pi} E_{\mathbf{k}}^{(0)} = \mathfrak{F} - 4\rho^2 \prod_{i=1}^{3} \gamma\left(\tfrac{a_i}{2}\right)$$

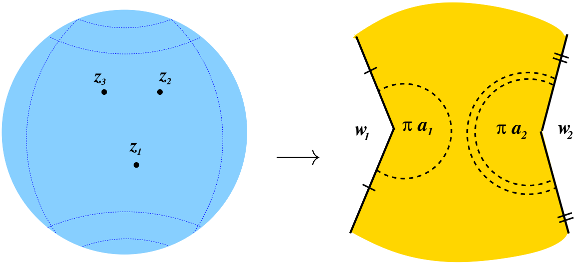
$$|k_i| = l_i + \frac{1}{2}, \ \mu R = 2\rho.$$

$$w = \rho \int \mathrm{d}z \; z^{\frac{a_1}{2}-1}(1-z)^{\frac{a_2}{2}-1}$$ – Schwarz-Christoffel mapping.

- $a_1 > 0, \; a_2 > 0 \;, \quad a_3 = 2 - a_1 - a_2 > 0$

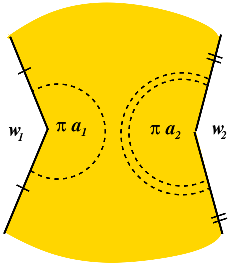


- $a_1 > 0, \; a_2 > 0 \;, \quad a_3 = 2 - a_1 - a_2 < 0$



Unitary regime

Bazhanov, Lukyanov, Kotousov'14



$$\partial_w \partial_{\bar w} \hat\eta - e^{2\hat\eta} + e^{-2\hat\eta} = 0$$

$$\hat\eta \to 2\, l_i \ \log|w - w_i| + O(1) \ \ (i = 1, 2)$$

$$\hat\eta \to 0 \qquad \text{as} \qquad w \to \infty$$

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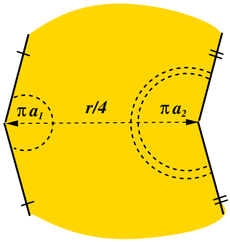
$$|k_i| = l_i + \tfrac{1}{2} \ \ (i = 1, 2); \qquad \mu R = 2\rho$$

BL regime ($a_3 = 0,\ a_{1,2} > 0,\ a_1 + a_2 = 2$)



$$\partial_w \partial_{\bar{w}} \hat{\eta} - \mathrm{e}^{2\hat{\eta}} + \mathrm{e}^{-2\hat{\eta}} = 0$$

$$\hat{\eta} \to 2\, l_i \ \log |w - w_i| + O(1) \ (i = 1, 2)$$

$$\hat{\eta} \to 0 \quad \text{as} \quad w \to \infty$$

$$\mathfrak{F}_{\rm BL} = -\mathfrak{f}_{\rm B}(2r\cos(\pi\delta/2)) - \frac{8}{\pi} \int \mathrm{d}^2 w \ \sinh^2(\hat{\eta}) + \sum_{i=1}^{2} a_i l_i^2$$

$$\mathfrak{f}_{\rm B}(\beta) = \frac{\beta}{2\pi^2} \int_{-\infty}^{\infty} \mathrm{d}\theta \ \cosh(\theta) \ \log\left(1 - \mathrm{e}^{-\beta\cosh(\theta)}\right)$$

$$|k_i| = l_i + \frac{1}{2} \ \ (i = 1, 2); \qquad \mu R = 2\rho$$

The exact instanton counting

- The sinh-Gordon model is a classical integrable system which can be studied by the inverse scattering transformation method. This allows one to rederive the NLIE for the BL model

  $$\tilde{\mathcal{L}}_{\rm BL} = \frac{1}{16\pi}\,\big((\partial\varphi_1)^2 + (\partial\varphi_2)^2\big) + 4\mu\,\cosh\big(\tfrac{\sqrt{\delta-1}}{2}\varphi_1\big)\cos\big(\tfrac{\sqrt{\delta+1}}{2}\varphi_2\big)$$

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- The Quantum-Classical duality can be applied to the sausage regime ($\delta > 1$)
- Resurgence of the Trans-Series expansions in $O(3)$ NLSM $\longrightarrow$ Resurgence of the Trans-Series expansions in the classical sinh-Gordon ODE.

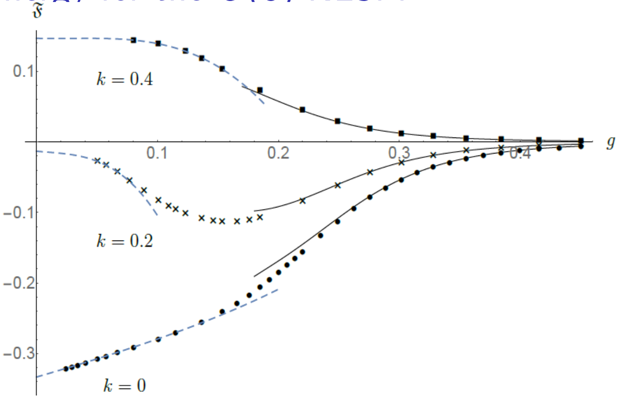
NLIE for O(3) model [Bazhanov,Kotousov,Lukyanov'17]

- $$\varepsilon(\theta - \mathrm{i}\gamma) = r\,\sinh(\theta - \mathrm{i}\gamma) - 2\pi k + \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta'}{2\pi\mathrm{i}} \Big[G(\theta - \theta' - 2\mathrm{i}\gamma)\big(L(\theta' - \mathrm{i}\gamma)$$
$$- G(\theta - \theta')\; L(\theta' - \mathrm{i}\gamma)\Big] + \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta'}{2\pi}\; G_1(\theta - \theta' - \mathrm{i}\gamma)\; \log\big(1 + \mathrm{e}^{-\omega(\theta')}\big)$$
- $$\omega(\theta) = r\,\cosh(\theta) + \Im m\Big[\int_{-\infty}^{\infty} \frac{\mathrm{d}\theta'}{\pi}\; G_1(\theta - \theta' + \mathrm{i}\gamma)\; L(\theta' - \mathrm{i}\gamma)\Big]$$
$$- \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta'}{\pi}\; G_2(\theta - \theta')\; \log\big(1 + \mathrm{e}^{-\omega(\theta')}\big)$$
- $$L(\theta) = \log\big(1 + \mathrm{e}^{-\mathrm{i}\varepsilon(\theta)}\big)$$

$$G(\theta) = \frac{2\pi}{(\theta + \mathrm{i}\pi)(\theta - \mathrm{i}\pi)}\;, \quad G_1(\theta) = \frac{4\pi^2}{(\theta + \frac{\mathrm{i}\pi}{2})(\theta - \frac{\mathrm{i}\pi}{2})(\theta + \frac{3\mathrm{i}\pi}{2})(\theta - \frac{3ri\pi}{2})}$$

$$G_2(\theta) = G(\theta) - \frac{2\pi}{(\theta + 2\mathrm{i}\pi)(\theta - 2\mathrm{i}\pi)}\;.$$

Vacuum energy for the $O(3)$ NLSM



A plot of $\mathfrak{F}(r,k) = ER/\pi$ with $k = 0$, 0.2 and 0.4 as a function of the running coupling constant g for the $O(3)$ sigma model. Note that the smallest value of the running coupling that we reached is $g = 0.0242\ldots$ (for $k = 0$), whereas the largest value is $g = 0.449\ldots$. These correspond to $MR = 10^{-15}$ and $MR = 5$, respectively.

Summary

- NLIE for the BL model
  - Renormalized Perturbation
  - Conformal Perturbation Theory
  - Saleur's NLIE
- Quantum/Classical correspondence:
  - There a connection between the theory of Integrable Models in two dimensions and the spectral theory of ordinary differential equations, Dorey-Tateo'98, BLZ'98, including massive QFT (Lukyanov-Zamolodchikov'10),
  - For the ground state of the BL model there is a precise match between Bethe Ansatz and "modified sinh-Gordon" equation approach.
- NLIE for $O(3)$ NLSM. Exact result for the vacuum energy.