Forced integrable systems: the case of sine-Gordon equation

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Revisiting the inhomogeneously driven sine–Gordon equation

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Wave Motion 73 (2017) 76–85

Motivation: Mathematical Models for Earthquakes

Geophys. J. R. astr. Soc. (1981) **64**, 151–161

Dynamics of a one-dimensional crack with variable friction

J. A. Landoni and L. Knopoff *Institute of Geophysics and Planetary Physics, University of California, Los Angeles, California 90024, USA*

2 Linear stress-drop model

We consider a uniform stress distribution prior to the onset of faulting. The differential equation for the relative displacement u on a one-dimensional fault is

$$\frac{u_{tt}}{c^2} - u_{xx} = \frac{f(x, t)}{\mu}, \tag{1}$$

where c is the velocity of sound, μ is the elastic modulus and $f(x, t)$ is the dynamical stress drop (Knopoff *et al.* 1973). We use conventional subscript notation to denote partial differentiation. For the case illustrated in Fig. 1

$$\frac{f(x, t)}{\mu} = \frac{f_0}{\mu}\{\epsilon H(t - x/c) + (1 - \epsilon)u/u_0\} \tag{2}$$

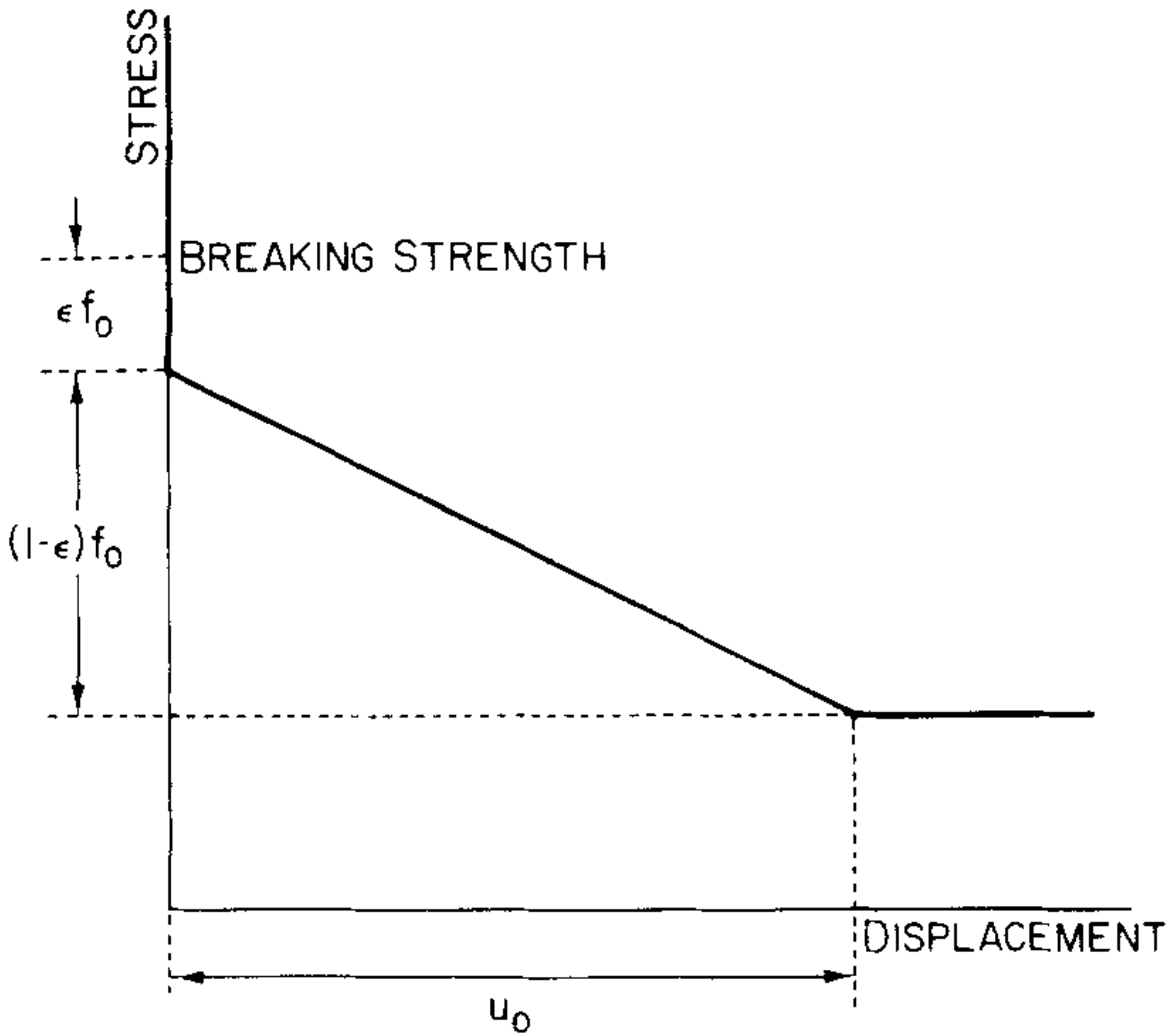


Figure 1. Modification of Andrews' model for stress drop.

J Seismol (2014) 18:497–510
DOI 10.1007/s10950-014-9422-7

ORIGINAL ARTICLE

Sine-Gordon equation and its application to tectonic stress transfer

Victor G. Bykov

Inspired us to replace u by sin(u) and consider

$$u_{tt} - u_{xx} + \sin u = F(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+$$

$$F(x, t) = AH(t - x).$$

One dimensional wave equation with forcing $F(x,t)$ is given as

$$u_{tt} - u_{xx} = F(x,t), \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$$

with following initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x)$$

and boundary conditions

$$u(0,t) = 0, \quad \lim_{x \to \infty} u_x(x,t) = 0$$

- Taking $F(x,t) = A$ (a constant). The exact solution is

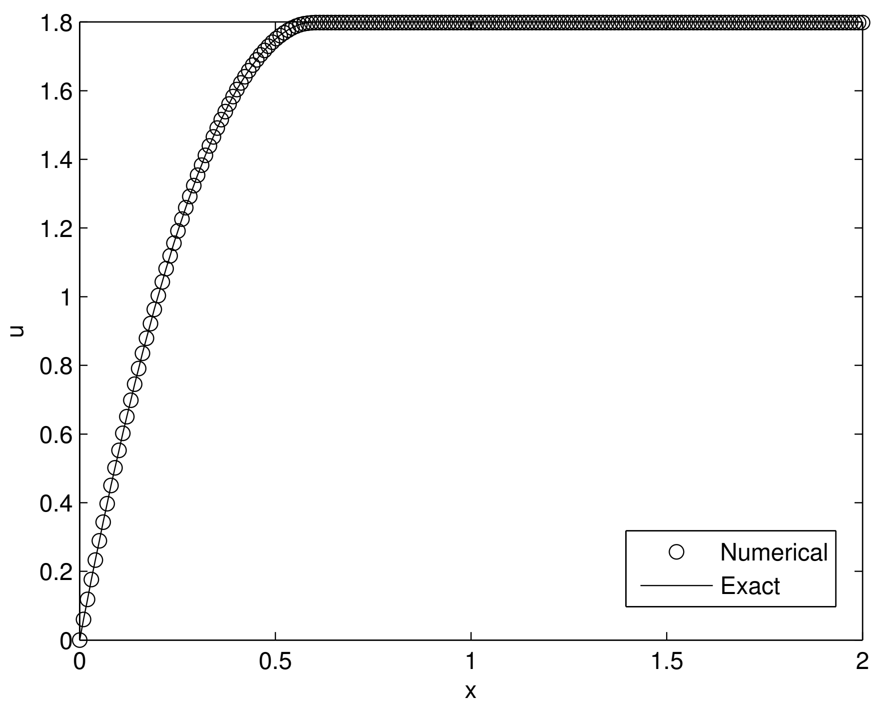
$$u = \frac{1}{2}[u_0(x+t) + u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} u_1(\tau)\, d\tau + \frac{A}{2}[t^2 - H(t-x)(t-x)^2]$$

where $H(t-x)$ is a Heaviside function.

- Next we take $F(x,t) = AH(t-x)$. The exact solution is

$$u = \frac{1}{2}[u_0(x+t)+u_0(x-t)]+\frac{1}{2}\int_{x-t}^{x+t} u_1(\tau)\, d\tau+A\frac{x}{2}H(t-x)(t-x)$$

Both second order time and space derivatives are discretized using second order central difference scheme. Figure 1 shows the comparison of exact solution of 1D Wave Equation with Heaviside forcing and numerical solution. In both cases final time is 0.6 and number of grid points are 200.



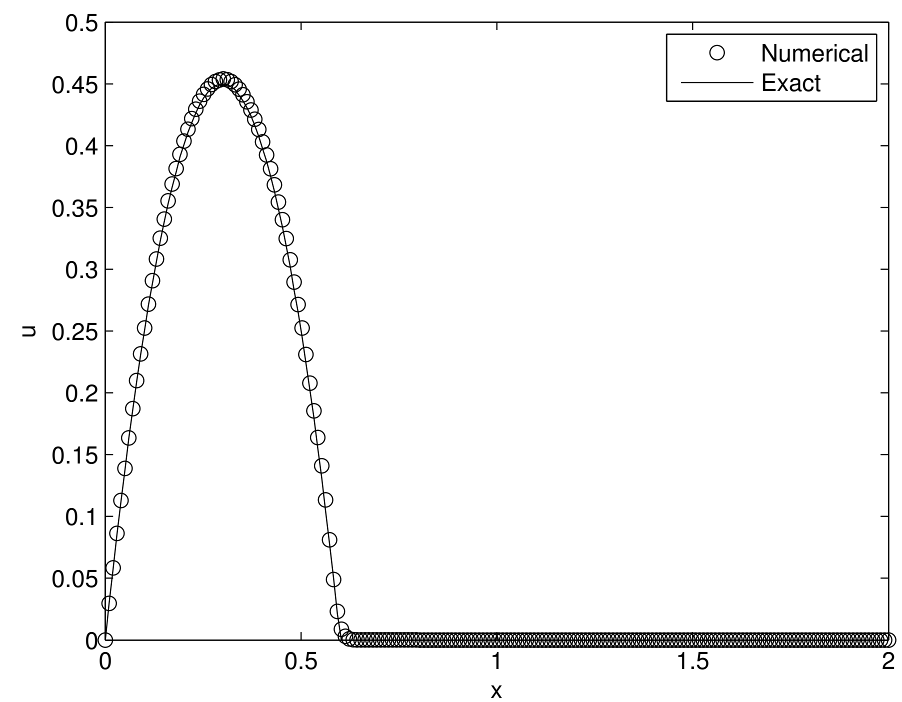


Figure: Solution of wave equation with zero initial conditions and with constant forcing A = 10 (left) and Heaviside forcing AH(t-x) with A = 10 (right).

$$u_{tt} - u_{xx} + \sin u = F(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+$$

where $F(x,t) = AH(t-x)$. The fully discretized sine-Gordon equation with Heaviside forcing is

$$u_j^{n+1} = -u_j^{n-1} + \frac{\Delta t^2}{\Delta x^2}(u_{j+1}^n + u_{j-1}^n) + 2\left(1 - \frac{\Delta t^2}{\Delta x^2}\right) u_j^n - \Delta t^2[\sin(u_j^n) - F(x,t)]$$

Time step is restricted by CFL condition.

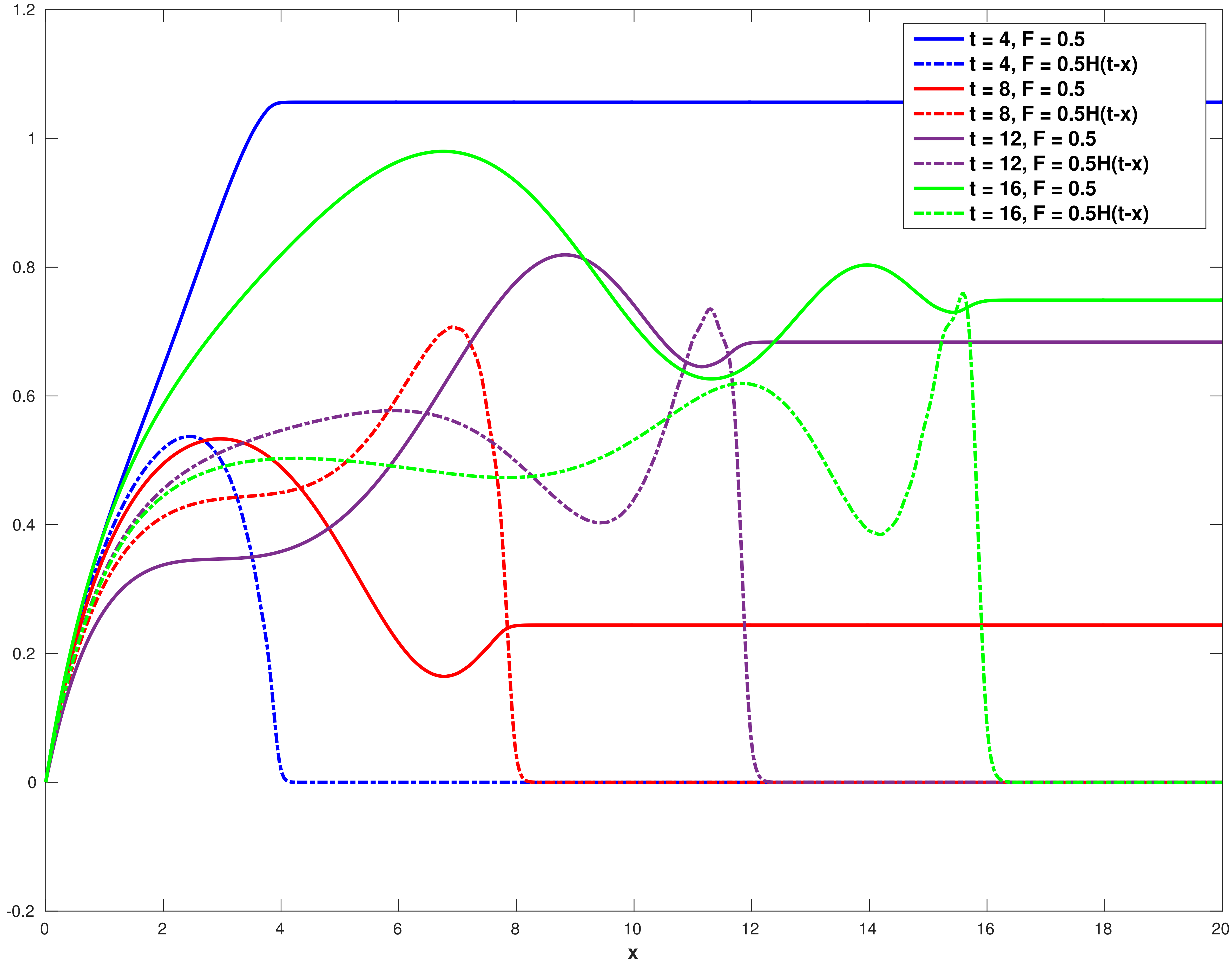


Figure 2: *Solution of inhomogeneous sG equation with constant and Heaviside forcing for different times.*

Simple oscillatory forcing function $F = A\cos \omega t$ was used to study equation [1]. These results are reproduced here.

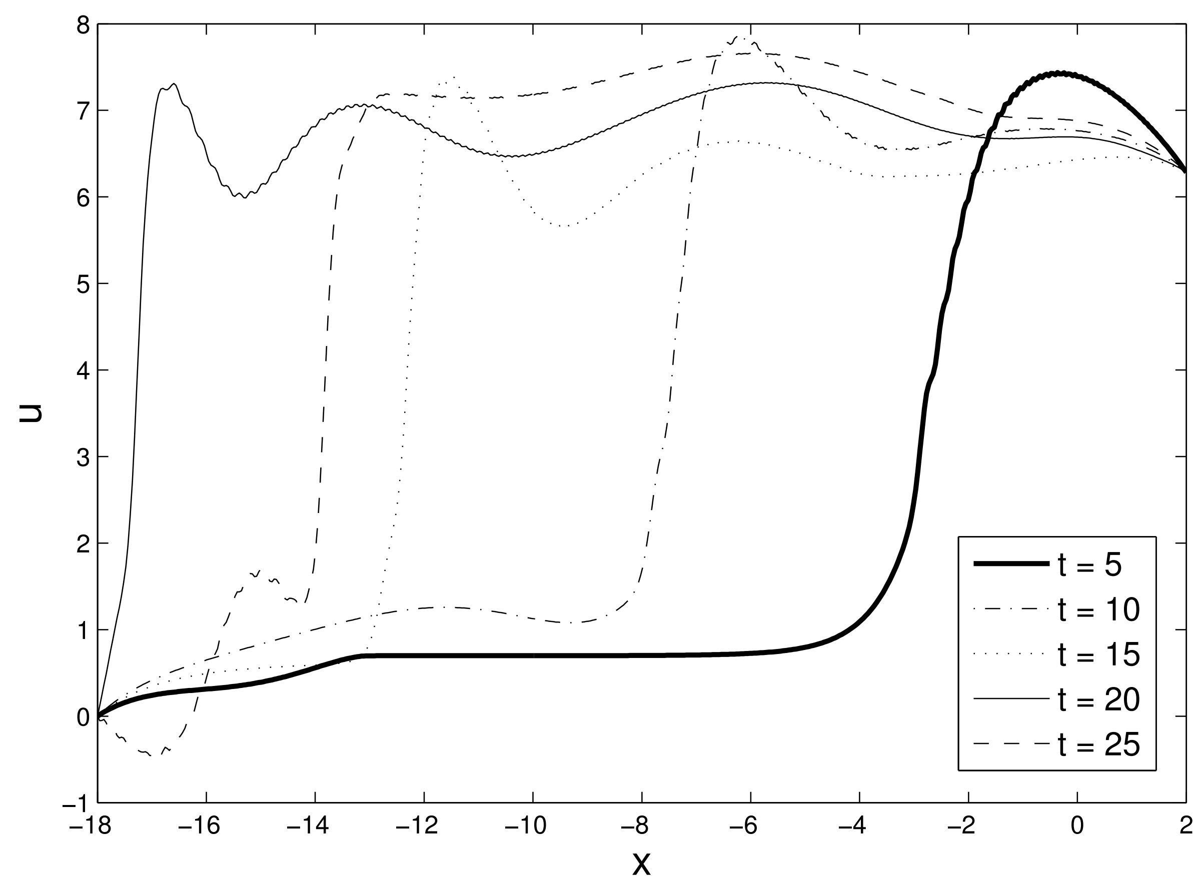


Figure: Solution of sine-Gordon equation for forcing function with $A = 0.5$ and $\omega = 0$

[1] Jiang S., Motion of a single kink in an applied field, *Chines Phys. Lett.* Vol. 3 No. 11, 1986.

In order to study the effect of discontinuous forcing term on sG equation two test cases are considered which are kink and breather. Initial conditions for kink are

$$u(x,0) = 4\ \arctan\left(\exp\left[\frac{x}{\sqrt{1-c^2}}\right]\right)$$
$$u_t(x,0) = -2\frac{c}{\sqrt{1-c^2}}\ \mathrm{sech}\left(\frac{x}{\sqrt{1-c^2}}\right)$$

and initial conditions for breather are

$$u(x,0) = 0$$
$$u_t(x,0) = 4\sqrt{1-c^2}\mathrm{sech}\left(x\sqrt{1-c^2}\right)$$

for $c = 0.2$. Domain size in both test cases is $[-L,\ L]$. Both no flux $u_x|_{\pm L} = 0$ and non-reflecting $u_x|_{\pm L} = \mp u_t$ boundary conditions are used.

Kink with No Flux Boundary Conditions:

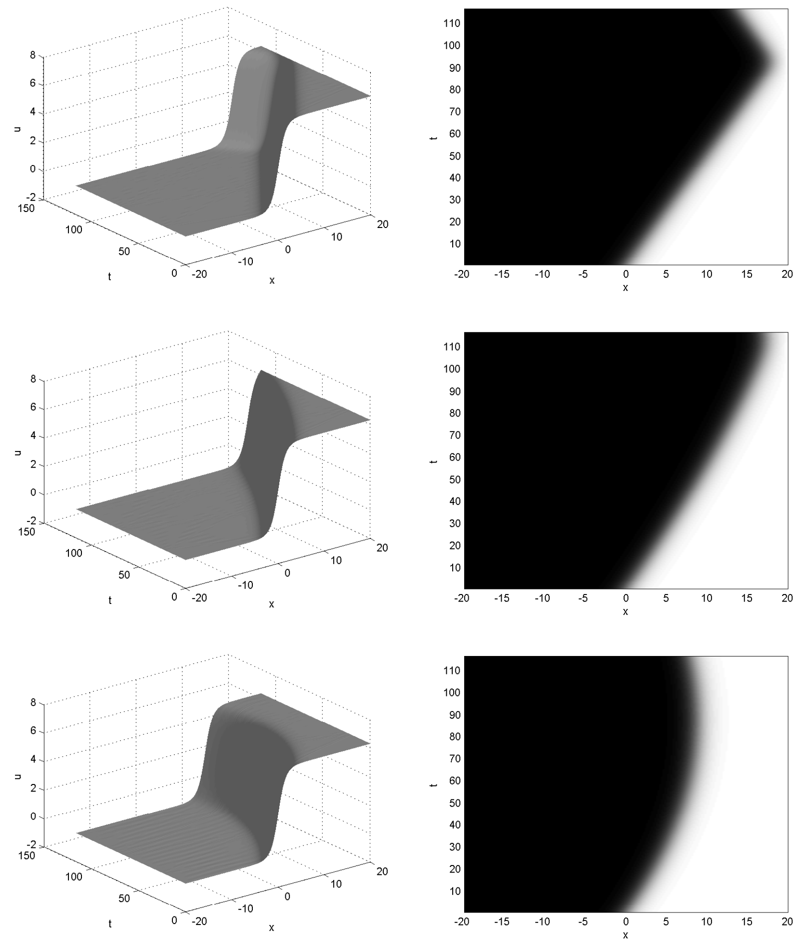


Figure: Solution of sine-Gordon equation using no flux boundary conditions with A = 0, 0.001, 0.003.

Kink with No Flux Boundary Conditions:

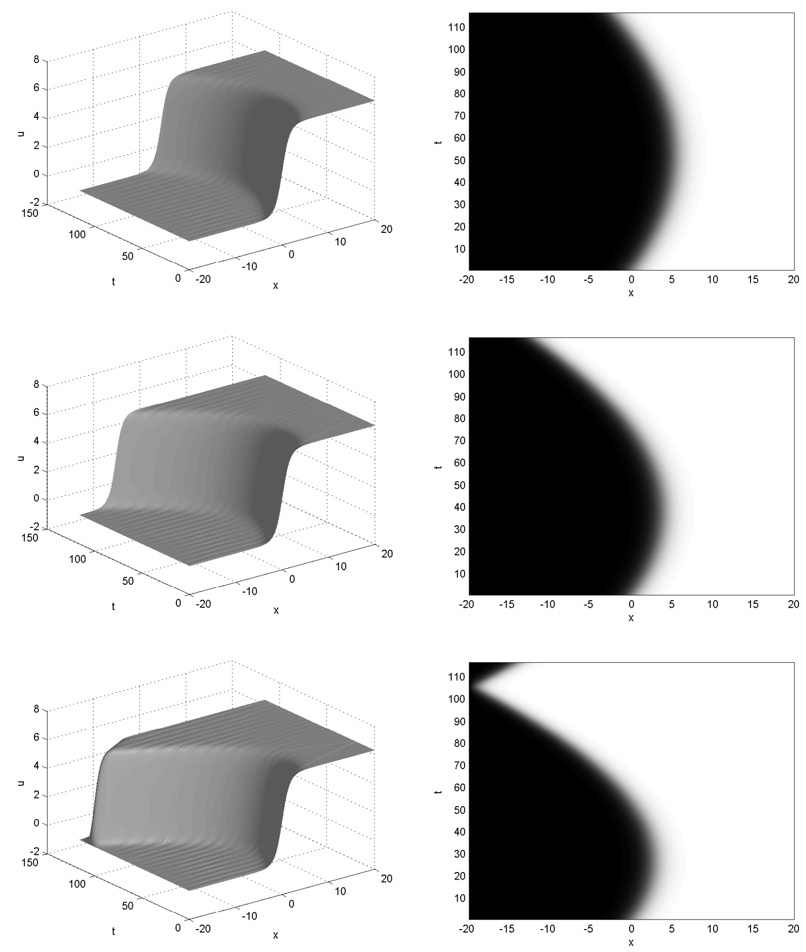


Figure: Solution of sine-Gordon equation using no flux boundary conditions with A = 0.005, 0.007, 0.01.

1. Numerical solution of kink without any forcing term *i.e.* A=0 exhibits soliton like behavior which does not change in shape and size. This wave moves with constant velocity and reflects back from the right boundary.
2. For A=0.001 solution still exhibits soliton behavior. But there is a slight change in velocity of the profile due to deceleration.
3. For $A = 0.003$ the deceleration increases which in turn increases the curvature of trajectory of the wave.
4. For higher values of A which are 0.005, 0.007 and 0.01 the wave moves in the backward direction towards the left boundary.

Kink with No Flux Boundary Conditions:

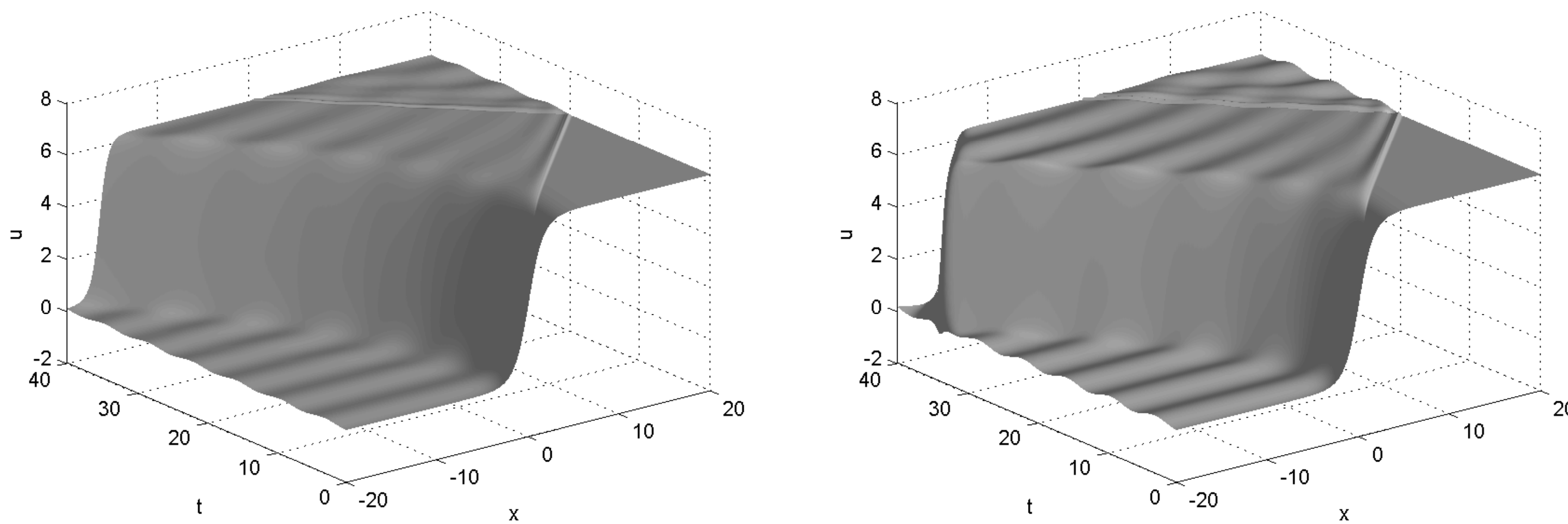


Figure: Solution of sine-Gordon equation using no flux boundary conditions with $A = 0.05,\ 0.1$

1. Further increasing the value of A results in backward movement of the initial profile. $A = 0.05, 0.1$.
2. This changes the shape of the wave due to space-time oscillations even before reflecting from the left boundary. Hence, after particular value of A solution does not exhibit soliton like behavior. This value is called as critical value of A denoted by A_{critical}.

Kink with Non-reflecting Boundary Conditions: In this case all the observation made in the previous case are same except that the boundary conditions are non-reflecting. Also A_{critical} value is same as before.

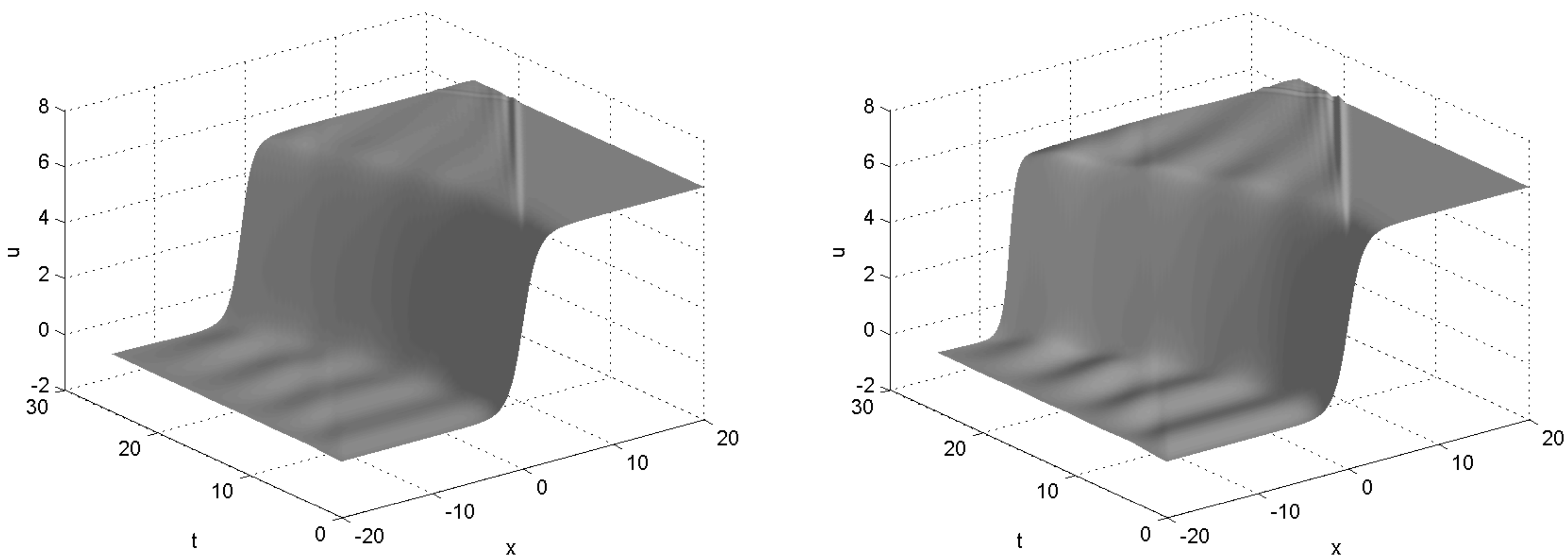


Figure: Solution of sine-Gordon equation using non-reflecting boundary conditions with $A = 0.05, 0.1$

A_{critical} vs c Variation over different domain size:

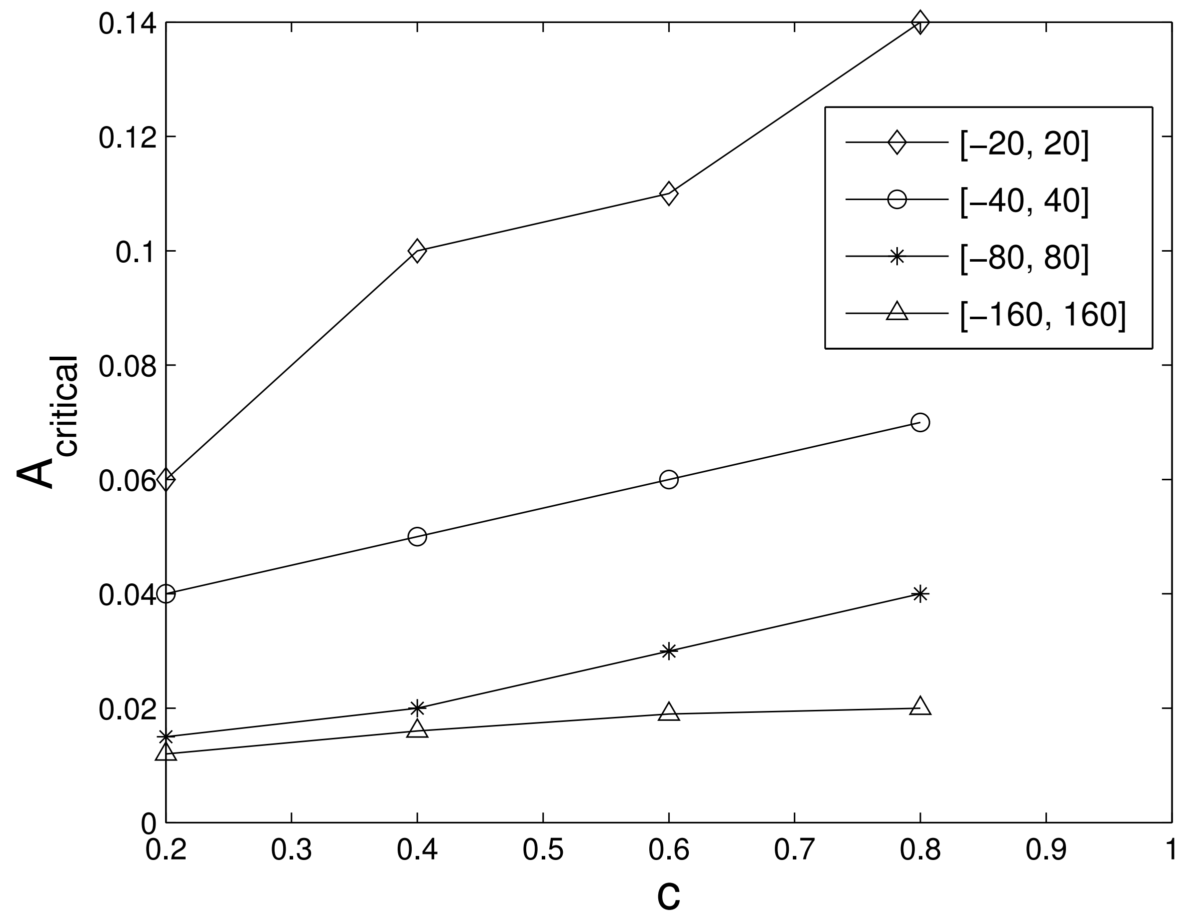


Figure: Variation of A_{critical} vs c values over different domain size

Breather with Non-Reflecting Boundary Conditions:

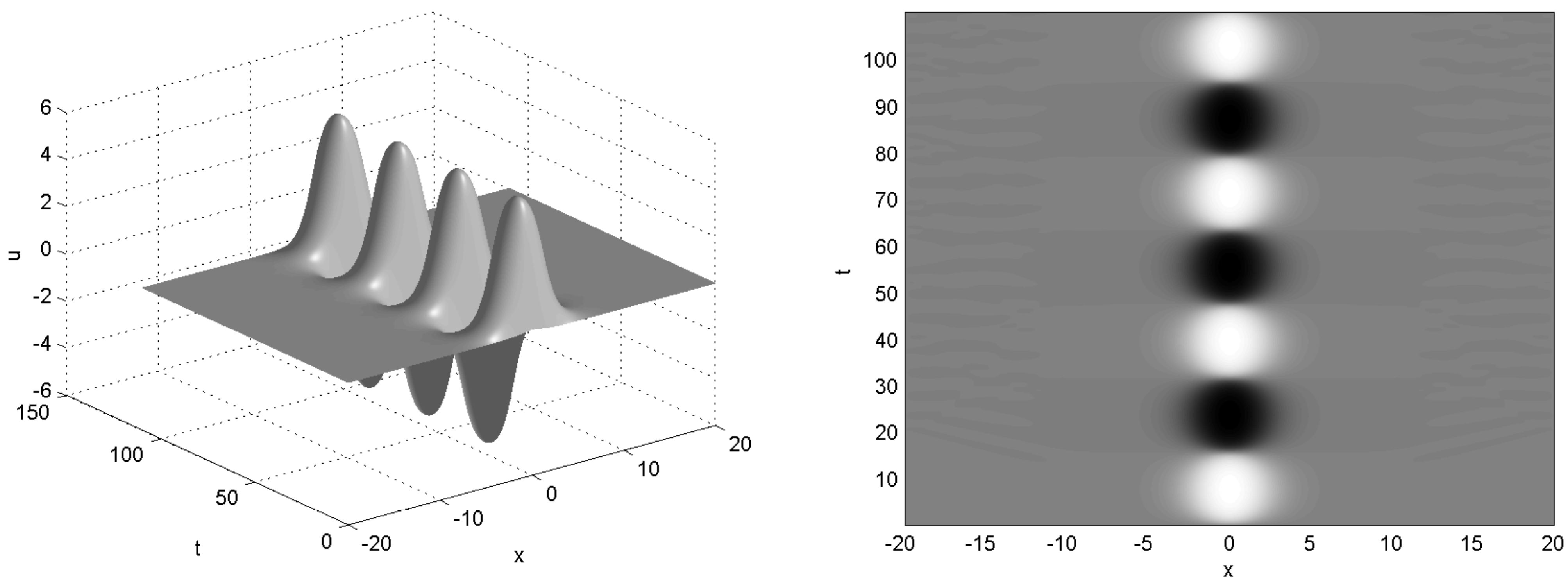


Figure: Solution of sine-Gordon equation with $A = 0$.

Breather with Non-Reflecting Boundary Conditions:

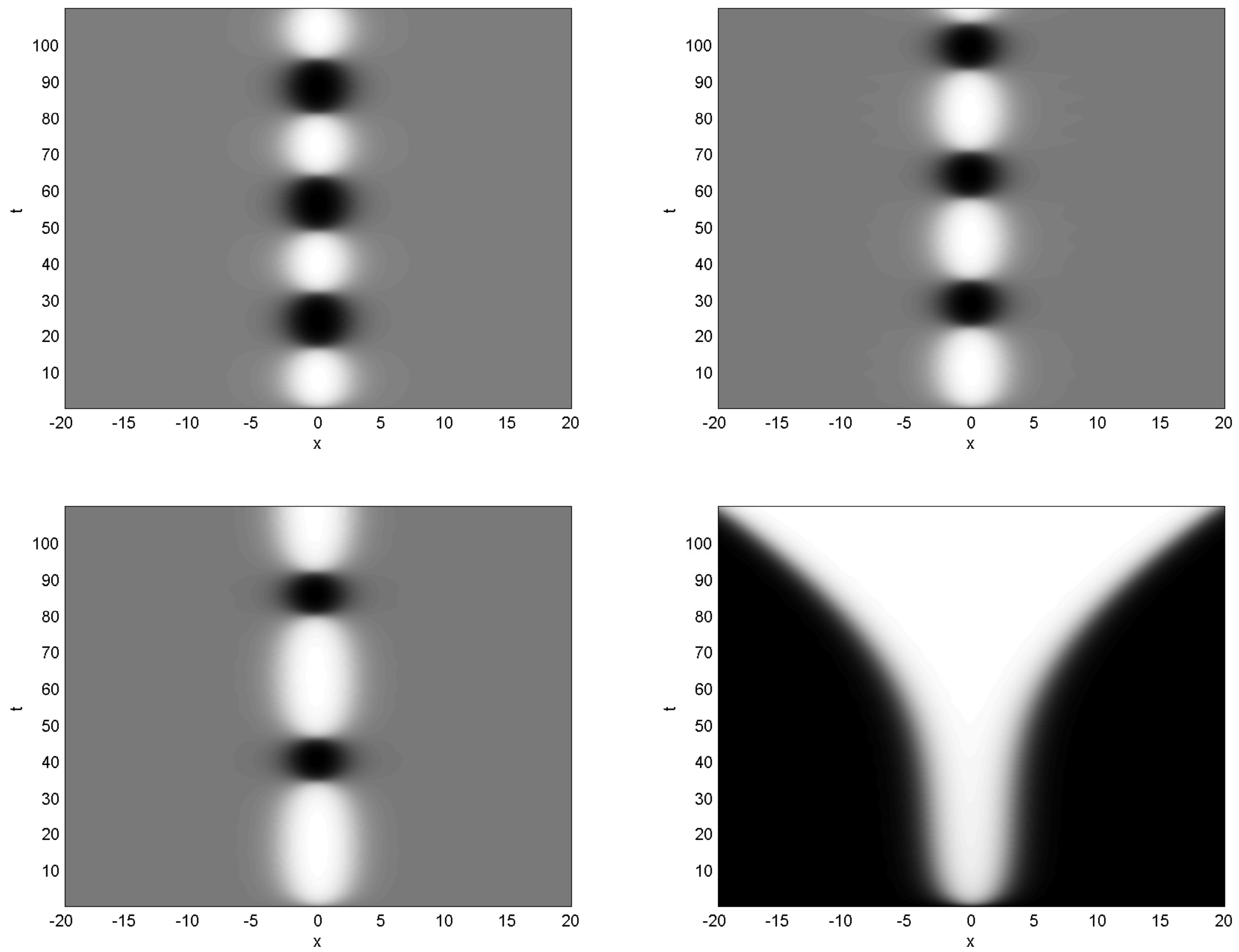


Figure: Breather solution of sine-Gordon equation using non-reflecting boundary conditions with A = 0.001, 0.005, 0.007, 0.0076

1. For $A = 0.001$, there is a little change in the solution compared with $A = 0$.
2. For $A = 0.005$, the oscillating frequency of the breather decreases which can be seen form the contour plot where white region becomes larger than black region. This trend continues for larger values of A.
3. For $A = 0.0076$ the breather solution completely destroys. This makes 0.0076 as the critical value of A.

Breather with No Flux Boundary Conditions: Breather solution with non-zero A values follow similar trend as discussed previously even for no flux boundary conditions. In case of A = 0.0076 which is the critical value of A one can see reflection from boundary due to no flux boundary condition.

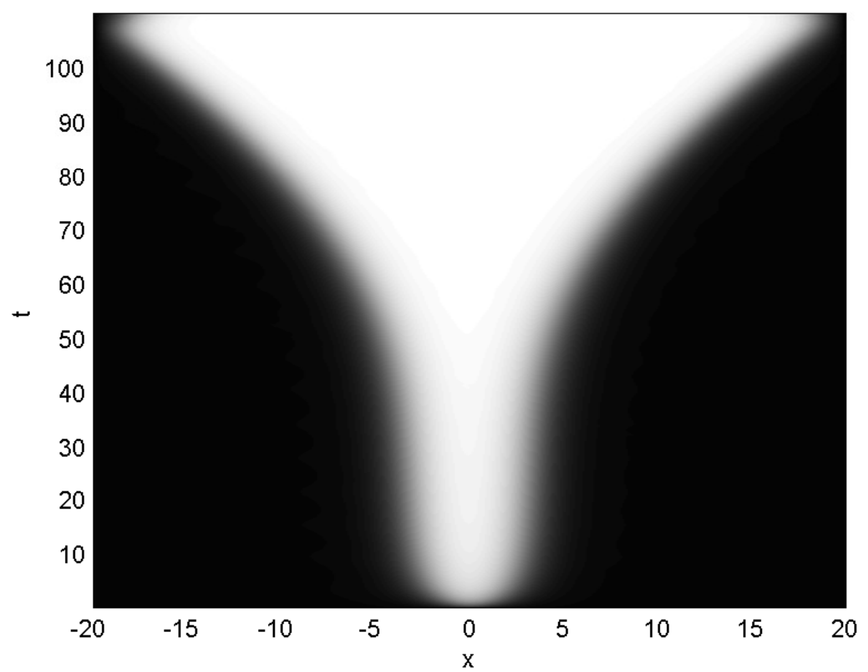


Figure: Breather solution of sine-Gordon equation using no flux boundary conditions with A = 0.0076

Variation of A_{critical} with c:

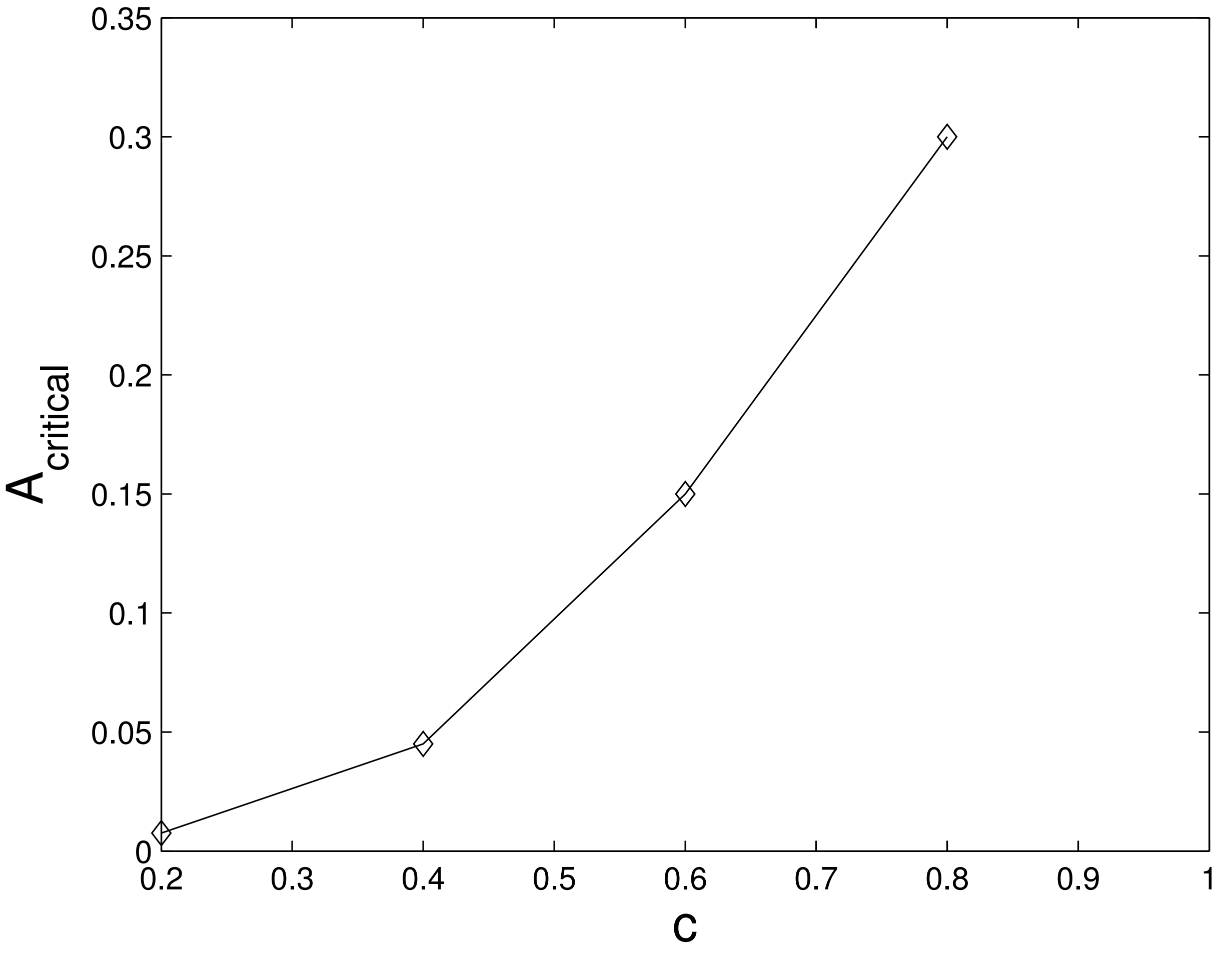


Figure: Variation of A_{critical} with c

One more interesting feature observed in the solution of breather. By increasing the value of c temporal frequency of oscillation increases but at the same time if value of A increases (but remains below A_{critical}), these oscillations shift towards left boundary

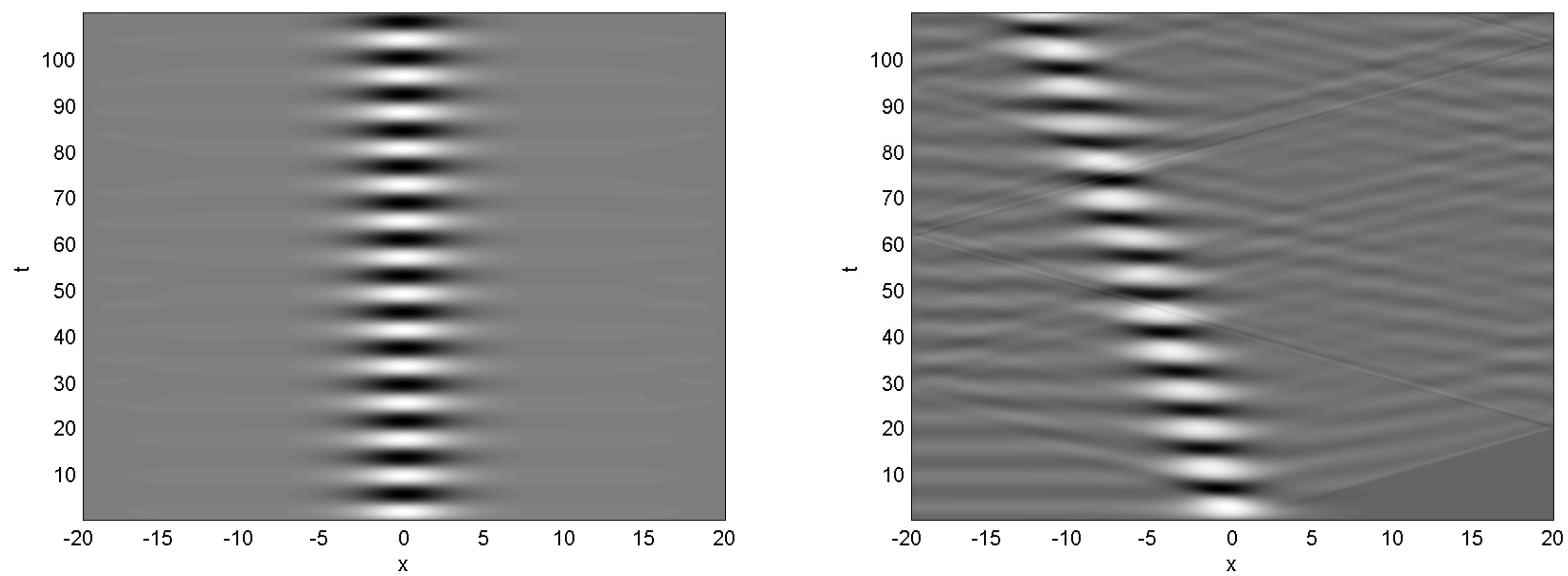


Figure: Breather solution of sine-Gordon equation with c $=$ 0.8, A $=$0 (left) and 0.25 (right)

Conclusions

In this work numerical solution of inhomogeneous sG equation is investigated with discontinuous forcing term AH(t x).

Such discontinuous forcing in sG equation can represent, for example, a dynamical stress drop in one-dimensional crack propagation problem.

Kink and breather test cases are solved using no flux as well as non-reflecting boundary conditions.

In kink test case, for small amplitude of *A* the solution still exhibit soliton like behaviour but changes the direction of propagation as well as velocity. Higher values of *A* completely destroys the soliton behaviour.

Future?

Numerical solution of PDE's on unbounded domains.

Heat Equation?

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x > 0,\ t > 0.$$

$$u(x,0) = u_0(x),$$

$$u(0,t) = g(t),$$

$$u(x \to \infty, t) = 0.$$

$$
\begin{aligned}
&u_t = u_{xx} + f(x,t), && \text{for } x \in (0,1), \quad t > 0 \\
&u(0,t) = b(t), && \text{for } t > 0, \\
&u_x(1,t) + Gu(1,t) = g(t), && \text{for } t > 0, \\
&u(x,0) = v(x), && \text{for } x \in (0,1),
\end{aligned}
$$

$$
Gu(t) = Ju_t(t), \quad \text{where } Jw(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{w(s)}{\sqrt{t-s}} ds.
$$

Journal of Computational Mathematics
Vol.33, No.1, 2015, 17–32.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1406-m4443

FINITE DIFFERENCE METHODS FOR THE HEAT EQUATION WITH A NONLOCAL BOUNDARY CONDITION*

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We consider first, in Section 3 below, the forward Euler approximation

$$\partial_t U_m^n - \partial_x \overline{\partial}_x U_m^n = f_m^n, \quad \text{for } m = 1, \ldots, M, \ n \geq 0, \quad f_m^n = f(x_m, t_n), \tag{1.4}$$

with the left side boundary values and initial values given by

$$\begin{aligned} U_0^n &= b^n, \quad \text{for } n \geq 1, \quad b^n = b(t_n), \\ U_m^0 &= v_m, \quad \text{for } m = 0. \ldots, M+1, \quad v_m = v(x_m). \end{aligned} \tag{1.5}$$

$$\partial_x U_M^n + G_k U_{M'}^n = g^n, \quad \text{for } n \geq 1, \quad \text{where } U_{M'}^n = \tfrac{1}{2}(U_M^n + U_{M+1}^n). \tag{1.9}$$

$$\lambda = k/h^2$$

Theorem 3.1. *Let U^n be the solution of (1.4), (1.5), (1.9), with $b^n = 0$, for $n \geq 1$. Assume that $1/\pi < \lambda_0 \leq \lambda \leq 1/2$ and $\nu > 1$. Then we have, with $C = C(\lambda_0)$,*

$$\|U^n\| \leq Ce^{\nu t_n}\|U^0\| + Ck \sum_{j=0}^{n-1} e^{\nu t_{n-1-j}} \|f^j\|_0 + C \max_{j \leq n}(e^{\nu t_{n-j}}|g^j|).$$

Nonlinear problems?

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) \quad ; \quad x \in \mathbb{R}, \; t > 0$$

$$u(x,0) = u_0(x)$$

$$u(x \to -\infty, t) = 1$$

$$u(x \to +\infty, t) = 0$$

Need to compute travelling wave solns

(Motivation: population biology)

$$v(x-ct)$$

Find c

Theory

All +ve u_0 ↓ exponentially $x \to \infty$

evolves into a ! TW with

$$c \geq 2$$

$$u_0(x) \sim e^{-\beta x} \qquad x \to \infty$$

$$c(\beta) = \begin{cases} \dfrac{1+\beta^2}{\beta} & \beta \leq 1 \\ 2 & \beta \geq 1 \end{cases}$$

Numerical soln with abc

$$u(-L,t) = 1$$

$$u(L,t) = 0$$

by Murray & others $\qquad c = 2 \;\; \forall \beta$!!

Keller & Hagstrom (1986)

ABC

$$-\int_0^t e^{(1-c^2/4)(t-s)}\left[\frac{1}{\sqrt{\pi(t-s)}} - \frac{c}{2}\, e^{c^2(t-s)/4}\, \mathrm{Erfc}\left(\frac{c}{2}\sqrt{t-s}\right)\right]$$

$$u_x(L,s)\,ds$$

$$= u(L,t) + \ldots \text{fns of } \mathrm{Erfc}\left(\frac{c}{2}\sqrt{t}\right)$$

The technique

- Truncate
- Linearize outside the bdd domain
- "Solve" the linearized problem and "match" it with the interior problem
- Hope (or prove) that the resulting IBVP is well-posed.

Need to do the same for the sine-Gordon equation.

SIAM J. SCI. COMPUT.
Vol. 29, No. 6, pp. 2494–2506

NUMERICAL SOLUTION TO THE SINE-GORDON EQUATION DEFINED ON THE WHOLE REAL AXIS*

CHUNXIONG ZHENG†

Abstract. Numerical simulation of the solution to the sine-Gordon equation on the whole real axis is considered in this paper. Based on nonlinear spectral analysis, exact nonreflecting boundary conditions are derived at two artificially introduced boundary points. Then a numerical scheme of second order is proposed to approximate the solution. In the end, some numerical examples are provided to demonstrate the effectiveness of the proposed scheme.

$$q_{tt} - q_{xx} + \sin q = 0, \;\; x \in \mathbf{R}, \;\; 0 < t \leq T,$$
$$q(x,0) = q_0(x), \;\; q_t(x,0) = q_1(x), \;\; x \in \mathbf{R}.$$

$$q_\nu(a,t) = \mathcal{K}(t; q(a,\cdot), q_\nu(a,\cdot))$$

$$q_\nu(b,t) = \mathcal{K}(t; q(b,\cdot), q_\nu(b,\cdot))$$

$$g_1(t) = \mathcal{K}(t; g_0(\cdot), g_1(\cdot)).$$

$$g_1(t) = -\dot{g}_0(t) - \int_0^t \mathcal{Z}(t-\tau) \left(\cos\left[\frac{g_0(t)}{2}\right] M_1(t, 2\tau - t) \right.$$
$$\left. + \sin\left[\frac{g_0(t)}{2}\right] M_2(t, 2\tau - t) \right) d\tau,$$

$$\mathcal{Z}(t) \overset{def}{=} \frac{1}{i\pi} \int_{\partial D} \frac{1}{k} \left(1 - \frac{1}{k^2}\right) e^{-\frac{i}{2}\left(k + \frac{1}{k}\right)t} dk$$
$$= -\frac{4}{\pi} \int_1^{+\infty} (x - \sqrt{x^2 - 1}) \sin(xt) dx - \frac{2J_1(t)}{t} + \frac{4}{\pi} \frac{t \cos t - \sin t}{t^2}.$$

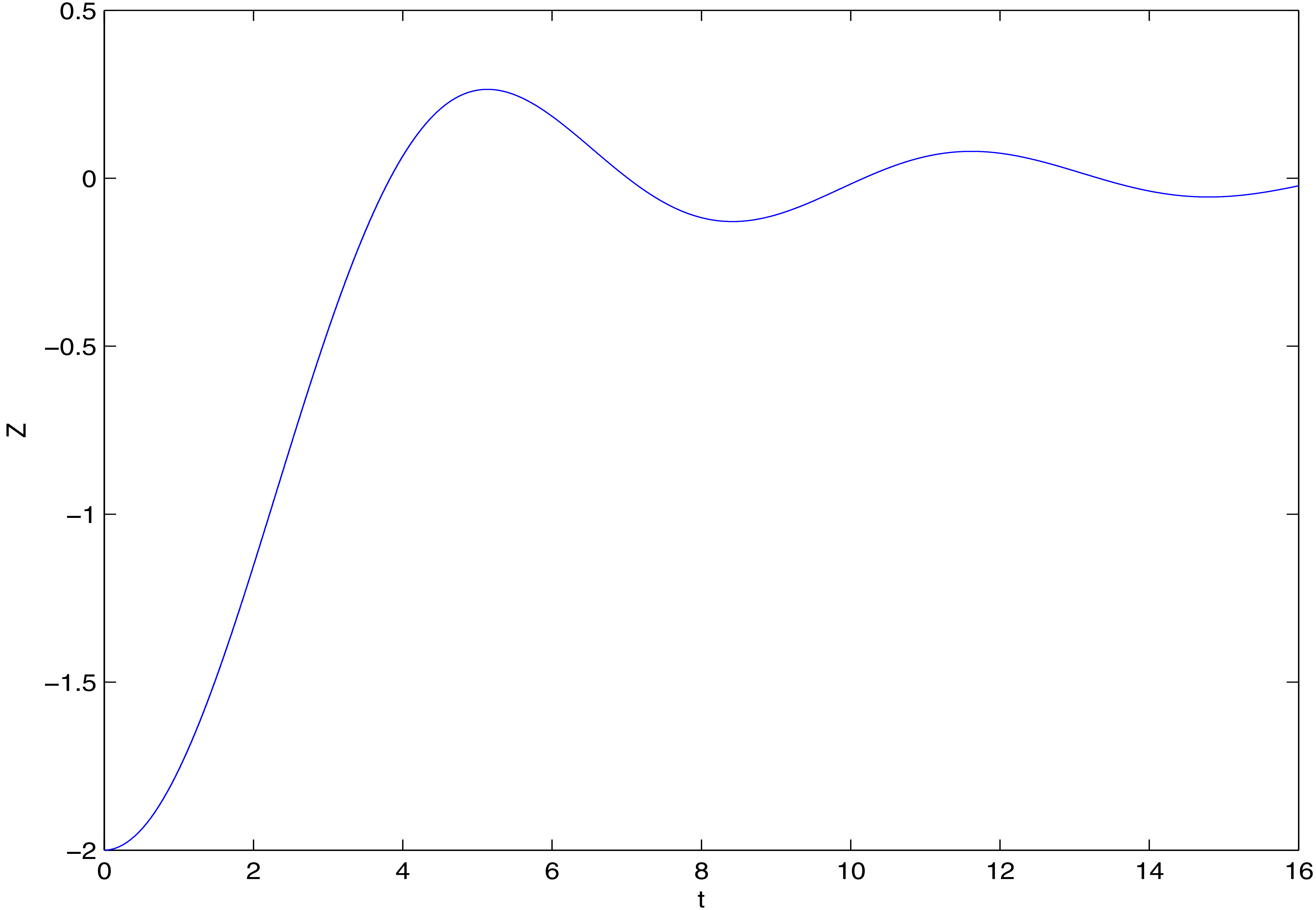

FIG. 1. *Plot of function $\mathcal{Z}(t)$.*

$$L_1(t,t) = -\frac{g_1(t) + \dot{g}_0(t)}{8}, \;\; M_1(t,t) = -\frac{1}{2}\sin\left[\frac{g_0(t)}{2}\right],$$

$$L_2(t,-t) = M_2(t,-t) = 0,$$

$$L_{1_t} - L_{1_s} = -\frac{g_1(t) + \dot{g}_0(t)}{4} L_2 + \frac{1}{4}\sin\left[\frac{g_0(t)}{2}\right] M_2,$$

$$L_{2_t} + L_{2_s} = \frac{g_1(t) + \dot{g}_0(t)}{4} L_1 - \frac{1}{4}\sin\left[\frac{g_0(t)}{2}\right] M_1,$$

$$M_{1_t} - M_{1_s} = -\sin\left[\frac{g_0(t)}{2}\right] L_2 + \frac{g_1(t) - \dot{g}_0(t)}{4} M_2,$$

$$M_{2_t} + M_{2_s} = \sin\left[\frac{g_0(t)}{2}\right] L_1 - \frac{g_1(t) - \dot{g}_0(t)}{4} M_1.$$

The Generalized Dirichlet-to-Neumann Map for Certain Nonlinear Evolution PDEs

A. S. FOKAS

Communications on Pure and Applied Mathematics, Vol. LVIII, 0639–0670 (2005)

Can we have simpler bc's?

Do solutions converge to the infinite domain problem?

Example

$$(\theta_x - \alpha \sin \frac{\theta}{2})|_{x=0} = 0\,, \quad \alpha \in \mathbb{R}\,.$$

Sklyanin, Boundary conditions for integrable equations, *Funct. Anal. Appli.*, 21:2 (1987)

Thank you