Bethe Ansatz Solution for Bloch electrons in magnetic field

P. Wiegmann,

University of Chicago

July 26, 2018

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- Magnetic field is good for electrons!

Bloch electrons in magnetic field

Lattice electrons in magnetic field

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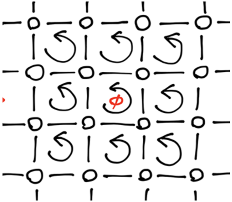

- Magnetic translation $T_{\mathbf{n}} T_{\mathbf{m}} = q^{-\mathbf{n}\times\mathbf{m}} T_{\mathbf{n}+\mathbf{m}}$

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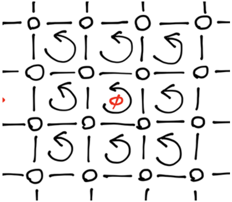

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- In Landau gauge: $T_x|\mathbf{n}\rangle = |\mathbf{n} + \mathbf{1_x}\rangle,\ T_y = q^2, \quad \psi_{\mathbf{n}} = e^{ikn_y}\psi_{n_x}(k)$

Schroedinger eq (Harper eq, Almost Mathieu eq, quasicrystal, etc)

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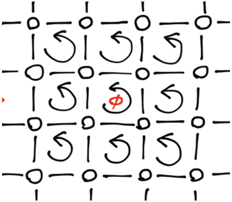

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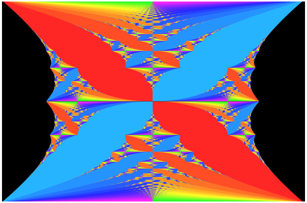
- One of the most celebrated problem of spectral theory, with applications to localization theory, chaos, quantum Hall effect, etc.

$$\psi_{n+1} + \psi_{n-1} + 2\cos(k + 2\pi\Phi)\,\psi_n = E\psi_n$$

- If the flux is a rational number $\Phi = P/Q$, the spectrum consists of Q bands - metal

- If $\Phi =$ irrational, the spectrum is a peculiar Cantor set - *singular continuous* (a set without isolated points but with zero measure). The total bandwidth is zero



- incomplete list of works before 1990:

  Zak 1964, Azbel 1964, Hofstadter 1976, Wannier 1978, Aubry-Andre 1980, Zak-Avron 1985, Bellissard-Simon 1980-1990, Thouless-Kohmoto 1982, Wilkinson 1987

Hall conductance=First Chern class

Hall conductance or the First Chern class is the topological characteristic of the spectrum.

The Hall conductance σ_k of the kth gap is the solution of the Diophantine equation

$$\boxed{P\sigma_k = k \pmod{Q}}$$

Example: $\frac{P}{Q} = \frac{4}{15}$,

$$\sigma_k = 4, -7, -3, 1, 5, -6, -2, 2, 6, -5, -1, 3, 7, 4$$

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- in 1997 together with Alexander Abanov and J. Talstra we determined the hierarchical structure of the spectrum

Hofstadter problem is integrable! Spectrum

$$\boxed{\psi_{n+1} + \psi_{n-1} + 2\cos(k + 2\pi\Phi)\,\psi_n = E\psi_n}$$

Equations for the middle of the bands $k = 0$,

$$q = e^{i\pi\Phi}, \qquad \Phi = \frac{P}{Q}$$

$$E = 2\,(-1)^P \sin(\pi\Phi) \sum_{l=1}^{Q-1} z_l$$

Roots $z_1, \ldots z_{Q-1}$ obeys the Bethe Ansatz equations

$$\boxed{\frac{z_l^2 + q}{q\,z_l^2 + 1} = (-1)^P \prod_{m \neq l}^{Q-1} \frac{q\,z_l - z_m}{z_l - q z_m}}$$

Hofstadter problem is integrable! Wave functions

Polynomials

$$\Psi(z) = \prod_{l=1}^{Q-1} (z - z_l)$$

play a special role. Some of them have names: $\Psi_{E=0}(z)$ - is q-Legendre polynomial.

The wave function

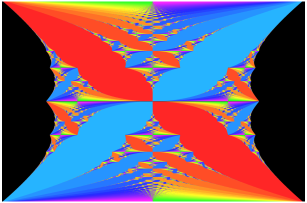
$$\psi_n = \sum_{m=1}^{Q-1} c_{nm} \Psi(z) \Big|_{z=q^m}$$

The coefficients are *quantum di-logarithm*

$$c_{nm} = q^{2nm+\frac{m}{2}} \prod_{j=0}^{m-1} \frac{1+q^{-j-\frac{1}{2}}}{1-q^{j+\frac{1}{2}}}$$

- The Bethe Ansatz is equivalent to a Heisenberg spin chain on only two sites but with large spin equal to the number of bands $Q-1$

$$\frac{z_l^2+q}{qz_l^2+1} = (-1)^P \prod_{m\neq l}^{Q-1} \frac{qz_l - z_m}{z_l - qz_m}$$



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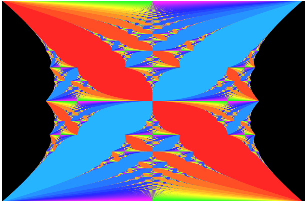

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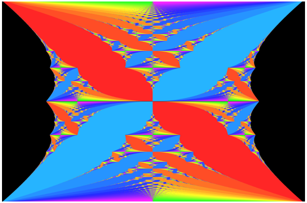

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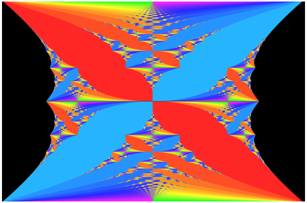
- ✓ How to obtain these equations?
- How to solve them? We are interested in $Q\to\infty$, $P\to\infty$, $\Phi=P/Q$ - fixed
- What does the limit mean in such complicated spectrum?



Chiral gauge

- Chiral gauge $\quad t_{\mathbf{n},\mathbf{n}+\mathbf{1_x}} = e^{-\mathrm{i}\frac{\Phi}{2}n_+}, \quad t_{\mathbf{n},\mathbf{n}+\mathbf{1_y}} = e^{+\mathrm{i}\frac{\Phi}{2}(n_+ +1)}, \quad n_+ = n_x + n_y, \quad \Psi_{\mathbf{n}} = e^{\mathrm{i}\mathbf{p}\mathbf{n}}\Psi_{n_+}$

$$\mathrm{i}q^{-1/2}(1+q^{2n+1})\Psi_{n+1} - \mathrm{i}q^{1/2}(1+q^{-2n+1})\Psi_{n-1} = E\Psi_n, \quad \mathbf{p} = (\frac{\pi}{2}, \frac{\pi}{2})$$

- What is the advantage this gauge?

Consider a difference equation, such that $\Psi_n = \Psi(z)\Big|_{z=q^n}$

$$i(z^{-1}+qz)\Psi(qz) - i(z^{-1}+q^{-1}z)\Psi(q^{-1}z) = E\Psi(z)$$

- All solutions of the difference equation are polynomials

$$\Psi(z) = \prod_{l=1}^{Q-1}(z-z_l)$$

- Equation for the roots

$$\frac{z_l^2+q}{qz_l^2+1} = (-1)^P \prod_{m\neq l}^{Q-1} \frac{qz_l - z_m}{z_l - qz_m}$$

- Q: When all solutions of the second order ODE are polynomials?

$$H\Psi = \left[a(z)\frac{d^2}{dz^2} + b(z)\frac{d}{dz} + c(z) \right] \Psi(z) = E\Psi(z)$$

A: If the operator is equivalent to the Euler top

$$H = \sum_{i,j=1,2,3} \alpha_{ij} S_i S_j + \sum_{i=1,2,3} \beta_i S_i$$

where

$$S_3 = z\frac{d}{dz} - j, \;\; S_+ = z\left(2j - z\frac{d}{dz}\right), \;\; S_- = \frac{d}{dz}$$

are finite dimension representation of $SU(2)$

- When all solutions of the difference equation

$$a(z)\Psi(q^2 z) + d(z)\Psi(q^{-2} z) + \mathrm{v}(z)\Psi(z) = E\Psi(z)$$

are polynomials $\Psi(z) = \prod_l (z - z_l)$?

Setting $z = q^n$ we obtain solvable discrete equation

$$a_n \psi_{n+1} + d_n \psi_{n-1} + \mathrm{v}_n \psi_n = E\psi_n$$

- Lie group → quantum group $SL(2) \to SL_q(2)$

$$\{\mathbf{1}, S_x, S_y, S_z\} \to \{A, B, C, D\}$$

$$[S_a, S_b] = \mathrm{i}\epsilon^{abc} S_c$$

$$AB = qBA,\ BD = qDB,$$
$$DC = qCD,\ CA = qAC,$$
$$AD = 1,\ [B, C] = \frac{A^2 - D^2}{q - q^{-1}}$$

- Universal R-matrix, obeying Yang-Baxter equation

$$R(u) = \begin{bmatrix} \frac{uA-u^{-1}D}{q-q^{-1}} & C \\ B & \frac{uD-u^{-1}A}{q-q^{-1}} \end{bmatrix}$$

Magnetic translations embedded into $SL_q(2)$

- Hamiltonian happens to be equal

$$H = \sum_{\mathbf{nm}} t_{\mathbf{nm}} c^{\dagger}_{\mathbf{n}} c_{\mathbf{m}} = \mathrm{Tr}\left[\sigma_1 R\right] = B + C$$

- How did we find this?
- Embedding

$$T_{\mathbf{n}} T_{\mathbf{m}} = q^{-\mathbf{n}\times\mathbf{m}} T_{\mathbf{n}+\mathbf{m}}$$

$$\begin{aligned} T_{-x} + T_{-y} &= B, & T_x + T_y &= C, \\ T_{-y} T_x &= q^{-1} A^2, & T_{-x} T_y &= q D^2 \end{aligned} \quad (1)$$

Strings

- How to solve the Bethe Ansatz equations?

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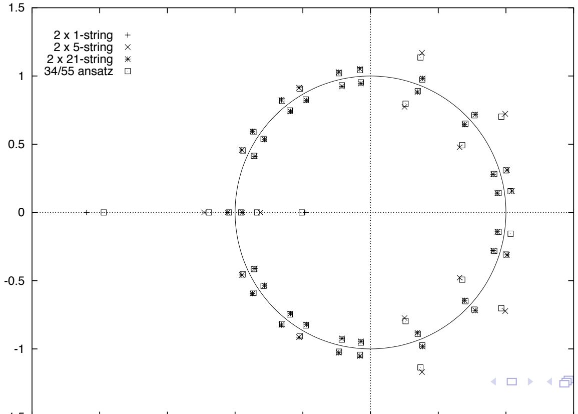
STRINGS

- How to solve the Bethe Ansatz equations?

$$\frac{z_l^2+q}{qz_l^2+1} = (-1)^P \prod_{m\neq l}^{Q-1} \frac{qz_l - z_m}{z_l - qz_m}$$

- At large Q solutions consist of collections of *strings*

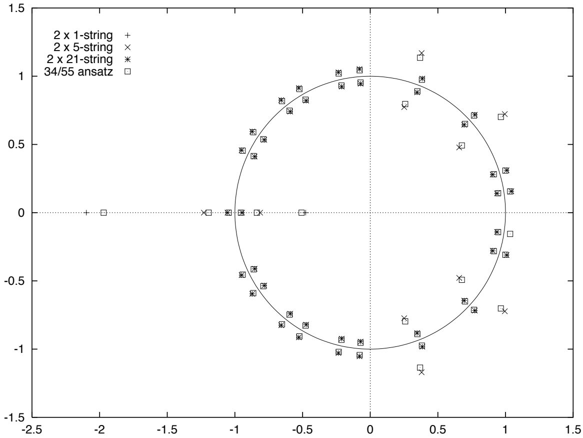
A string of spin l centered at x_l is a set roots of unity $z^{(l)} = x_l \times \{e^{i\pi k/l}\}, \quad k = 1, \ldots, l$



String and Hall conductance

- The length of the longest string of a given band is the **Hall conductance** of the band:

$$(2l+1)_{\text{max}} = |\text{Hall conductance}|$$



- Each solution is labeled by a content of strings $\{l_j, l_{j-1}, \ldots\}$
- The length of the longest string of a given band is the **Hall conductance** of the band:

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- The remaining roots of the state is a solution of the Bethe equation for the parent state

$$\Psi^{\text{daugther}}(z) \approx \prod_{m=-l}^{l}(z - x_l q_l^m)\Psi^{\text{parent}}(z)$$

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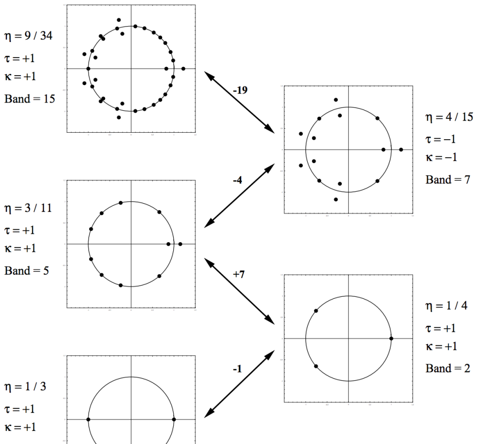


Fibonacci Tree: Example

- Golden mean $\Phi = \frac{\sqrt{5}-1}{2}$
- The sequence of rational approximants is given by ratios of subsequent Fibonacci numbers

$$\Phi_i = \frac{F_{i-1}}{F_i}: \qquad F_i = F_{i-2} + F_{i-1} = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$$

- The set of Hall conductances (lengths of strings) are again Fibonacci numbers: F_{k-1}. The wave function of this state is

$$\Psi\left(z|\Phi_k = \frac{F_{3k-1}}{F_{3k}}\right) \approx \prod_{n=0}^{k-1} \prod_{j=-\frac{1}{2}(F_{3n}-1)}^{\frac{1}{2}(F_{3n}-1)} \left(z - e^{i\pi \frac{F_{3n-1}}{F_{3n}} j}\right)^2$$

Scaling Hypothesis (Abanov, Talstra, P.W.)

✓ A gap width scales along generations as Q^{-2} (proven)

► Mid band energy scales along generations

$$|E_{\text{daughter band}} - E_{\text{parent band}}| \sim Q^{-2+\Delta_{\text{band}}}$$

$$\Delta_{\text{band}} - \text{scaling dimension - function of flux and chemical potential}$$

$$\eta = \frac{4}{15} : \begin{cases} \sigma(k) &= 4, -11, 4, 4, 4, -11, 4, 4, 4, -11, 4, 4, 4, -11, 4 \\ \sigma_k &= 4, -7, -3, 1, 5, -6, -2, 2, 6, -5, -1, 3, 7, 4 \end{cases}$$

$$\eta_6 = \frac{3}{11} : \begin{cases} \sigma(k) &= 4, -7, 4, 4, -7, 4, -7, 4, 4, -7, 4, \\ \sigma_k &= 4, -3, 1, 5, -2, 2, -5, -1, 3, -4 \end{cases}$$

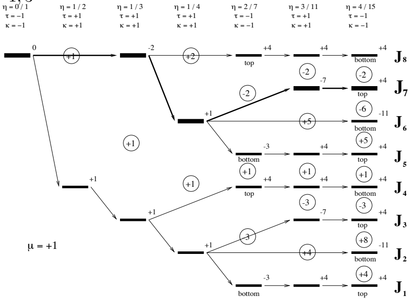
$$\eta_5 = \frac{2}{7} : \begin{cases} \sigma(k) &= -3, 4, -3, 4, -3, 4, -3 \\ \sigma_k &= -3, 1, -2, 2, -1, 3 \end{cases}$$

$$\eta_4 = \frac{1}{4} : \begin{cases} \sigma(k) &= 1, 1, -3, 1 \\ \sigma_k &= 1, 2, -1 \end{cases}$$

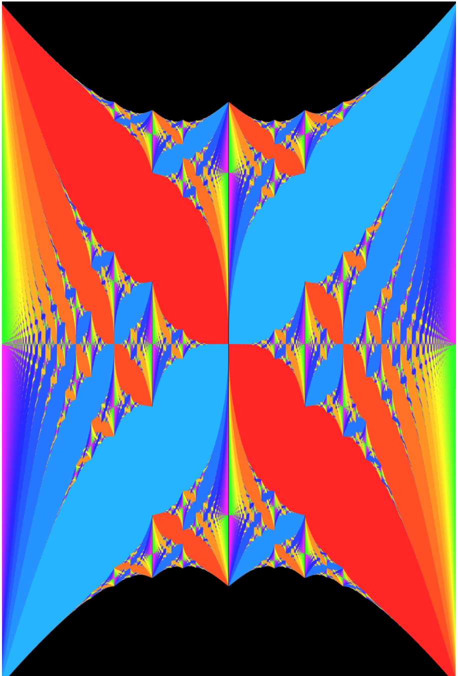
$$\eta_3 = \frac{1}{3} : \begin{cases} \sigma(k) &= 1, -2, 1 \\ \sigma_k &= 1, -1 \end{cases}$$

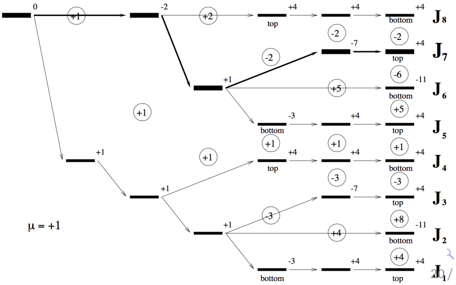
$$\eta_2 = \frac{1}{2} : \begin{cases} \sigma(k) &= 1, -1 \\ \sigma_k &= 1 \end{cases}$$

FIGURES



Guided tours inside the Butterfly

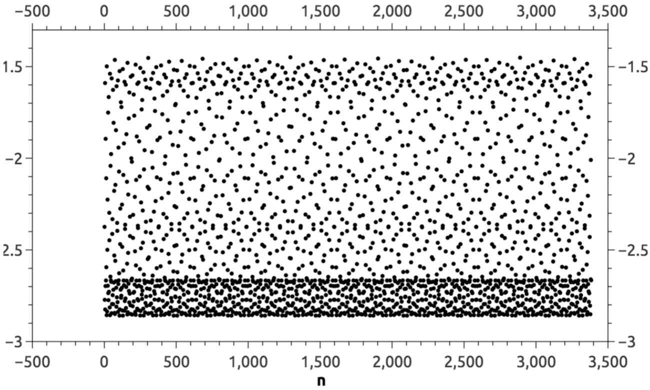




Scaling Dimensions

- Once we know the dimensions, we know physics of singular continuum state of matter e.g., a.c. conductance, specific heat, etc. of

- The main problem: what are the anomalous dimensions remains unsolved!



V. Nabokov "Gift" Chapter 4

Truth bends her head to fingers curved cupwise;
And with a smile and care
Examines something she is holding there
Concealed by her from our eyes.