Alternating Sign Triangles

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Joint with Roger E. Behrend and Ilse Fischer

Plan of the Talk

1. A brief history of Alternating Sign Matrices (ASMs)
2. Diagonally Symmetric and Antisymmetric ASMs (DASASMs)
3. Extreme DASASMs and Alternating Sign Triangles (ASTs)
4. QASTs and OOSASMs
5. Six-vertex models on triangular subsets of the square grid
6. Proof ideas

Dodgson condensation

- Let M be an $n \times n$ matrix (of integers, say).
- For $I, J \subset \{1, \ldots, n\}$ with $\#I = \#J$, let M_J^I denote the matrix with the rows in I and columns in J removed.
- Then, the determinant of M can be computed recursively as

$$\det M = \frac{\det M_1^1 \ \det M_n^n - \det M_n^1 \ \det M_1^n}{\det M_{1,n}^{1,n}},$$

where $\det \varnothing = 1$.
- Discovered by Rev. C. L. Dodgson, *Proceedings of the Royal Society of London*, **15** (1866-1867), 150-155.

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- Rev. C. L. Dodgson = Lewis Carroll!

The λ-determinant

A generalisation

$$\det_\lambda M := \frac{\det_\lambda M_1^1 \ \det_\lambda M_n^n + \lambda \det_\lambda M_n^1 \ \det_\lambda M_1^n}{\det_\lambda M_{1,n}^{1,n}},$$

with $\det_\lambda \varnothing = 1$ and $\det_\lambda (x) = x$.
We want to give an explicit formula for $\det_\lambda M$.

Definition

An *alternating sign matrix* (ASM) of order n is an $n \times n$ matrix A with entries in $\{0, \pm 1\}$ such that all row and column sums of A are 1 and nonzero entries in every row and column alternate in sign.

Alternating Sign Matrices

For A an ASM, let $n_-(A)$ be the number of -1's in A and $\mathrm{inv}(A) = \sum_{i<k,j>\ell} A_{i,j}A_{k,\ell}$ be the number of inversions of A.

Theorem (Robbins and Rumsey, '86)

Let M be an $n \times n$ matrix. Then

$$\det_\lambda M = \sum_{A \in \mathcal{A}_n} \lambda^{\mathrm{inv}(A)-n_-(A)} (1+\lambda)^{n_-(A)} \prod_{1\le i,j\le n} M_{i,j}^{A_{i,j}}.$$

The number of ASMs of order n is given by

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Proved by Zeilberger in '95 and Kuperberg in '96.
The sequence starts $1, 2, 7, 42, 429, \ldots$

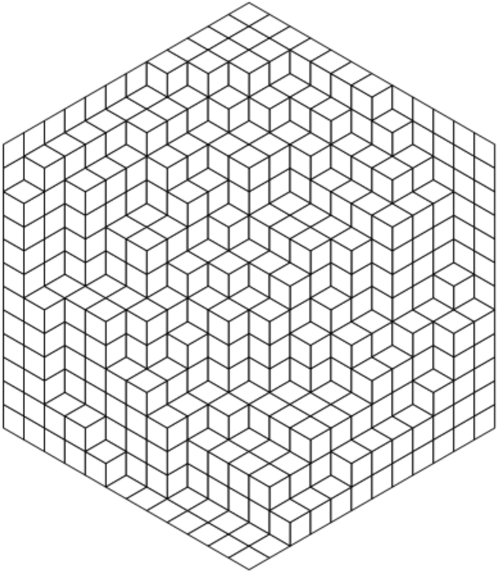
Example: $n = 3$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

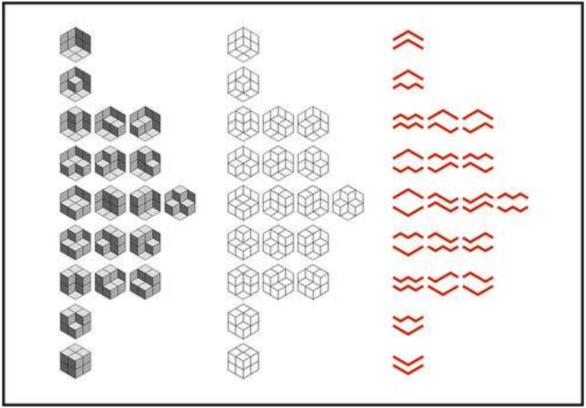
$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

Digression: Plane Partitions



Digression: Plane Partitions



Digression: Plane Partitions

Theorem (MacMahon 1916)

The number of plane partitions contained in an $m \times n \times p$ box is given by

$$\prod_{i=1}^{m} \prod_{j=1}^{n} \prod_{k=1}^{p} \frac{i+j+k-1}{i+j+k-2}.$$

Digression: Symmetry classes

Several symmetry classes of plane partitions also have simple enumerations:

- Cyclically symmetric plane partitions (CSPP)
- Totally symmetric plane partitions (TSPP)
- Self-complementary plane partitions (SCPP)
- Cyclically symmetric self-complementary plane partitions (CSSCPP)
- Totally symmetric self-complementary plane partitions (TSSCPP)

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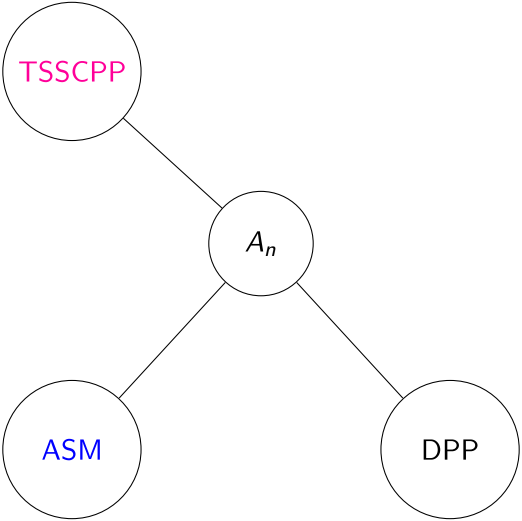
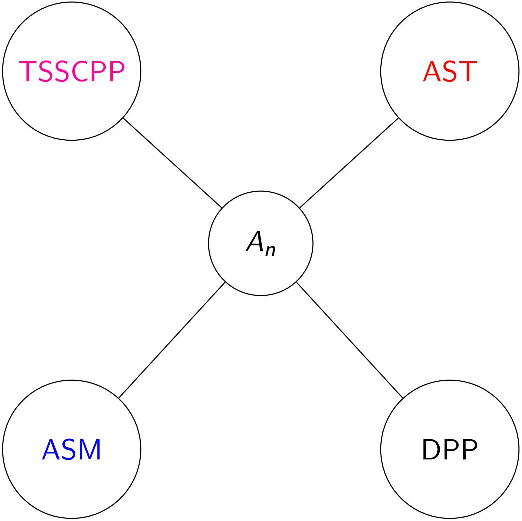
- Totally symmetric plane partitions (TSPP)
- Self-complementary plane partitions (SCPP)
- Cyclically symmetric self-complementary plane partitions (CSSCPP)
- Totally symmetric self-complementary plane partitions (TSSCPP)
- Number of TSSCPPs in a $2n \times 2n \times 2n$ box is A_n.

Objects with the same enumeration as ASMs



Objects with the same enumeration as ASMs



Symmetry Classes of ASMs

- Motivated by the study of plane partitions, R. Stanley suggested looking at symmetry classes.
- Experimentation by D. Robbins led to several classes with nice conjectured "round" formulas
- G. Kuperberg gave a proof for many classes using six-vertex model

Examples of symmetry classes

- VSASMs - Vertically Symmetric ASMs

$$\mathrm{vsasm}(2n+1) = \prod_{i=1}^{n} \frac{(6i-2)!}{(2n+2i)!}.$$

- VHSASMs - Vertically and Horizontally Symmetric ASMs

$$\mathrm{vhsasm}(4n+1) = \prod_{i=0}^{n-1} \frac{(3i+2)!(3n+3i)!}{(2n+i)!(3n+i)!},$$

$$\mathrm{vsasm}(4n+3) = \prod_{i=1}^{n} \frac{(3i-1)!(3n+3i)!}{(2n+i)!(3n+i+1)!}.$$

- DSASMs - Diagonally Symmetric ASMs (no formula known)

Examples of symmetry classes

- HTSASM - Half Turn Symmetric ASMs

$$\mathrm{htsasm}(2n) = (-3)^{\binom{n}{2}} A(n) \prod_{i,j=1}^{n} \frac{3(j-i)+2}{j-i+n}$$

- QTSASM - Quarter Turn Symmetric ASMs

$$\mathrm{qtsasm}(4n) = \mathrm{htsasm}(2n) A(n)^2.$$

- OSASM - DSASMs with zeros on the main diagonal

$$\mathrm{osasm}(2n) = \mathrm{vsasm}(2n+1)$$

DASASM

- DASASMs - Diagonally and Antidiagonally Symmetric ASMs
- The last "round" conjecture of Robbins from the 80s

Theorem (R. Behrend, I. Fischer, M. Konvalinka, *Adv. Math.* (2017))

The number of DASASMs of size $2n+1$ is given by

$$dasasm(2n+1) = \prod_{i=0}^{n} \frac{(3i)!}{(n+i)!}.$$

No formula for even DASASMs is known.

Example of size 13

$$
\begin{pmatrix}
{\color{red}0} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{blue}0} \\
0 & {\color{red}1} & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & {\color{blue}0} & 0 \\
0 & 0 & {\color{red}0} & 1 & 0 & 0 & 0 & -1 & 0 & 1 & {\color{blue}0} & 0 & 0 \\
1 & -1 & 1 & {\color{red}-1} & 1 & 0 & 0 & 0 & 0 & {\color{blue}-1} & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & {\color{red}-1} & 1 & 0 & 0 & {\color{blue}0} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & {\color{red}-1} & 0 & {\color{blue}1} & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & {\color{green}1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & {\color{blue}1} & 0 & {\color{red}-1} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & {\color{blue}0} & 0 & 0 & 1 & {\color{red}-1} & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & {\color{blue}-1} & 0 & 0 & 0 & 0 & 1 & {\color{red}-1} & 1 & -1 & 1 \\
0 & 0 & {\color{blue}0} & 1 & 0 & -1 & 0 & 0 & 0 & 1 & {\color{red}0} & 0 & 0 \\
0 & {\color{blue}0} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & {\color{red}1} & 0 \\
{\color{blue}0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & {\color{red}0}
\end{pmatrix}
$$

DASASM triangle of order 6

Look at the “fundamental region” of the DASASM.

$$
\begin{array}{ccccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\
 & & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & & \\
 & & & -1 & 1 & 0 & 0 & 0 & 0 & -1 & & & \\
 & & & & -1 & 1 & 0 & 0 & 0 & & & & \\
 & & & & & -1 & 0 & 1 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}
$$

DASASM triangles of size 5

$$\begin{array}{ccccc} 1&0&0&0&0\\ &1&0&0&\\ &&1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 1&0&0&0&0\\ &0&1&0&\\ &&-1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 1&0&0&0&0\\ &0&0&1&\\ &&1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 0&1&0&0&0\\ &0&0&0&\\ &&1&& \end{array}$$

$$\begin{array}{ccccc} 0&1&0&0&0\\ &-1&1&0&\\ &&-1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 0&1&0&0&0\\ &-1&0&1&\\ &&1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 0&0&1&0&0\\ &1&0&0&\\ &&-1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 0&0&1&0&0\\ &1&-1&1&\\ &&1&& \end{array}$$

$$\begin{array}{ccccc} 0&0&1&0&0\\ &0&0&1&\\ &&-1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 0&0&0&1&0\\ &1&0&-1&\\ &&1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 0&0&0&1&0\\ &0&1&-1&\\ &&-1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 0&0&0&1&0\\ &0&0&0&\\ &&1&& \end{array}$$

$$\begin{array}{ccccc} 0&0&0&0&1\\ &1&0&0&\\ &&1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 0&0&0&0&1\\ &0&1&0&\\ &&-1&& \end{array} \;\Bigg|\; \begin{array}{ccccc} 0&0&0&0&1\\ &0&0&1&\\ &&1&& \end{array}$$

Enumeration according to the number of zeros at the boundary

$2n+1$	1	2	3	4	5	6	7	8
3	2	1						
5		7	6	2				
7			42	48	30	6		
9				429	594	528	198	33

Enumeration according to the number of zeros at the boundary

$2n+1$	1	2	3	4	5	6	7	8
3	2	1						
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7			42	48	30	6		
9				429	594	528	198	33

A_{n+1} - number of ASMs of size $n+1$

vhsasm$(2n+3)$ - number of vertically and horizontally symmetric ASMs (VHSASMs) of size $2n+3$

Enumeration according to the number of +1's at the boundary

$2n+1$	0	1	2	3	4	5
3	1	0	2			
5	2	6	2	5		
7	13	32	43	18	20	
9	128	377	498	463	184	132

Enumeration according to the number of +1's at the boundary

$2n+1$	0	1	2	3	4	5
3	1	0	2			
5	2	6	2	5		
7	13	32	43	18	20	
9	128	377	498	463	184	132

cspp(n) - number of cyclically symmetric plane partitions (CSPPs) in an $n \times n \times n$ box.

Enumeration according to the number of −1's at the boundary

$2n+1$	0	1	2	3	4
3	2	1			
5	7	6	2		
7	40	53	26	7	
9	395	666	495	184	42

Enumeration according to the number of −1's at the boundary

$2n+1$	0	1	2	3	4
3	2	1			
5	7	6	2		
7	40	53	26	7	
9	395	666	495	184	42

A_n - number of ASMs of size n

DASASM triangles of size 7 with 3 −1's

$$\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & -1 & 1 & 0 & 0 & 0 & \\ & & -1 & 1 & 0 & & \\ & & & -1 & & & \end{array}$$

$$\begin{array}{ccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 1 & -1 & 1 & 0 & \\ & & -1 & 1 & -1 & & \\ & & & -1 & & & \end{array}$$

$$\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & -1 & 0 & 0 & 1 & 0 & \\ & & 0 & 1 & -1 & & \\ & & & -1 & & & \end{array}$$

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 & 0 & 0 & \\ & & -1 & 1 & -1 & & \\ & & & -1 & & & \end{array}$$

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 & -1 & \\ & & 0 & 1 & -1 & & \\ & & & -1 & & & \end{array}$$

$$\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 & \\ & & -1 & 1 & -1 & & \\ & & & -1 & & & \end{array}$$

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 0 & -1 & \\ & & -1 & 1 & 0 & & \\ & & & -1 & & & \end{array}$$

Inequalities for boundaries

For $\alpha \in \{0, 1, -1\}$, let

$$N_\alpha(A) = \# \text{ of } \alpha\text{'s along the portions of the diagonals of } A \text{ that lie in the fundamental triangle.}$$

Proposition

For any $(2n+1) \times (2n+1)$ DASASM A, *the statistics* $N_\alpha(A)$ *lie in the following intervals.*

1. $0 \leq N_{-1}(A) \leq n$
2. $0 \leq N_1(A) \leq n+1$
3. $n \leq N_0(A) \leq 2n$

All inequalities are sharp.

Remarks

Let A be a DASASM triangle of order n.

- We have $N_{-1}(A) = n$ if and only if the sum of entries of A is minimal.
- We have $N_{-1}(A) = n$ if and only if each column sum of A is zero.
- We have $N_{-1}(A) = n$ if and only if, for each row except the bottom row, the sum of entries is 1 when disregarding the left and right boundary entries.

Alternating Sign Triangles

Delete the diagonals of such an odd DASASM-triangle A.

Definition

An *alternating sign triangle* (AST) of order n is a triangular array $(a_{i,j})_{1 \leq i \leq n, i \leq j \leq 2n-i}$ in which each entry is 0, 1 or -1 and the following conditions are fulfilled.

1. The non-zero entries alternate in each row and each column.
2. All row sums are 1.
3. The topmost non-zero entry of each column is 1 (if it exists).

The set of order n AST's is denoted by $\mathrm{ast}(n)$.

Examples of size 3

$$\begin{array}{ccccc} 1&0&0&0&0\\ &1&0&0&\\ &&1&& \end{array} \quad\Bigg|\quad \begin{array}{ccccc} 0&0&0&1&0\\ &1&0&0&\\ &&1&& \end{array} \quad\Bigg|\quad \begin{array}{ccccc} 0&0&0&0&1\\ &1&0&0&\\ &&1&& \end{array}$$

$$\begin{array}{ccccc} 1&0&0&0&0\\ &0&0&1&\\ &&1&& \end{array} \quad\Bigg|\quad \begin{array}{ccccc} 0&1&0&0&0\\ &0&0&1&\\ &&1&& \end{array} \quad\Bigg|\quad \begin{array}{ccccc} 0&0&0&0&1\\ &0&0&1&\\ &&1&& \end{array}$$

$$\begin{array}{ccccc} 0&0&1&0&0\\ &1&-1&1&\\ &&1&& \end{array}$$

Refined enumeration

For $A \in \mathrm{ASM}(n)$, let $\mu(A)$ be the number of -1's in A.
For $T \in \mathrm{AST}(n)$, let $\mu_\nabla(T)$ be the number of -1's in T.

Theorem (A., R. Behrend, I. Fischer, arXiv:1611.03823)

For all integers n and m, we have

$$\#\{A \in \mathrm{ASM}(n) \mid \mu(A) = m\} = \#\{T \in \mathrm{AST}(n) \mid \mu_\nabla(T) = m\}.$$

Corollary

The number of alternating sign triangles of size n is A_n.

The CSPP case: $N_1(A) = n + 1$

Definition

A *quasi alternating sign triangle* (QAST) of order n is a triangular array $(a_{i,j})_{1 \le i \le n, i \le j \le 2n-i}$ in which each entry is 0, 1 or -1 and the following conditions are fulfilled.

1. The non-zero entries alternate in each row and column.
2. The row sums are 1 for rows $1, 2, \ldots, n-1$, and 0 or 1 for row n.
3. The topmost non-zero entry in each column is 1 (if it exists).

The set of order n QASTs is denoted by $\mathrm{QAST}(n)$.

Examples of size 2

$$\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ & 0 & & & 1 & & & 0 & & & 0 & & & 1 & \end{array}$$

Enumeration of QASTs

Theorem (A., R. Behrend, I. Fischer, arXiv:1611.03823)

The number of QASTs of order n is given by

$$\# \mathrm{QAST}(n) = cspp(n) = \prod_{i=0}^{n-1} \frac{(3i+2)(3i)!}{(n+i)!}.$$

The OOSASM Case: $N_0(A) = 2n$

- OOSASMs of order $2n+1$ are DASASMs of order $2n+1$ with $2n$ zeroes on the boundary.
- In this case, the only non-zero entry on the boundary of the DASASM triangle is at the bottom.
- As close as one can get to OOSASMs for odd order.

Examples of size 4

$$
\begin{array}{ccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & \\
 & & 0 & 1 & 0 & & \\
 & & & -1 & & &
\end{array}
\quad\Bigg|\quad
\begin{array}{ccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 0 & \\
 & & 0 & 0 & 0 & & \\
 & & & -1 & & &
\end{array}
\quad\Bigg|\quad
\begin{array}{ccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & 0 & 1 & 0 & 0 & 0 & \\
 & & 0 & 0 & 0 & & \\
 & & & -1 & & &
\end{array}
$$

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & \\
 & & 0 & 1 & 0 & & \\
 & & & -1 & & &
\end{array}
\quad\Bigg|\quad
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 0 & \\
 & & 0 & 0 & 0 & & \\
 & & & -1 & & &
\end{array}
\quad\Bigg|\quad
\begin{array}{ccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 1 & 0 & \\
 & & 0 & 0 & 0 & & \\
 & & & -1 & & &
\end{array}
$$

Enumeration of OOSASMs

Theorem (A., R. Behrend, I. Fischer, arXiv:1611.03823)

The number of OOSASMs of order $4n-1$ is given by

$$\#\,\mathrm{OOSASM}(4n-1) = \prod_{i=0}^{n-1} \frac{(3i+2)!(3n+3i)!}{(2n+i)!(3n+i)!},$$

and this is also the number of $(4n+1)\times(4n+1)$ VHSASMs.
The number of OOSASMs of order $4n+1$ is given by

$$\#\,\mathrm{OOSASM}(4n+1) = \prod_{i=1}^{n} \frac{(3i-1)!(3n+3i)!}{(2n+i)!(3n+i+1)!},$$

and this is also the number of $(4n+3)\times(4n+3)$ VHSASMs.

DASASM Triangles

Let an odd DASASM triangle A of order n be a triangular array

$$\begin{array}{ccccccccc}
A_{11} & A_{12} & A_{13} & \ldots & A_{1,n+1} & \ldots & A_{1,2n-1} & A_{1,2n} & A_{1,2n+1} \\
 & A_{22} & A_{23} & \ldots & A_{2,n+1} & \ldots & A_{2,2n-1} & A_{2,2n} & \\
 & & \ddots & & \vdots & & \iddots & & \\
 & & & A_{nn} & A_{n,n+1} & A_{n,n+2} & & & \\
 & & & & A_{n+1,n+1}, & & & &
\end{array}$$

such that each entry is 0, 1 or -1 and, for each $i = 1, \ldots, n+1$, the nonzero entries along the sequence

$$\begin{array}{ccccc}
A_{1i} & & & & A_{1,2n+2-i} \\
A_{2i} & & & & A_{2,2n+2-i} \\
\vdots & & & & \vdots \\
A_{i-1,i} & & & & A_{i-1,2n+2-i} \\
A_{ii} & A_{i,i+1} & \ldots & A_{i,2n+1-i} & A_{i,2n+2-i}
\end{array}$$

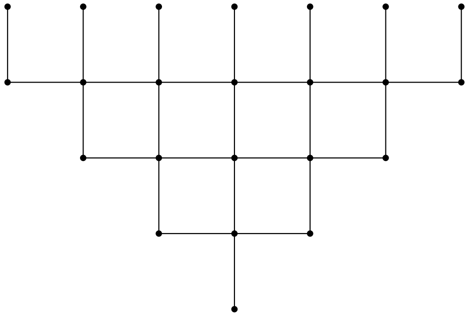
alternate in sign and have a sum of 1.

DASASM triangle of order 6

0	0	0	1	0	0	0	0	0	0	0	0	0
	1	0	−1	0	0	0	1	0	0	0	0	
		0	1	0	0	0	−1	0	1	0		
			−1	1	0	0	0	0	−1			
				−1	1	0	0	0				
					−1	0	1					
						1						

Six-vertex model configurations

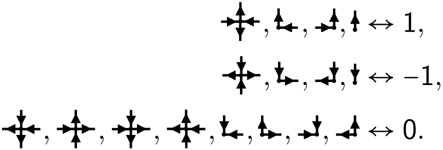
Define a grid graph on a triangle T_n as



Now assign arrows to all edges.

Six-vertex model configurations

- Each top edge is directed upwards.
- For every bulk vertex, two edges are directed inwards, and two, outwards.
- Bijection with DASASM triangles



Running example

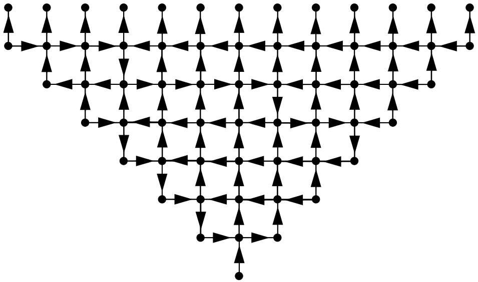
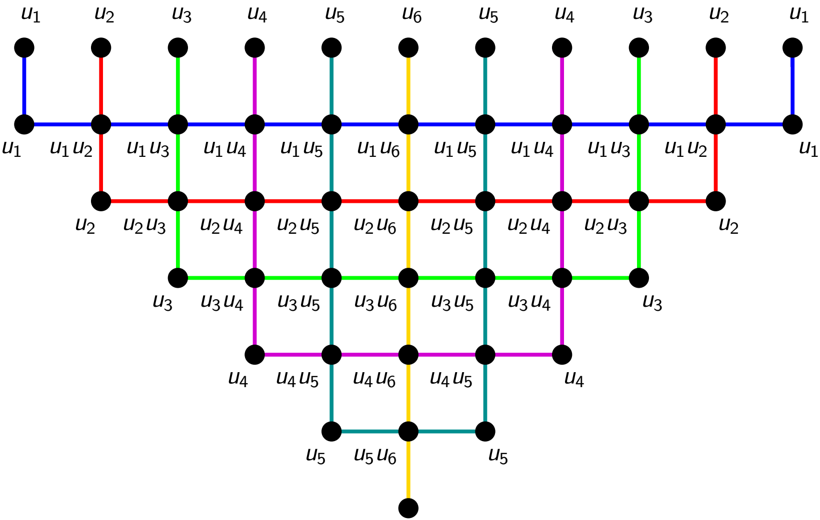


Figure: Triangular six-vertex configuration of order 6

Weights of configurations

- Given indeterminates $u_1, \ldots, u_{n+1}$ (spectral parameters) and q.
- The left boundary weights depend on the *left boundary constants* $\alpha_L, \beta_L, \gamma_L, \delta_L$.
- The right boundary weights depend on the *right boundary constants* $\alpha_R, \beta_R, \gamma_R, \delta_R$.
- An edge belonging to the j'th path has label u_j
- The label of a vertex is the product of the spectral parameters of the paths that contain the vertex

Illustration



Weights of configurations

- The weight $W(v, u)$ depends purely on the local configuration of edges incident to the vertex.
- The weights of a configuration is then the product of the weights of the vertices.
- Let $\bar{u} = u^{-1}$ and $\sigma(u) = u - \bar{u}$.

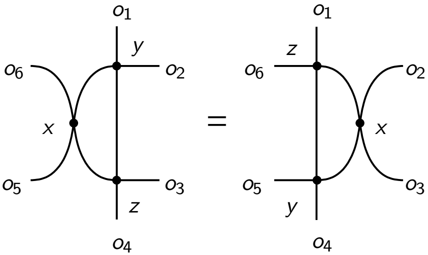
Bulk weights	Left boundary weights	Right boundary weights
W(, u) $= 1$	W(, u) $= \frac{\beta_L q u + \gamma_L \bar{q}\bar{u}}{\sigma(q^2)}$	W(, u) $= \frac{\beta_R q \bar{u} + \gamma_R \bar{q} u}{\sigma(q^2)}$
W(, u) $= 1$	W(, u) $= \frac{\gamma_L q u + \beta_L \bar{q}\bar{u}}{\sigma(q^2)}$	W(, u) $= \frac{\gamma_R q \bar{u} + \beta_R \bar{q} u}{\sigma(q^2)}$
W(, u) $= \frac{\sigma(q^2 u)}{\sigma(q^4)}$	W(, u) $= \alpha_L \frac{\sigma(q^2 u^2)}{\sigma(q^2)}$	
W(, u) $= \frac{\sigma(q^2 u)}{\sigma(q^4)}$	W(, u) $= \delta_L \frac{\sigma(q^2 u^2)}{\sigma(q^2)}$	
W(, u) $= \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$		W(, u) $= \alpha_R \frac{\sigma(q^2 \bar{u}^2)}{\sigma(q^2)}$
W(, u) $= \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$		W(, u) $= \delta_R \frac{\sigma(q^2 \bar{u}^2)}{\sigma(q^2)}$

The running example has weight given by

$$\frac{\delta_L^2\delta_R}{\sigma(q^2)^6\sigma(q^4)^7}\sigma(q^2u_1^2)(\beta_L\, qu_2+\gamma_L\,\bar{q}\bar{u}_2)\sigma(q^2u_3^2)\sigma(q^2\bar{u}_1^2)$$
$$\times(\beta_R\, q\bar{u}_2+\gamma_R\,\bar{q}u_2)(\beta_R\, q\bar{u}_3+\gamma_R\,\bar{q}u_3)$$
$$\times\sigma(q^2u_1u_2)\sigma(q^2u_1u_3)\sigma(q^2u_1u_4)\sigma(q^2\bar{u}_1\bar{u}_2)$$
$$\times\sigma(q^2\bar{u}_2\bar{u}_3)\sigma(q^2\bar{u}_2\bar{u}_4)\sigma(q^2u_3u_4).$$

Yang-Baxter equation

If $xyz = q^2$, then



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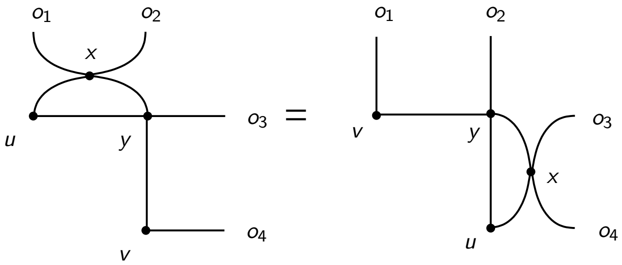
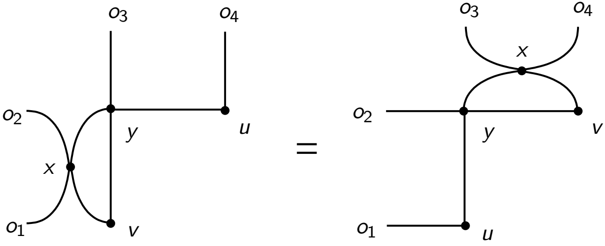
Left reflection equation

If $x = q^2 \bar{u} v, y = uv$, then



Right reflection equation

If $x = q^2 \bar{u} v, y = \bar{u}\bar{v}$, then



Symmetry of the partition function

The *partition function* is given by

$$Z_n(u_1, \dots, u_n; u_{n+1}) = \sum_C \mathrm{wt}(C),$$

where the sum runs over all DASASM configurations C and $\mathrm{wt}(C)$ is the product of the vertex weights $W(\cdot, \cdot)$.

Proposition

The partition function $Z_n(u_1, \dots, u_n; u_{n+1})$ is symmetric in $u_1, \dots, u_n$.

Boundary specialisation for ASTs

We want to count DASASM triangles of order n with n -1's on the boundary. Choose weights as follows.

α_L	β_L	γ_L	δ_L	α_R	β_R	γ_R	δ_R
0	$-\bar{p}\bar{q}$	pq	1	0	$-p\bar{q}$	$\bar{p}q$	1

Then the local weights satisfy

$$W(c,1)\Big|_{q=e^{i\pi/6}} = 1, \quad \text{for a bulk vertex in } c,$$

$$\text{and} \quad W(c,1) = 1, \quad \text{for a boundary vertex in } c.$$

Thus, one needs to evaluate $Z(u_1,\ldots,u_n,1)$ at $q = e^{i\pi/6}$.

Partition Function of ASTs

Let $Z_n(u_1,\dots,u_n;u_{n+1})_{\mathrm{AST}}$ be the partition function with the AST boundary weights.
Then

$$Z_n(1,\dots,1;1)_{\mathrm{AST}}\Big|_{p=1,q=e^{\frac{i\pi}{6}}} = \#\,\mathrm{AST}(n).$$

Theorem (A., R. Behrend, I. Fischer, arXiv:1611.03823)

The partition function $Z_n(u_1,\dots,u_n;u_{n+1})_{\mathrm{AST}}$ is given by

$$\frac{\prod_{j=1}^{n}\sigma(\bar{p}u_j)\,\prod_{i=1}^{n}\prod_{j=1}^{n+1}\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(q^2)^{2n}\,\sigma(q^4)^{n(n-1)}\,\prod_{i=1}^{n}\sigma(u_i\bar{u}_{n+1})\,\prod_{1\le i<j\le n}\sigma(u_i\bar{u}_j)^2}$$

$$\times\det_{1\le i,j\le n+1}\left(\begin{cases}\dfrac{1}{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}, & i\le n\\[2ex] 1-\dfrac{\sigma(\bar{u}_{n+1}u_j)}{\sigma(\bar{p}u_j)}, & i=n+1\end{cases}\right)$$

Proof Ideas

- Establish properties of the partition function:
  1. Yang-Baxter equation
  2. Left and Right reflection equations
- Establish symmetry.
- For special values of the spectral parameter, establish recurrences.
- Show that any Laurent polynomial of appropriate degrees with these properties is unique.
- Finally, show that the determinant satisfy these properties.

Specialisation at $u_{n+1} = p$

Corollary

The partition function $Z_n(u_1, \dots, u_n; p)_{\mathrm{AST}}$ *is given by*

$$\frac{\prod_{i=1}^{n} \sigma(q^2 p u_i)\sigma(q^2 \bar{p}\bar{u}_i) \prod_{i,j=1}^{n} \sigma(q^2 u_i u_j)\sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(q^2)^{2n} \sigma(q^4)^{n(n-1)} \prod_{1 \le i < j \le n} \sigma(u_i \bar{u}_j)^2}$$

$$\times \det_{1 \le i,j \le n} \left(\frac{1}{\sigma(q^2 u_i u_j)\, \sigma(q^2 \bar{u}_i \bar{u}_j)} \right)$$

Schur functions

- Schur functions s_λ form a basis in the ring of symmetric functions indexed by partitions λ.
- Let λ be of length $\ell(\lambda) \leq k$, and let $x_1, \ldots, x_k$ be commuting variables.
- Let $\mathrm{SSYT}_\lambda(k)$ be the set of semistandard Young tableaux of shape λ with entries from $\{1, \ldots, k\}$. Then

$$s_\lambda(x_1, \ldots, x_k) = \sum_{T \in \mathrm{SSYT}_\lambda(k)} x_1^{\#(1,T)} \ldots x_k^{\#(k,T)}$$

Schur functions

- A determinantal formula for Schur functions is

$$s_\lambda(x_1,\ldots,x_k)=\frac{\det\limits_{1\le i,j\le k}\left(x_i^{\lambda_j+k-j}\right)}{\prod\limits_{1\le i<j\le k}\left(x_i-x_j\right)}.$$

- The number of semistandard Young tableaux is

$$s_\lambda(\underbrace{1,\ldots,1}_{k})=\#\mathrm{SSYT}_\lambda(k)=\frac{\prod_{1\le i<j\le k}(\lambda_i-\lambda_j-i+j)}{\prod_{i=1}^{k-1} i!}.$$

Enumeration of ASTs

Theorem (Okada 2006, Stroganov 2006)

At $q = e^{\frac{i\pi}{6}}$, $Z_n(u_1, \ldots, u_n; p)_{\mathrm{AST}}$ *is given by*

$$3^{-\binom{n+1}{2}} \prod_{i=1}^{n} (p^2 u_i^2 + 1 + \bar{p}^2 \bar{u}_i^2) s_{(n-1,n-1,n-2,n-2,\ldots,1,1)}(u_1^2, \bar{u}_1^2, \ldots, u_n^2, \bar{u}_n^2).$$

We thus obtain the same formula as for the ASMs.

Refined enumeration

Suppose $T \in \mathrm{AST}(n)$. Then let the inversion number of an T be

$\mathrm{inv}_{\triangledown}(T) = \frac{1}{2}\Big(\#$  $\in C(T) + \#$  $\in C(T)\Big),$

and $\mu_{\triangledown}(T)$ be the number of -1's in an AST.

Theorem (A., R. Behrend, I. Fischer, arXiv:1611.03823)

The joint distribution of the statistics μ *and* inv *on the set* $\mathrm{ASM}(n)$ *is equal to the joint distribution of the statistics* $\mu_{\triangledown}$ *and* $\mathrm{inv}_{\triangledown}$ *on the set* $\mathrm{AST}(n)$*, i.e., for all integers* m *and* i *we have*

$$\#\{A \in \mathrm{ASM}(n) \mid \mu(A) = m, \mathrm{inv}(A) = i\} = \#\{T \in \mathrm{AST}(n) \mid \mu_{\triangledown}(T) = m, \mathrm{inv}_{\triangledown}(T) = i\}.$$

Boundary specialisation for QASTs

We choose the boundary parameters as follows.

α_L	β_L	γ_L	δ_L	α_R	β_R	γ_R	δ_R
0	pq	$-\bar{p}\bar{q}$	1	0	$\bar{p}q$	$-p\bar{q}$	1

Then the relevant partition function is $Z_n^{\uparrow}(u_1, \ldots, u_n; u_{n+1})_{\mathrm{QAST}}$.

$$Z_n^{\uparrow}(1, \ldots, 1; 1)_{\mathrm{QAST}}\Big|_{p=1, q=e^{\frac{i\pi}{6}}} = \#\,\mathrm{QAST}(n).$$

Determinant formula

Theorem (A., R. Behrend, I. Fischer, arXiv:1611.03823)

We have the following determinant formula for the partition function $Z_n^{\uparrow}(u_1,\ldots,u_n;u_{n+1})_{\mathrm{QAST}}$.

$$\frac{\prod_{j=1}^{n}\sigma(\bar{p}u_j)\prod_{i=1}^{n}\prod_{j=1}^{n+1}\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(q^2)^{2n}\sigma(q^4)^{n^2}\prod_{i=1}^{n}\sigma(u_i\bar{u}_{n+1})\prod_{1\le i<j\le n}\sigma(u_i\bar{u}_j)^2}$$
$$\times\det_{1\le i,j\le n+1}\left(\begin{cases}\frac{1}{\sigma(q^2u_iu_j)}+\frac{1}{\sigma(q^2\bar{u}_i\bar{u}_j)}, & i\le n\\ \frac{\sigma(\bar{p}u_{n+1})}{\sigma(\bar{p}u_j)}, & i=n+1\end{cases}\right)$$

Specialisation at $u_{n+1} = p$

Corollary

The partition function $Z_n^{\uparrow}(u_1, \ldots, u_n; p)_{\mathrm{QAST}}$ *is given by*

$$\frac{\prod_{i=1}^{n} \sigma(q^2 p u_i)\sigma(q^2 \bar{p}\bar{u}_i) \prod_{i,j=1}^{n} \sigma(q^2 u_i u_j)\sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(q^2)^{2n} \sigma(q^4)^{n^2} \prod_{1 \le i < j \le n} \sigma(u_i \bar{u}_j)^2} \times \det_{1 \le i,j \le n} \left(\frac{1}{\sigma(q^2 u_i u_j)} + \frac{1}{\sigma(q^2 \bar{u}_i \bar{u}_j)} \right)$$

Enumeration of QASTs

Theorem (Okada 2006, Stroganov 2006)

At $q = e^{\frac{i\pi}{6}}$, $Z_n^{\uparrow}(u_1, \ldots, u_n; p)_{\mathrm{QAST}}$ *is given by*

$$3^{-\binom{n+1}{2}} \prod_{i=1}^{n} (p^2 u_i^2 + 1 + \bar{p}^2 \bar{u}_i^2) s_{(n,n-1,n-1,n-2,n-2,\ldots,1,1)}(u_1^2, \bar{u}_1^2, \ldots, u_n^2, \bar{u}_n^2)$$

When $u_i = p = 1$, *we obtain*

$$cspp(n) = 3^{-\binom{n}{2}} s_{(n,n-1,n-1,n-2,n-2,\ldots,1,1)}(1^{2n}) = \prod_{i=0}^{n-1} \frac{(3i+2)(3i)!}{(n+i)!}$$

Boundary specialisation for OOSASMs

We choose the boundary parameters as follows.

$\alpha_L = \alpha_R$	$\beta_L = \beta_R$	$\gamma_L = \gamma_R$	$\delta_L = \delta_R$
1	0	0	1

Then Therefore,

$$Z_n(1,\ldots,1;1)_{\mathrm{OOSASM}}\Big|_{q=e^{\frac{i\pi}{6}}} = \#\,\mathrm{OOSASM}(2n+1).$$

Pfaffian formula

Let

$$P_m(u_1,\ldots,u_{2m}) = \sigma(q^4)^{-(m-1)2m} \prod_{1\le i<j\le 2m} \frac{\sigma(q^2 u_i u_j)\,\sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \times \operatorname*{Pf}_{1\le i<j\le 2m}\left(\frac{\sigma(u_i\bar{u}_j)}{\sigma(q^2 u_i u_j)\,\sigma(q^2 \bar{u}_i \bar{u}_j)}\right)$$

and

$$Q_m(u_1,\ldots,u_{2m-1}) = \sigma(q^4)^{-(m-1)(2m-1)} \prod_{1\le i<j\le 2m-1} \frac{\sigma(q^2 u_i u_j)\,\sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \times \operatorname*{Pf}_{1\le i<j\le 2m}\left(\begin{cases} \frac{\sigma(u_i\bar{u}_j)}{\sigma(q^2 u_i u_j)} + \frac{\sigma(u_i\bar{u}_j)}{\sigma(q^2 \bar{u}_i \bar{u}_j)}, & j<2m \\ 1, & j=2m \end{cases}\right).$$

Pfaffian formula

Theorem (A., R. Behrend, I. Fischer, arXiv:1611.03823)

We have the following Pfaffian formula for the OOSASM *partition function*

$$Z_n(u_1,\ldots,u_n;u_{n+1})_{\text{OOSASM}} = P_{\lceil \frac{n}{2} \rceil}(u_1,\ldots,u_{2\lceil \frac{n}{2} \rceil}) \times Q_{\lceil \frac{n+1}{2} \rceil}(u_1,\ldots,u_{2\lceil \frac{n+1}{2} \rceil -1}),$$

Specialisation at $q = e^{\frac{i\pi}{6}}$

Corollary

We have the following formula for $Z_{2n-1}(u_1, \dots, u_{2n-1}; u_{2n})_{\mathrm{OOSASM}}$ *in terms of symplectic characters*

$$3^{-(n-1)(2n-1)} sp_{(n-1,n-1,n-2,n-2,\dots,1,1)}(u_1^2, \dots, u_{2n}^2) \times sp_{(n-1,n-2,n-2,n-3,n-3,\dots,1,1)}(u_1^2, \dots, u_{2n-1}^2)$$

and for $Z_{2n}(u_1, \dots, u_{2n}; u_{2n+1})_{\mathrm{OOSASM}}$

$$3^{-n(2n-1)} sp_{(n-1,n-1,n-2,n-2,\dots,1,1)}(u_1^2, \dots, u_{2n}^2) \times sp_{(n,n-1,n-1,n-2,n-2,\dots,1,1)}(u_1^2, \dots, u_{2n+1}^2).$$

Thank you!