A computational theory for Riemann-Hilbert problems

Tom Trogdon
UC Irvine
ttrogdon@math.uci.edu

Outline

- Cauchy integrals and their computation
- Riemann-Hilbert problems
- Applications to integrable systems

Cauchy integrals

Basic theory

Let $\Gamma \subset \mathbb{C}$ be a contour. Define the Cauchy integral of $f : \Gamma \to \mathbb{C}$ by

$$\mathscr{C}_\Gamma f(z) = \int_\Gamma \frac{f(s)}{s-z} đs, \quad đs = \frac{\mathrm{d}s}{2\pi \mathrm{i}}, \quad z \in \mathbb{C} \setminus \Gamma,$$

whenever the integral exists.

A big component of the theory is determining for which classes of contours Γ and classes of functions f the integral $\mathscr{C}_\Gamma f(z)$ exists.

For Γ, locally rectifiable, equip it with arclength measure $|\mathrm{d}s|$. Assume there exists $a \in \mathbb{C} \setminus \Gamma$ such that

$$\int_\Gamma \frac{1}{|s-a|^2} |\mathrm{d}s| < \infty.$$

Then the Cauchy integral exists for all $f \in L^2(\Gamma)$.

Boundary values

An equally important question is about the existence of boundary values for $\mathscr{C}_\Gamma f(z)$.

If Γ is absolutely continuous, we have a tangent at $|\mathrm{d}s|$-almost every $s \in \Gamma$.

Let $\gamma(s)$ be the positively-oriented unit normal vector at $s \in \Gamma$ (when it exists).

Define

$$\mathscr{C}_\Gamma^{\pm} f(s) = \lim_{\epsilon \downarrow 0} \mathscr{C}_\Gamma(s \pm \epsilon\gamma(s)),$$

when this limit exists.

For $f \in L^2(\Gamma)$, it can be shown:

1. This limit exists for a.e. s when $\Gamma = \mathbb{R}$, and $\mathscr{C}_\Gamma^{\pm} \in L^2(\Gamma)$ (elementary)
2. This limit exists for a.e. s when Γ is a Lipschitz graph, and $\mathscr{C}_\Gamma^{\pm} \in L^2(\Gamma)$ (involved)
3. This limit exists for a.e. s and $\mathscr{C}_\Gamma^{\pm} \in L^2(\Gamma)$ if and only if Γ is a Carleson curve.

Carleson curves

All of the imposed conditions are nicely captured, and generalized, by the definition of a Carleson curve.

Assume $\Gamma = \cup_{j=1}^{m} \Gamma_j$ is the finite union of contours Γ_j:

$$\Gamma_j = \{p_j(t) : t \in I_j \subset \mathbb{R}\}, \quad p_j \text{ of bounded variation,}$$

that can only intersect at their endpoints. Γ is a Carleson curve if

$$\sup_{s\in\Gamma} \sup_{r>0} \frac{|\Gamma \cap B(s,r)|}{r} = c(\Gamma) < \infty.$$

The $L^2(\Gamma)$ operator norm of $\mathscr{C}_\Gamma^\pm$ depends continuously on $c(\Gamma)$.

Albrecht Böttcher and Yuri I. Karlovich. Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators. Birkhäuser Basel, Basel, 1997

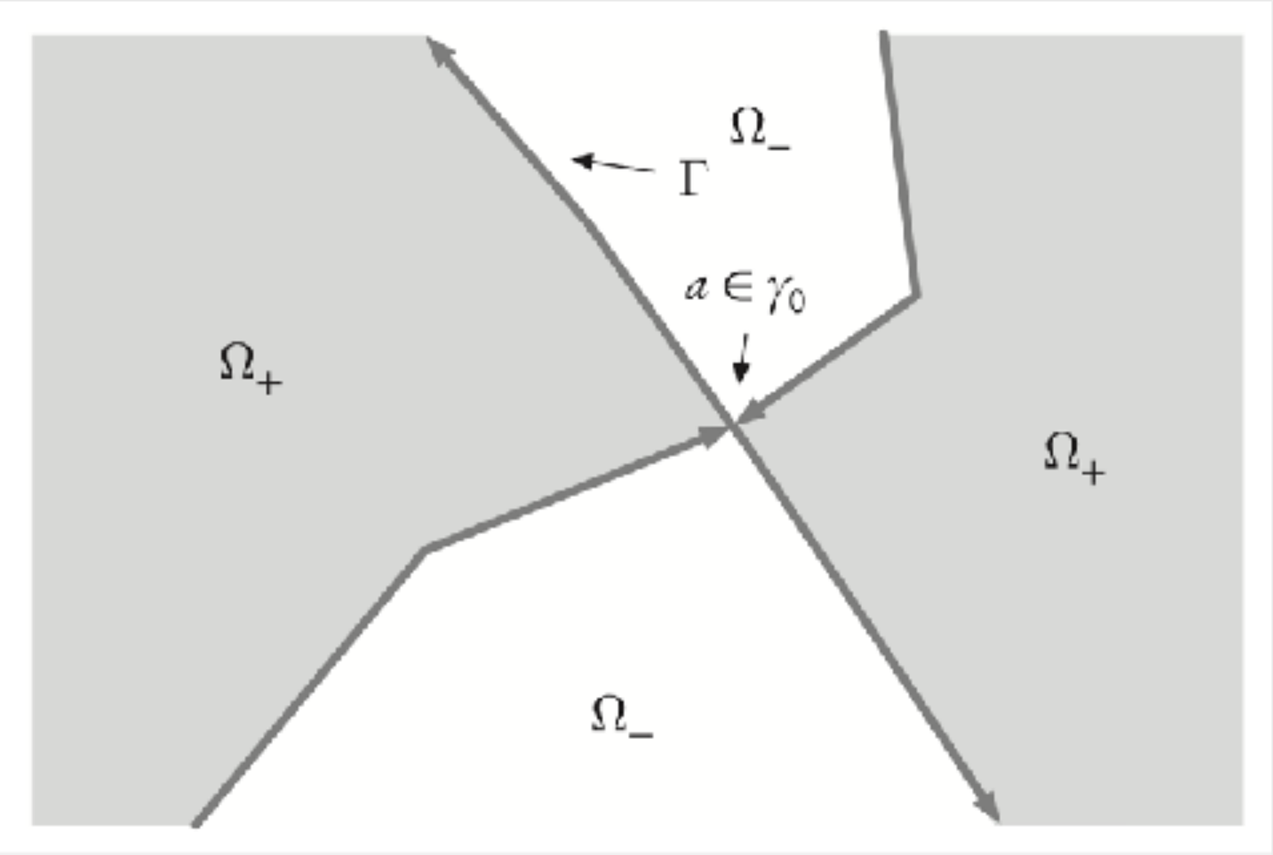
Hardy spaces

The L^2-based Hardy space of the upper-half plane $\mathscr{E}^2(\mathbb{C}^+)$ consists of functions f analytic in $\mathbb{C}^+$ such that

$$\sup_{r>0} \int_{\mathbb{R}} |f(s+\mathrm{i}r)|^2 ds < \infty.$$

Define $\mathscr{E}^{\pm}(\mathbb{R})$ to be the space of all functions $f : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ such that

$$f(\diamond), f(-\diamond) \in \mathscr{E}^2(\mathbb{C}^+).$$



For a general contour, we assume that $\mathbb{C} \setminus \Gamma = \Omega_+ \cup \Omega_-$ where $\Omega_- \cap \Omega_+ = \varnothing$ and Ω_+ (Ω_-) lies to the left (right) of Γ.

The space $\mathscr{E}^{\pm}(\Gamma)$ can be defined in an analogous way, requiring that $f \in \mathscr{E}^2(D)$ for each connected component of $\Omega_{\pm}$.

Additional properties

- $\mathscr{C}_\Gamma^+ f - \mathscr{C}_\Gamma^- = f \quad f \in L^2(\Gamma)$
- $\mathscr{C}_\Gamma^+ \mathscr{C}_\Gamma^+ f = \mathscr{C}_\Gamma^+ f \quad f \in L^2(\Gamma)$
- $\mathscr{C}_\Gamma^- \mathscr{C}_\Gamma^- f = -\mathscr{C}_\Gamma^- f \quad f \in L^2(\Gamma)$
- $\mathscr{C}_\Gamma L^2(\Gamma) = \mathscr{E}^\pm(\Gamma)$
- Each function f in $\mathscr{E}^\pm(\Gamma)$ has boundary values on Γ and f is given by the Cauchy integral of its boundary function.
- If $\mathscr{C}_\Gamma^+ f - \mathscr{C}_\Gamma^- f = 0$ then $f = 0$

Note: In the lectures I used the notation $\mathscr{E}^2(\Gamma) = \mathscr{E}^\pm(\Gamma)$

Computing Cauchy integrals

As a prototype for computing a Cauchy integral, consider two approaches to computing

$$\int_{-1}^{1} f(x)\mathrm{d}x.$$

1. Quadrature nodes and weights: Approximate

$$f(x) \approx \sum_j f(x_j) w_j \delta(x - x_j)$$

2. Basis approximation: Approximate

$$f(x) \approx \sum_j c_j \phi_j(x) \quad \text{where} \quad \int_{-1}^{1} \phi_j(x)\mathrm{d}x \quad \text{is known}$$

Computing Cauchy integrals

Case 1 is easy to extend to computing Cauchy integrals ($\mathbb{I} = [-1,1]$):

$$\mathscr{C}_{\mathbb{I}} f(z) \approx \frac{1}{2\pi \mathrm{i}} \sum_j \frac{f(x_j) w_j}{z - x_j}.$$

Case 2 is more difficult:

$$\mathscr{C}_{\mathbb{I}} f(z) \approx \sum_j c_j \mathscr{C}_{\mathbb{I}} \phi_j(z).$$

An additional method to compute $\mathscr{C}_{\mathbb{I}} \phi_j$ is needed.

$\sum_j c_j \mathscr{C}_{\mathbb{I}} \phi_j(z)$ may have well-defined boundary values on $\mathbb{I}$.

$\sum_j \frac{f(x_j) w_j}{z - x_j}$ does not.

A controlled basis

We choose the basis $\{\phi_j\}$ so that:

- $\sup_j ||\phi_j||_u < \infty$
- $\sup_{j>2} ||\mathscr{C}_{\mathbb{I}}\phi_j||_u$

Then the (uniform) convergence of

$$\sum_j c_j \mathscr{C}_{\mathbb{I}}\phi_j \quad \text{to} \quad \mathscr{C}_{\mathbb{I}} f$$

is effectively at the same rate as

$$\sum_j c_j \phi_j \quad \text{to} \quad f.$$

Let T_j be the jth-order Chebyshev polynomial of the first kind. We use

$$\phi_j(x) = T_j(x) - T_{j-2}(x), \quad j \geq 2,$$

with $\phi_1 = T_1$ and $\phi_2 = T_2$. The coefficients can be computed with the DCT.

Define the auxiliary functions

$$\mu_m(z) = \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{z^{2j-1}}{2j-1}, \quad \psi_0(z) = \frac{2}{\mathrm{i}\pi} \mathrm{arctanh} z,$$

$$\psi_m(z) = z^m \left(\psi_0(z) - \frac{2}{\mathrm{i}\pi} \begin{cases} \mu_{-m-1}(z) & m < 0 \\ \mu_m(1/z) & m > 0 \end{cases} \right).$$

Then

$$\mathscr{C}_{\mathbb{I}} T_k(x) = -\frac{1}{2} \left(\psi_k(T_+^{-1}(x)) + \psi_{-k}(T_+^{-1}(x)) \right),$$

and taking limits,

$$\mathscr{C}_{\mathbb{I}}^{+} T_k(x) = -\frac{1}{2} \left(\psi_k(T_\downarrow^{-1}(x)) + \psi_{-k}(T_\downarrow^{-1}(x)) \right)$$

$$= -\frac{2}{\mathrm{i}\pi} T_k(x) \mathrm{arctanh} T_\downarrow^{-1}(x) + \frac{1}{\mathrm{i}\pi} \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{T_{k-2j+1}(x)}{2j-1} \begin{cases} 1, & k-2j+1=0, \\ 2, \text{otherwise}, \end{cases}$$

$$\mathscr{C}_{\mathbb{I}}^{-} T_k(x) = -\frac{1}{2} \left(\psi_k(T_\uparrow^{-1}(x)) + \psi_{-k}(T_\uparrow^{-1}(x)) \right)$$

$$= -\frac{2}{\mathrm{i}\pi} T_k(x) \mathrm{arctanh} T_\uparrow^{-1}(x) + \frac{1}{\mathrm{i}\pi} \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{T_{k-2j+1}(x)}{2j-1} \begin{cases} 1, & k-2j+1=0, \\ 2, \text{otherwise}. \end{cases}$$

S Olver. A general framework for solving Riemann-Hilbert problems numerically. Numer. Math., 122(2):305–340, 2012

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Then

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See also:

J. A. C. Weideman and L. N. Trefethen. The kink phenomenon in Fejér and Clenshaw-Curtis quadrature. Numer. Math., 107(4):707–727, 2007

$$= -\frac{2}{\mathrm{i}\pi} T_k(x) \mathrm{arctanh} T_\downarrow^{-1}(x) + \frac{1}{\mathrm{i}\pi} \sum_{j=1} \frac{T_{k-2j+1}(x)}{2j-1} \begin{cases} 1, & k-2j+1=0, \\ 2, \text{otherwise}, \end{cases}$$

$$\mathscr{C}_{\mathbb{I}}^- T_k(x) = -\frac{1}{2} \left(\psi_k(T_\uparrow^{-1}(x)) + \psi_{-k}(T_\uparrow^{-1}(x)) \right)$$

$$= -\frac{2}{\mathrm{i}\pi} T_k(x) \mathrm{arctanh} T_\uparrow^{-1}(x) + \frac{1}{\mathrm{i}\pi} \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{T_{k-2j+1}(x)}{2j-1} \begin{cases} 1, & k-2j+1=0, \\ 2, \text{otherwise}. \end{cases}$$

S Olver. A general framework for solving Riemann-Hilbert problems numerically. Numer. Math., 122(2):305–340, 2012

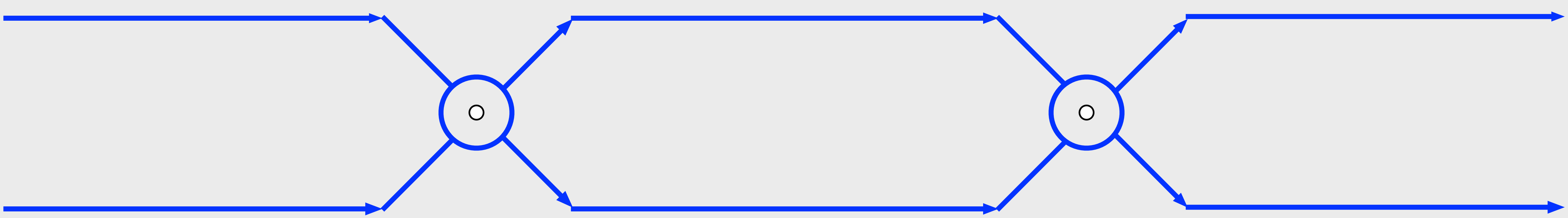
Generalizing the contours

The previous discussion can be used to treat

$$\Gamma = \cup_{j=1}^{m} \Gamma_j, \quad \Gamma_j = M_j(\mathbb{I}),$$

where M_j is a Möbius transformation.

So, Γ can be a finite union of arcs and lines of possibly infinite extent.



Riemann-Hilbert problems

A definition and a singular integral equation

Definition 1. *A function Φ is a $\mathscr{E}^2(\Gamma)$ solution of an RH problem $[G;\Gamma]$ if it satisfies the jump condition*

$$\Phi^+(s) = \Phi^-(s)G(s), \quad G:\Gamma \to \mathbb{C}^{2\times 2},$$

and $\Phi - I \in \mathscr{E}^{\pm}(\Gamma)$.

If $\Phi \in \mathscr{E}^{\pm}(\Gamma)$ then we call Φ a vanishing solution.

From this definition, it follows that

$$\Phi(z) = I + \mathscr{C}_\Gamma u(z), \quad \text{for some} \quad u \in L^2(\Gamma).$$

$$\begin{aligned}
I + \mathscr{C}_\Gamma^+ u(s) &= \left(I + \mathscr{C}_\Gamma^- u(s)\right) G(s) \\
\mathscr{C}_\Gamma^+ u(s) - \mathscr{C}_\Gamma^- u(s) &= \left(I + \mathscr{C}_\Gamma^- u(s)\right) G(s) - I - \mathscr{C}_\Gamma^- u(s) \\
u(s) &= \left(I + \mathscr{C}_\Gamma^- u(s)\right)(G(s) - I) \\
u(s) - \mathscr{C}_\Gamma^- u(s)(G(s) - I) &= G(s) - I \\
\mathscr{C}[G;\Gamma]u &= G - I
\end{aligned}$$

Sobolev spaces

Use a decomposition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$ (Γ_j smooth) to define Sobolev spaces:

$$H^k(\Gamma) = \bigoplus_{j=1}^{n} H^k(\Gamma_j),$$

$$H^k(\Gamma_j) = \{f \in L^2(\Gamma_j) : f^{(i)} \in L^2(\Gamma_j),\ \ i = 1, \ldots, k\}.$$

In these spaces, different smooth components of Γ are independent. We need some restriction.

Definition 2. *A function f is said to satisfy the kth-order zero-sum condition at a point s^* if*

$$\sum_{i=1}^{m} \sigma_i \lim_{\substack{s \to s^* \\ s \in \Gamma_i}} f^{(j)}(s) = 0, \ \ j = 0, \ldots, k,$$

where $\Gamma_1, \ldots, \Gamma_m$ is an ordering of components of Γ that contain s^*. Furthermore, $\sigma_i = +1(-1)$ if Γ_i is oriented away(towards) s^*.

Zero-sum space

Definition 3. *The zero-sum space* $H_z^k(\Gamma)$ *is the closed subspace of* $H^k(\Gamma)$ *consisting of functions that satisfy the* $(k-1)$*th-order zero-sum condition at all self-intersection points of* Γ.

The Cauchy operators behave nicely on these spaces:

Theorem 1 (TT). $\mathscr{C}_\Gamma^\pm$ *are bounded linear operators on* $H_z^k(\Gamma)$.

Regularity of the jump matrix

Definition 4. *A function G satisfies the kth-order product condition at a point s^* if using Taylor expansions we have*

$$\prod_{i=1}^{m}\left(G(s)|_{s\in\Gamma_i}\right)^{\sigma_i} = I + O(|s-s^*|^{k-1/2})$$

where $\Gamma_1,\ldots,\Gamma_m$ is a counter-clockwise ordering of components of Γ that contain s^*. Furthermore, $\sigma_i = +1(-1)$ if Γ_i is oriented away(towards) s^*.

Definition 5. *A RH problem $[G;\Gamma]$ is k-regular if $G - I \in H^k(\Gamma)$ and G satisfies the kth-order product condition at all self-intersection points of Γ.*

Associated operators

Theorem 2 (TT). *If an RH problem is k-regular there exists an operator $\mathscr{M}$, invertible on $H^k(\Gamma)$, such that*

- $\mathscr{M} \circ \mathscr{C}[G;\Gamma]$ *is bounded on* $H_z^k(\Gamma)$,
- $\mathscr{M} \circ \mathscr{C}[G;\Gamma]$ *is Fredholm on* $H_z^k(\Gamma)$ *with*

$$\begin{aligned}\operatorname{ind} \mathscr{M} \circ \mathscr{C}[G;\Gamma] &= \operatorname{codim} \operatorname{rng} \mathscr{M} \circ \mathscr{C}[G;\Gamma] - \dim \ker \mathscr{M} \circ \mathscr{C}[G;\Gamma] \\ &= \int_\Gamma đ \log \det G.\end{aligned}$$

This relies on:

X Zhou. The Riemann-Hilbert problem and inverse scattering. SIAM J. Math. Anal., 20(4):966–986, 1989

K F Clancey and I Gohberg. Factorization of matrix functions and singular integral operators, volume 3 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 1981

Smoothness

Theorem 3 (TT and Olver). *If an RH problem is k-regular and $\mathscr{M} \circ \mathscr{C}[G;\Gamma]$ is invertible on $H_z^k(\Gamma)$ then $\mathscr{C}[G;\Gamma]u = G - I$ implies*

$$||u||_{H^k(\Gamma)} \leq F_k(\mathscr{C}[G;\Gamma], \mathscr{C}[G;\Gamma]^{-1}, G, G', G'', \ldots)||G - I||_{H^k(\Gamma)}.$$

If G and Γ depend on a (large) parameter, and we can bound

- the Carleson constant $c(\Gamma)$,
- the $L^2(\Gamma)$ norms of $G - I, \ldots, G^{(k)}$, and
- (most difficult) $||\mathscr{C}[G;\Gamma]^{-1}||_{L^2(\Gamma)}$

uniformly with respect to this parameter, we obtain a uniform bound on $||u||_{H^k(\Gamma)}$.

Some notes on numerical solutions

A discretization $C_n[G;\Gamma]$ of $\mathscr{C}[G;\Gamma]$ can be computed using Chebyshev collocation. Let $G_n - I$ denote the discretized right-hand side.

Theorem 4 (Olver). *If G satisfies the first-order product condition, if there is a solution u_n of*

$$C_n[G;\Gamma]u_n = G_n - I$$

then u_n must satisfy the first-order zero-sum condition and therefore

$$\Phi_n(z) = I + \mathscr{C}_\Gamma u_n(z)$$

is bounded.

The numerical solution of Riemann-Hilbert problems

The convergence rate is closely tied to the smoothness of the solution.

References:

T T and S Olver. Riemann–Hilbert Problems, Their Numerical Solution and the Computation of Nonlinear Special Functions. SIAM, Philadelphia, PA, 2016.

S Olver. RHPackage. `http://www.maths.usyd.edu.au/u/olver/projects/RHPackage.html`, 2010.

T T. ISTPackage. `https://bitbucket.org/trogdon/istpackage`, 2013

The defocusing nonlinear Schrödinger equation

w/ Sheehan Olver

The initial value problem

Consider the numerical evaluation of the solution $q(x,t)$ of

$$-\mathrm{i}q_t + q_{xx} - 2q|q|^2 = 0, \quad q(x,t) = q_0(x), \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

The associated inverse scattering transform gives a nonlinear mapping:

$$R = \mathscr{S}(q_0), \quad R : \mathbb{R} \times [0,\infty) \to \mathbb{C}.$$

The function $R(s,t) = R(s,0)\mathrm{e}^{4\mathrm{i}s^2 t}$ is the so-called reflection coefficient.

Riemann–Hilbert Problem 1. *The RH problem for the defocusing nonlinear Schrödinger equation is*

$$\Phi^+(s) = \Phi^-(s) \begin{bmatrix} 1 - R(s,0)\overline{R(s,0)} & -\overline{R(s,t)}\mathrm{e}^{-2\mathrm{i}xz} \\ R(s,t)\mathrm{e}^{2\mathrm{i}xz} & 1 \end{bmatrix},$$

and the corresponding solution to the defocusing nonlinear Schrödinger equation is

$$q(x,t) = -2\mathrm{i} \lim_{z\to\infty} z\Phi_{21}(z).$$

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See:

V E Zakharov and A B Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Sov. Phys. JETP, 34:62–69, 1972

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An important calculation

Assume a RH problem $[G;\Gamma]$ is k-regular. Assume further that $G - I \in L^1(\Gamma)$. Then if

$$\begin{aligned} \mathscr{C}[G;\Gamma]u &= G - I, \\ u - \mathscr{C}_\Gamma^- u \cdot (G - I) &= G - I. \end{aligned}$$

If u is an $H^k(\Gamma)$ solution, then the solution Φ is given by

$$\begin{aligned} \Phi &= I + \mathscr{C}_\Gamma u, \\ u &= \mathscr{C}_\Gamma^- u \cdot (G - I) + G - I, \\ ||u||_{L^1(\Gamma)} &\leq ||\mathscr{C}_\Gamma^-||_{L^2(\Gamma)} ||u||_{L^2(\Gamma)} ||G - I||_{L^2(\Gamma)} + ||G - I||_{L^1(\Gamma)}. \end{aligned}$$

Thus $u \in L^1(\mathbb{R})$ and

$$\lim_{z \to \infty,\ d(z,\Gamma) > \delta} z(\Phi(z) - I) = -\int_\Gamma u(s) đs$$

Steepest descent

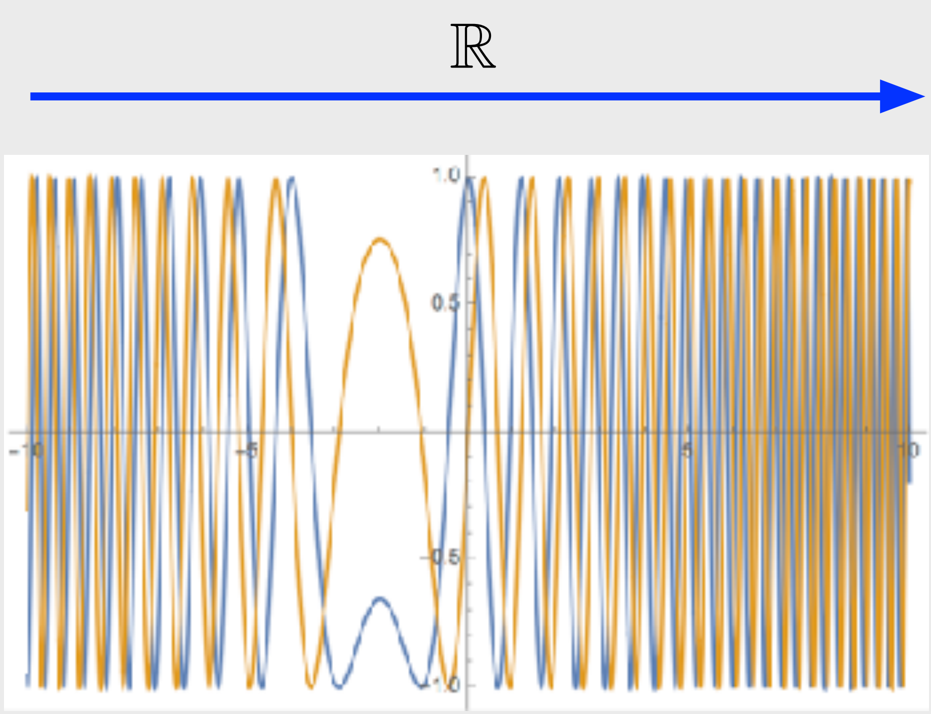
To illustrate the complications that arise in solving this RH problem for all x and t, consider solving the linearlized equation

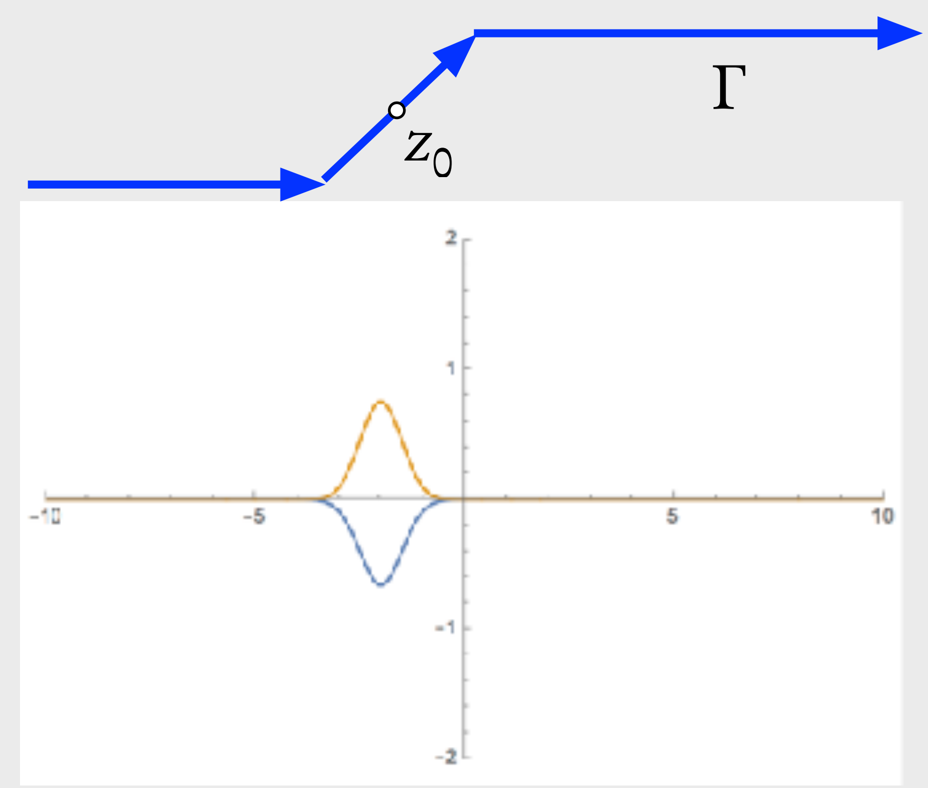
$$-\mathrm{i}q_t + q_{xx} = 0, \quad q(x,0) = q_0(x), \quad q(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sx+\mathrm{i}s^2t}\hat{q}_0(s)\mathrm{d}s.$$

$\mathrm{e}^{\mathrm{i}sx+\mathrm{i}s^2t}$ for $s \in \mathbb{R}$

$$z_0 = -\frac{x}{2t}$$

$\mathrm{e}^{\mathrm{i}sx+\mathrm{i}s^2t}$ for $s \in \Gamma$





A deformation

Assume that

$$q_0 \in L^1(\mathbb{R}, e^{2\gamma|x|}dx), \quad \gamma > 0.$$

Then $R(s)$ has an analytic extension to a strip of width 2γ that contains the real axis.

Now, assume that

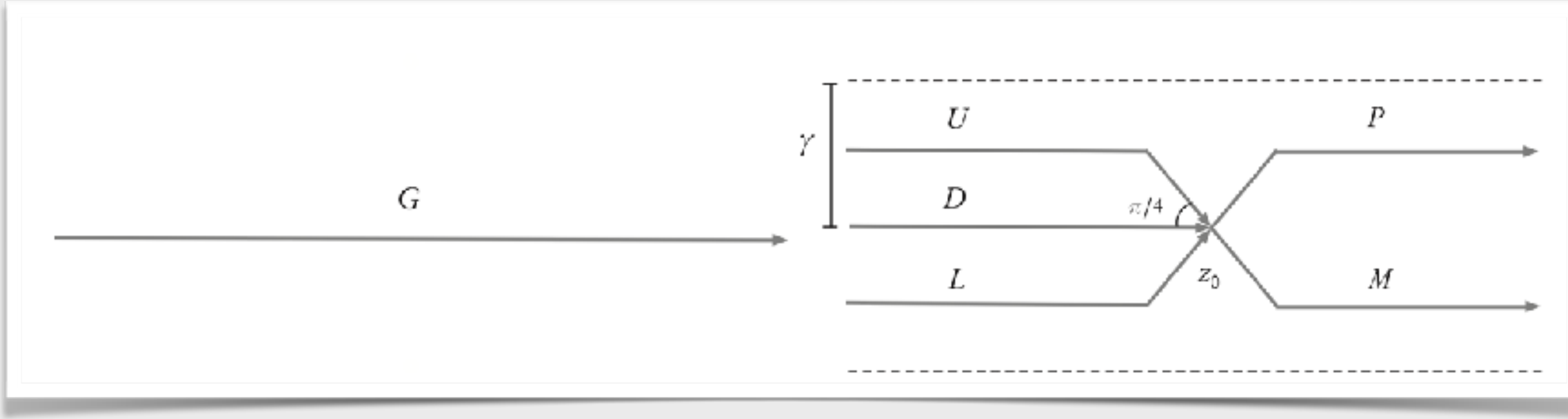
$$G(s) = M(s)P(s)$$

where M, P, $\det M = \det P = 1$, are analytic within this strip, continuous on the closure. Define

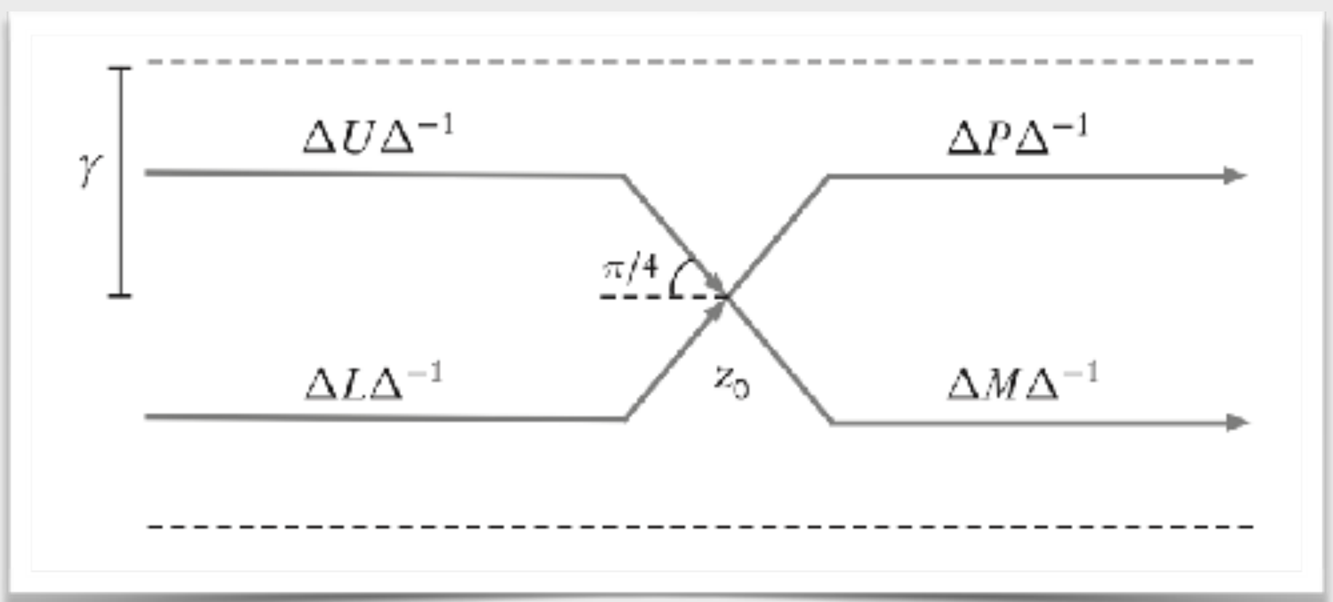
$$\Phi_1(z) = \begin{cases} \Phi(z)P^{-1}(z) & 0 < \operatorname{Im} z < \gamma, \\ \Phi(z)M(z) & -\gamma < \operatorname{Im} z < 0, \\ \Phi(z) & \text{otherwise.} \end{cases}$$

With appropriate decay assumptions, $\Phi_1(z)$ satisfies the RH problem

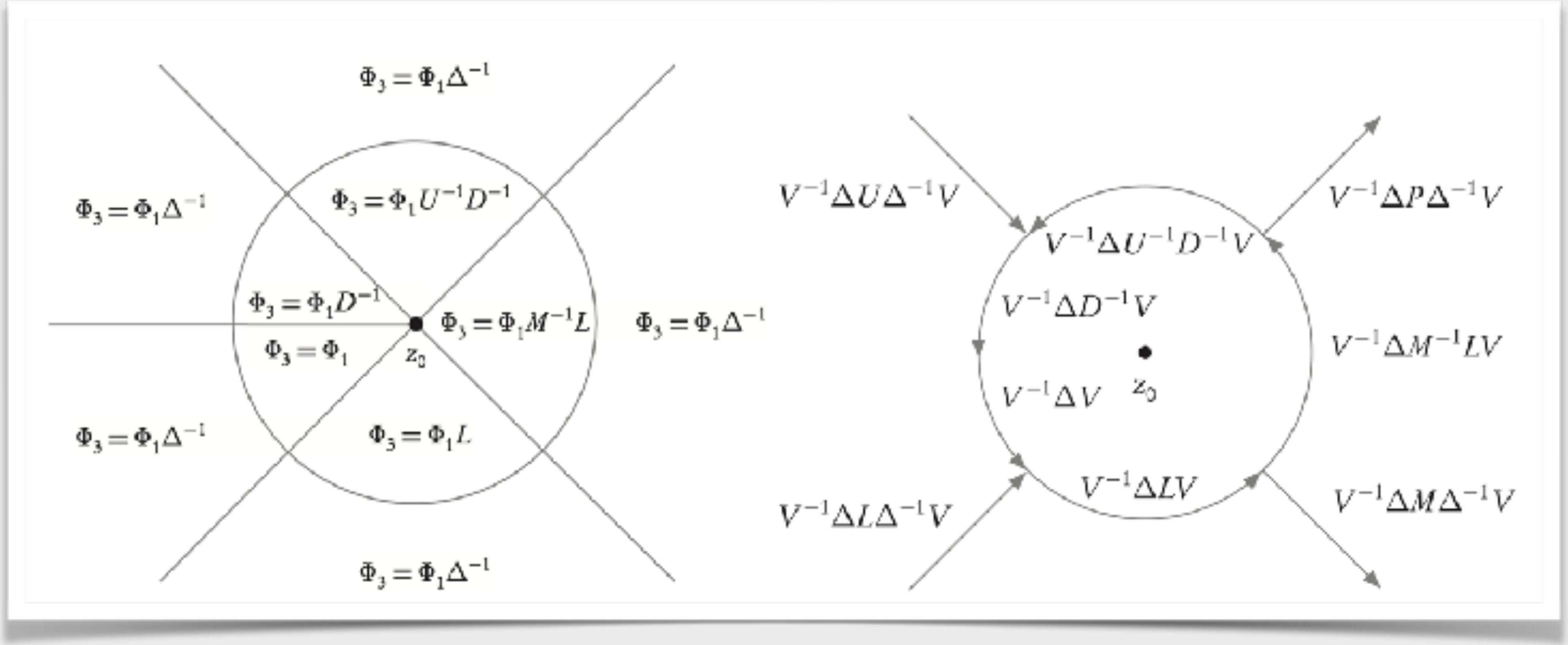
$$\Phi_1^+(s) = \Phi_1^-(s) \begin{cases} P(s) & \operatorname{Im} s = \gamma, \\ M(s) & \operatorname{Im} s = -\gamma. \end{cases}$$



$$z_0 = -\frac{x}{4t} \qquad \Delta^+(s) = \Delta^-(s)D(s)$$



Solution is not smooth at z_0



The radius of the circle is $\sim t^{-1/2}$

Theorem 5 (TT & Olver). *Assuming that the numerical method used to discretize and solve the RH problem for the NLS equation satisfies 4 common conditions, the convergence is uniform in x and t:*

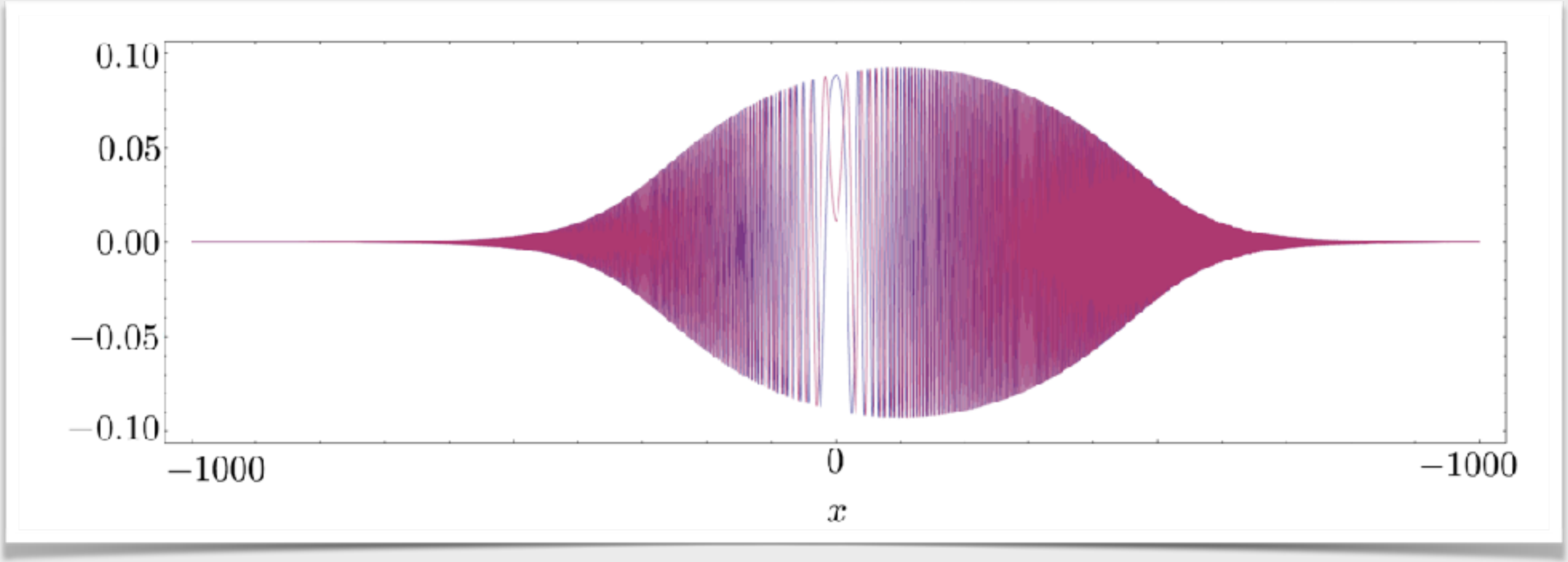
$$|q(x,t) - q_n(x,t)| \leq C_k n^{-k}, \quad \forall k > 0,$$

where n is the degree of approximation (i.e., # of collocation points).

S Olver and T T. Nonlinear steepest descent and numerical solution of Riemann–Hilbert problems. Commun. Pure Appl. Math., 67(8):1353–1389, aug 2014

$$-\mathrm{i}q_t + q_{xx} - 2|q|^2 q = 0$$

$$q_0(x) = 2\mathrm{e}^{-x^2+\mathrm{i}x} \qquad t = 50$$



The KdV equation

Now, consider the numerical solution of the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0, \quad q(x,0) = q_0(x), \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

We will assume that q_0 decays rapidly. There exists an analogous RH problem for the KdV equation (ignoring solitons):

$$\Phi^+(s) = \Phi^-(s) \begin{bmatrix} 1 - \rho(s)\rho(-s) & -\rho(-s)\mathrm{e}^{-2\mathrm{i}xs - 8\mathrm{i}s^3 t} \\ \rho(s)\mathrm{e}^{2\mathrm{i}xs + 8\mathrm{i}s^3 t} & 1 \end{bmatrix}$$

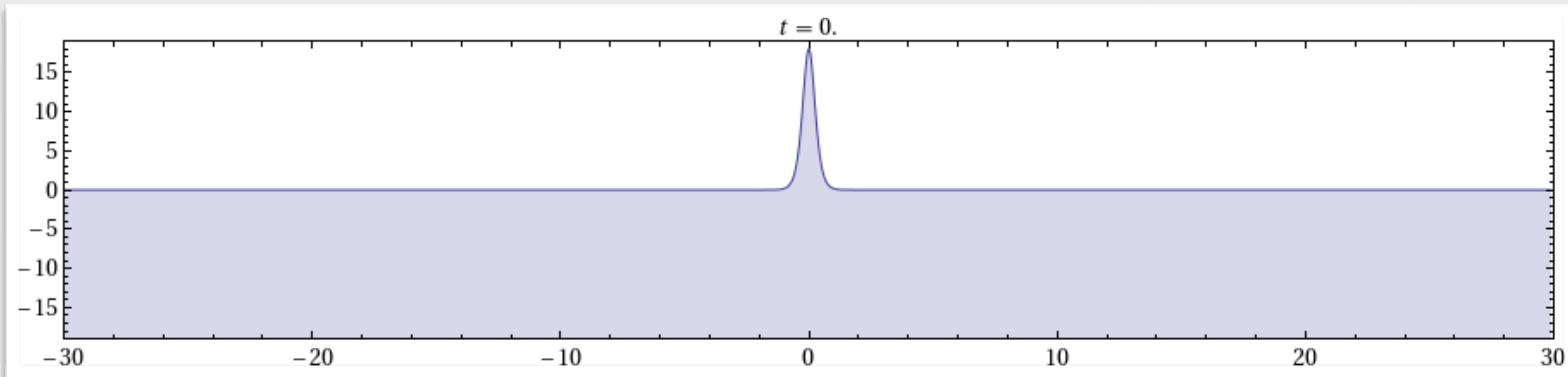
and

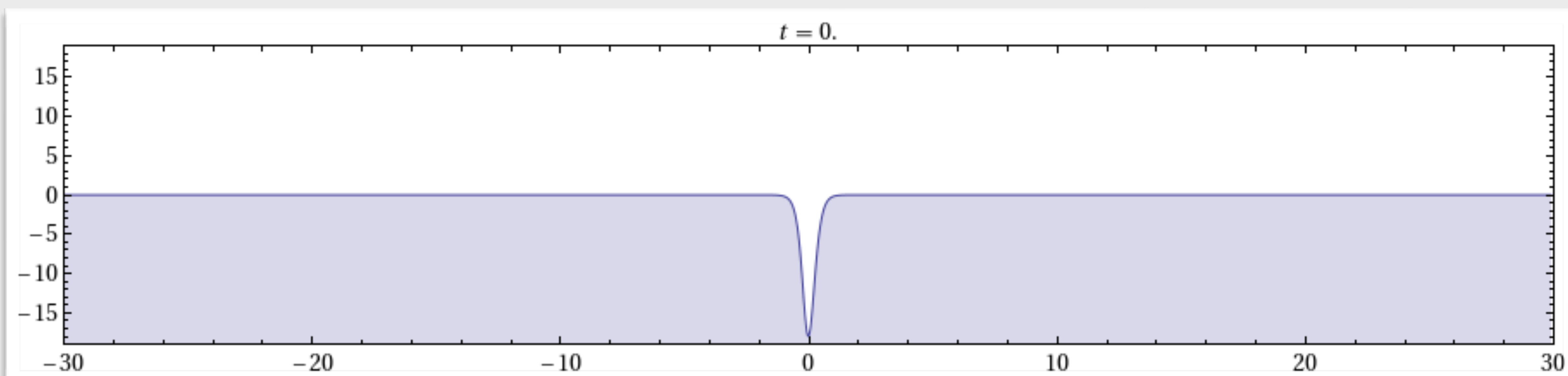
$$q(x,t) = 2\mathrm{i} \lim_{z\to\infty} z \partial_x \left(\Phi_{11}(z) + \Phi_{21}(z)\right).$$

(ignoring a number of technical issues)

The KdV equation with decaying data

Using simple Fourier methods





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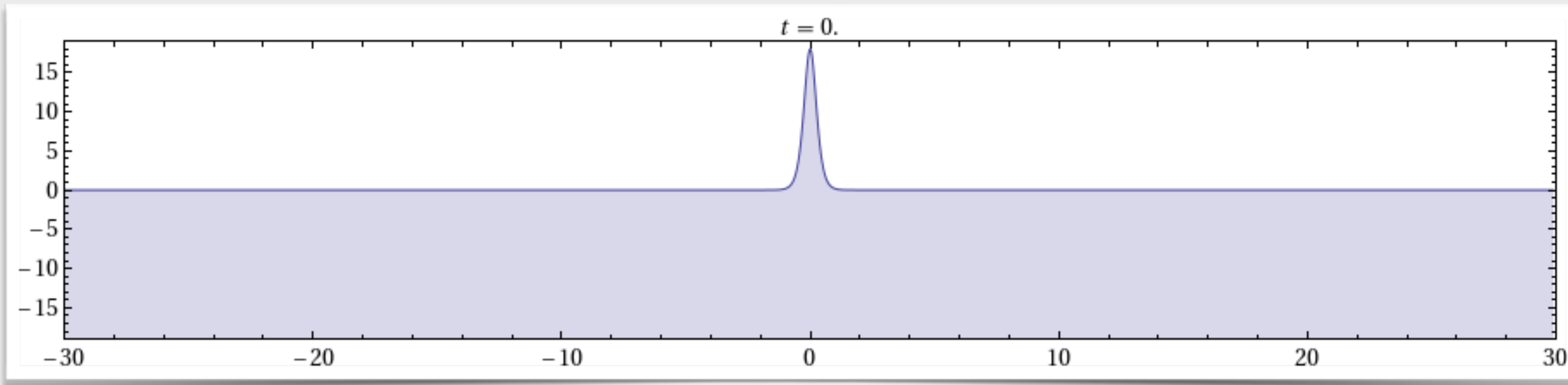
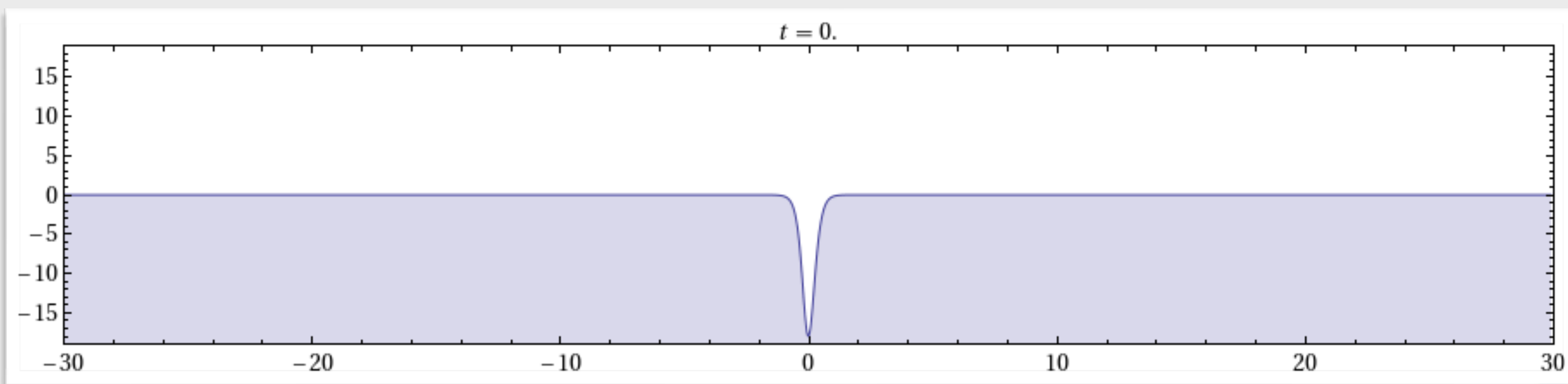
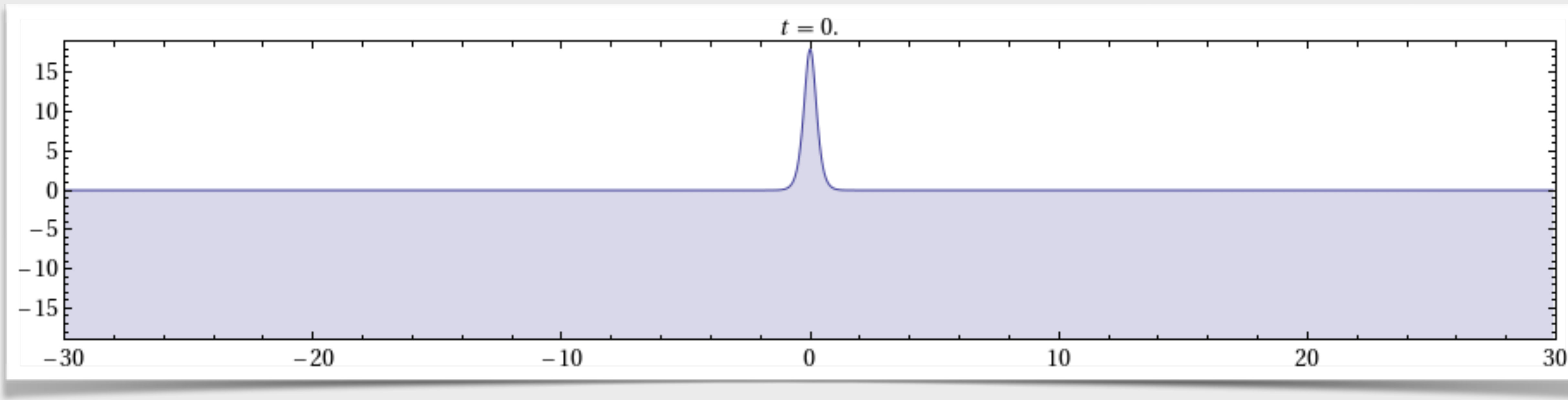
Using simple Fourier methods

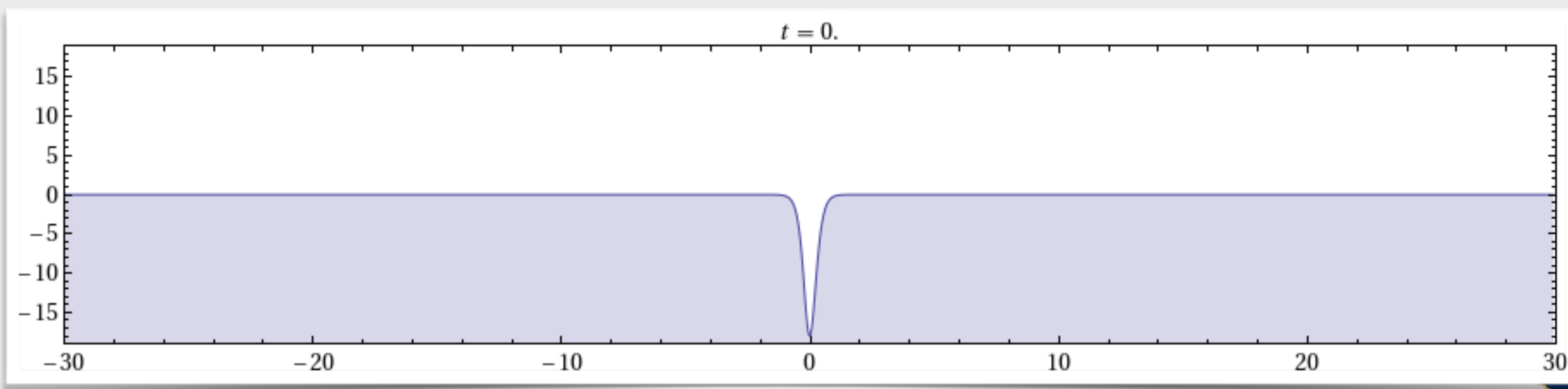




The KdV equation with decaying data

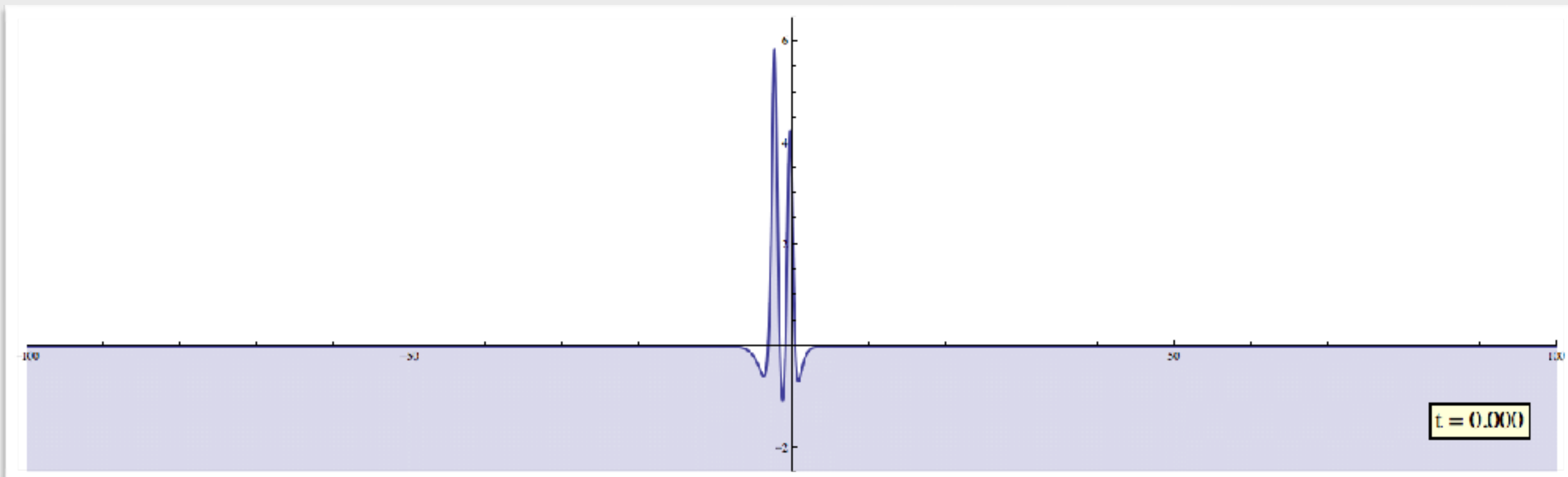
Using simple Fourier methods





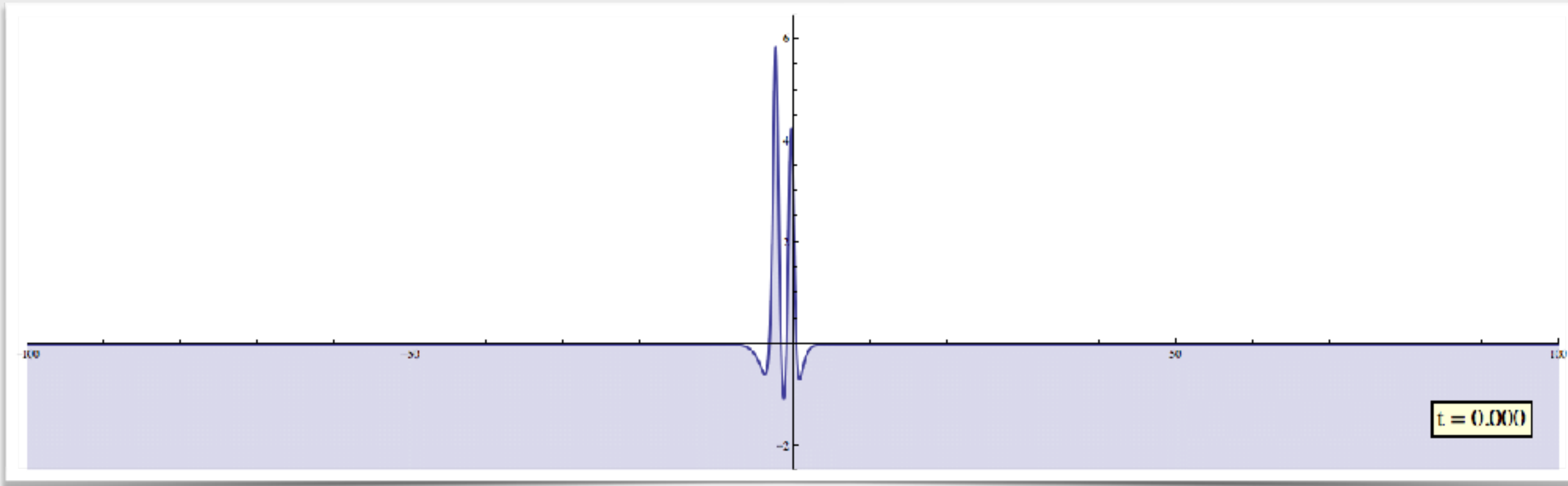
The KdV equation with decaying data

w/ Sheehan Olver and Bernard Deconinck

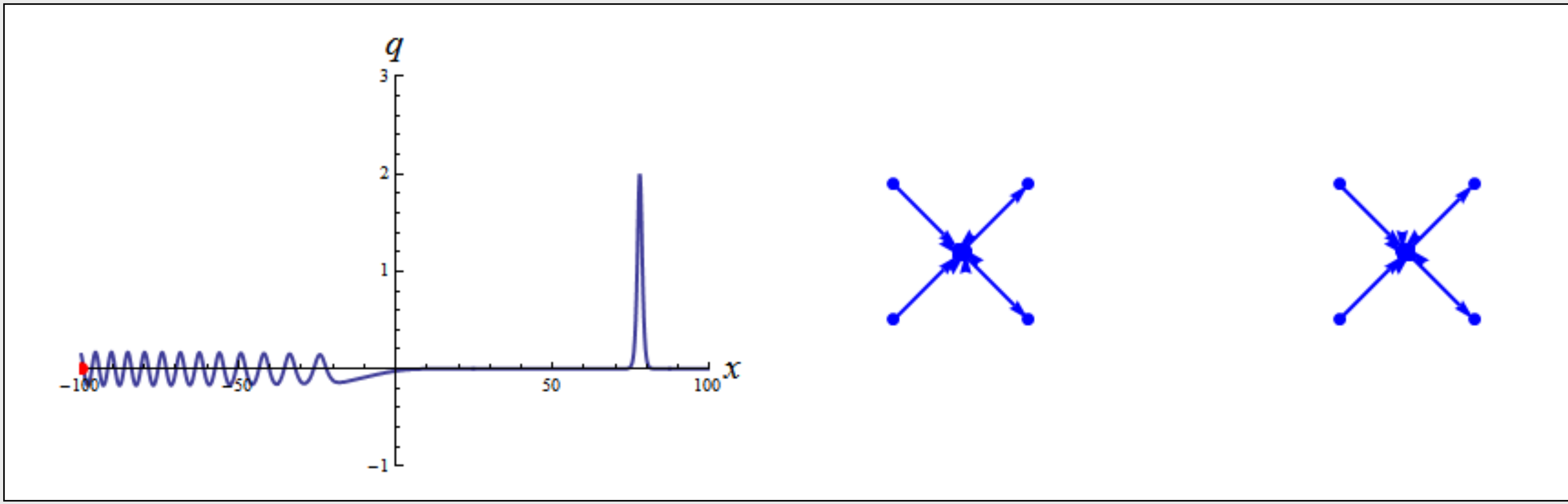


The KdV equation with decaying data

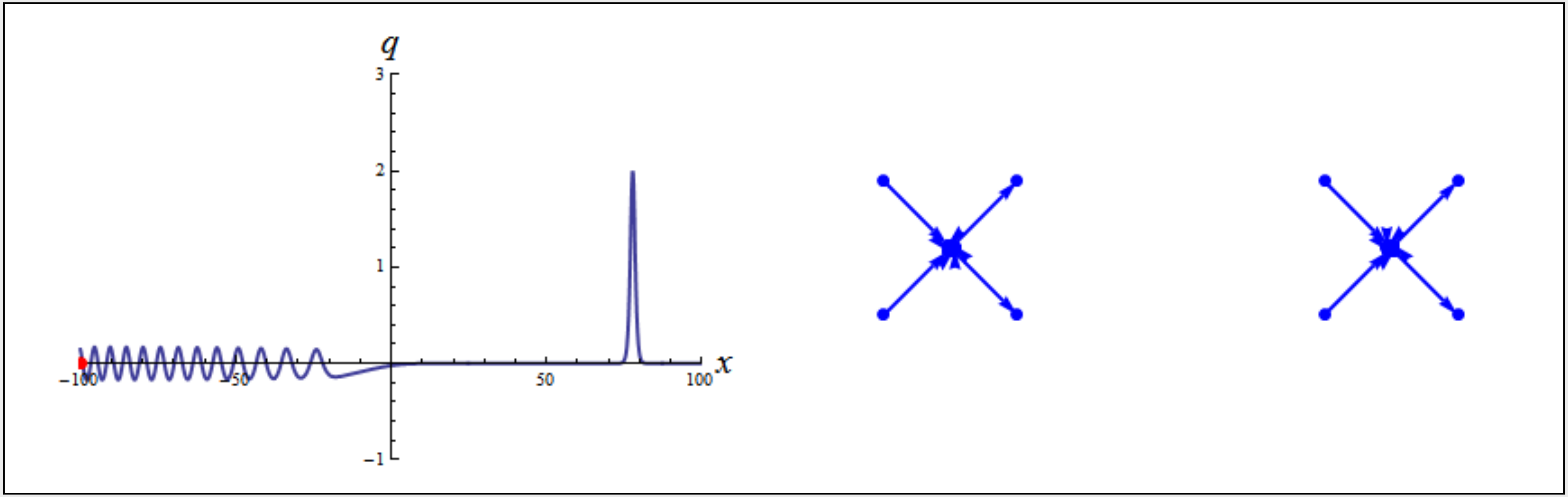
w/ Sheehan Olver and Bernard Deconinck



Contours at a fixed time

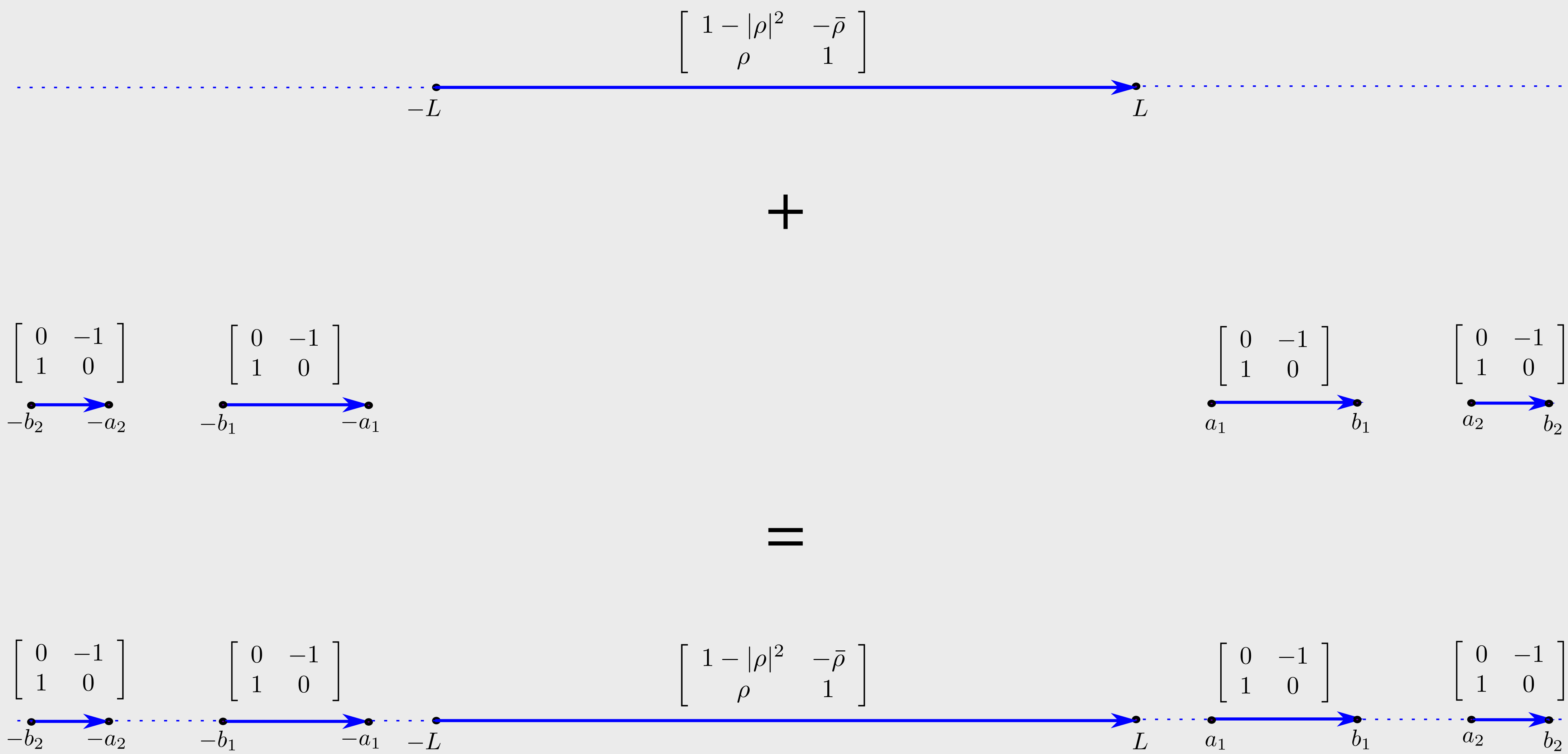


Contours at a fixed time

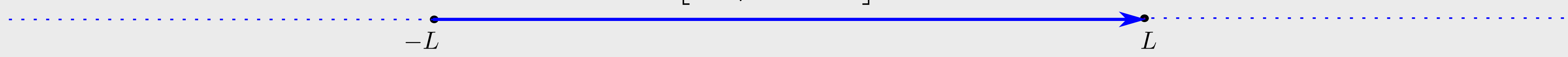


Nonlinear superposition

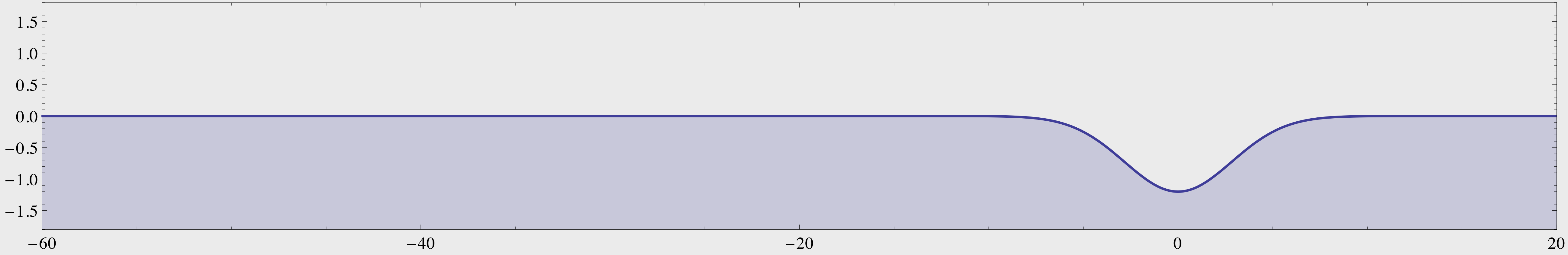
w/ Bernard Deconinck



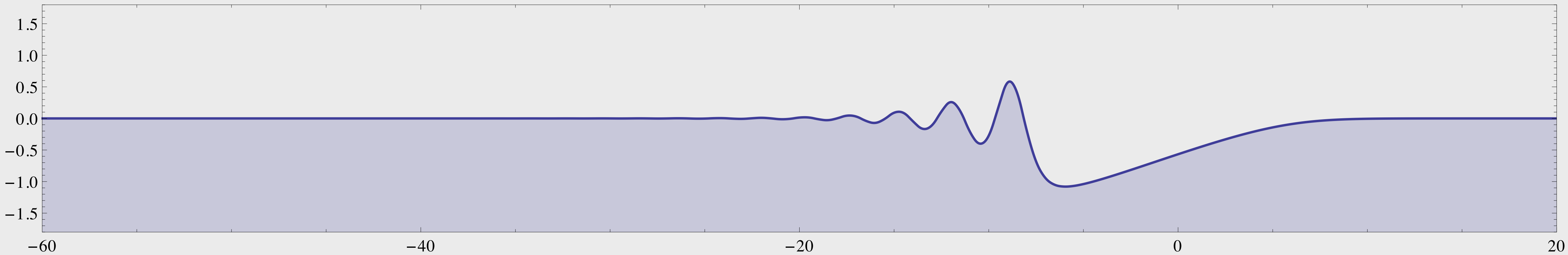
$$\begin{bmatrix} 1-|\rho|^2 & -\bar{\rho} \\ \rho & 1 \end{bmatrix}$$



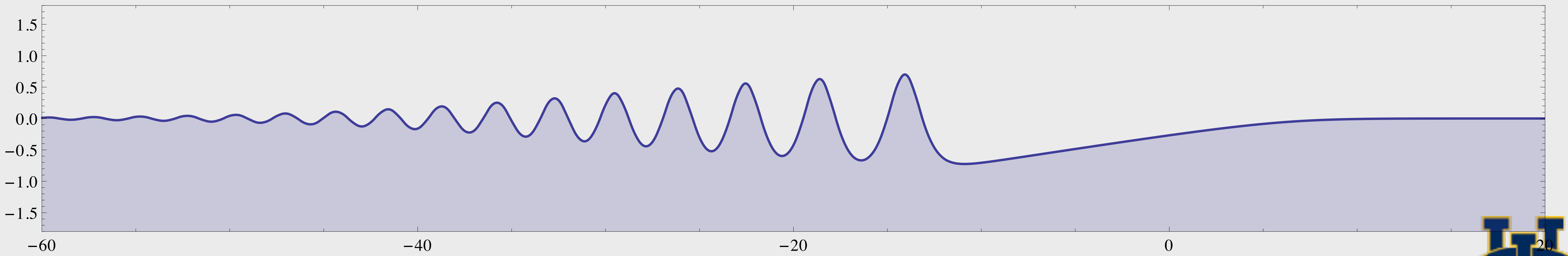
$t\ = 0$

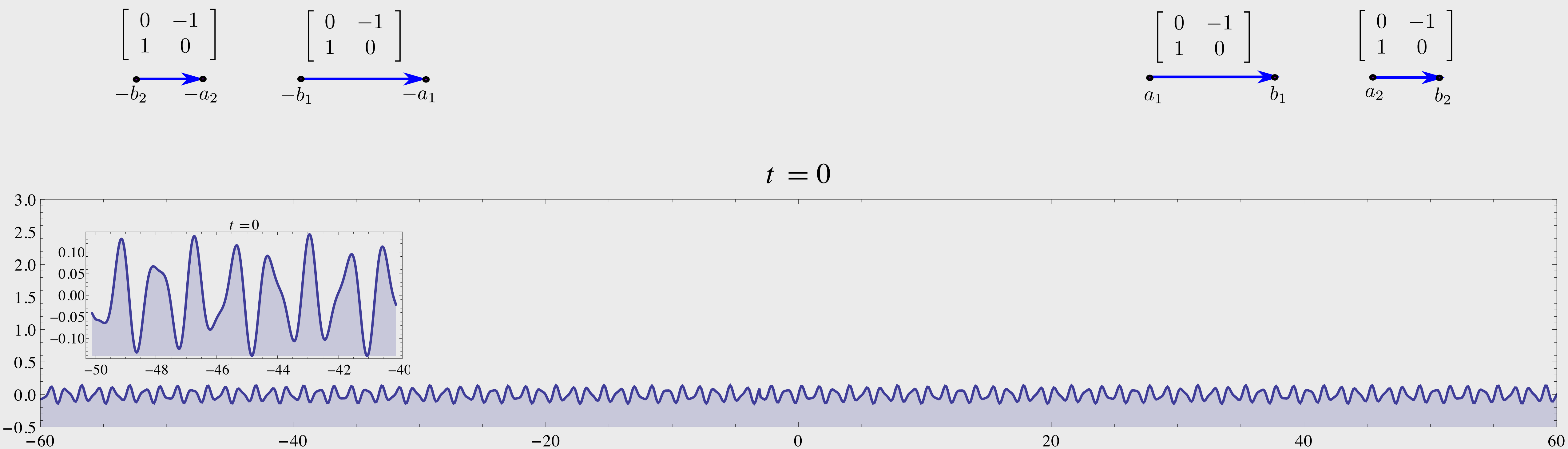


$t\ = 1$

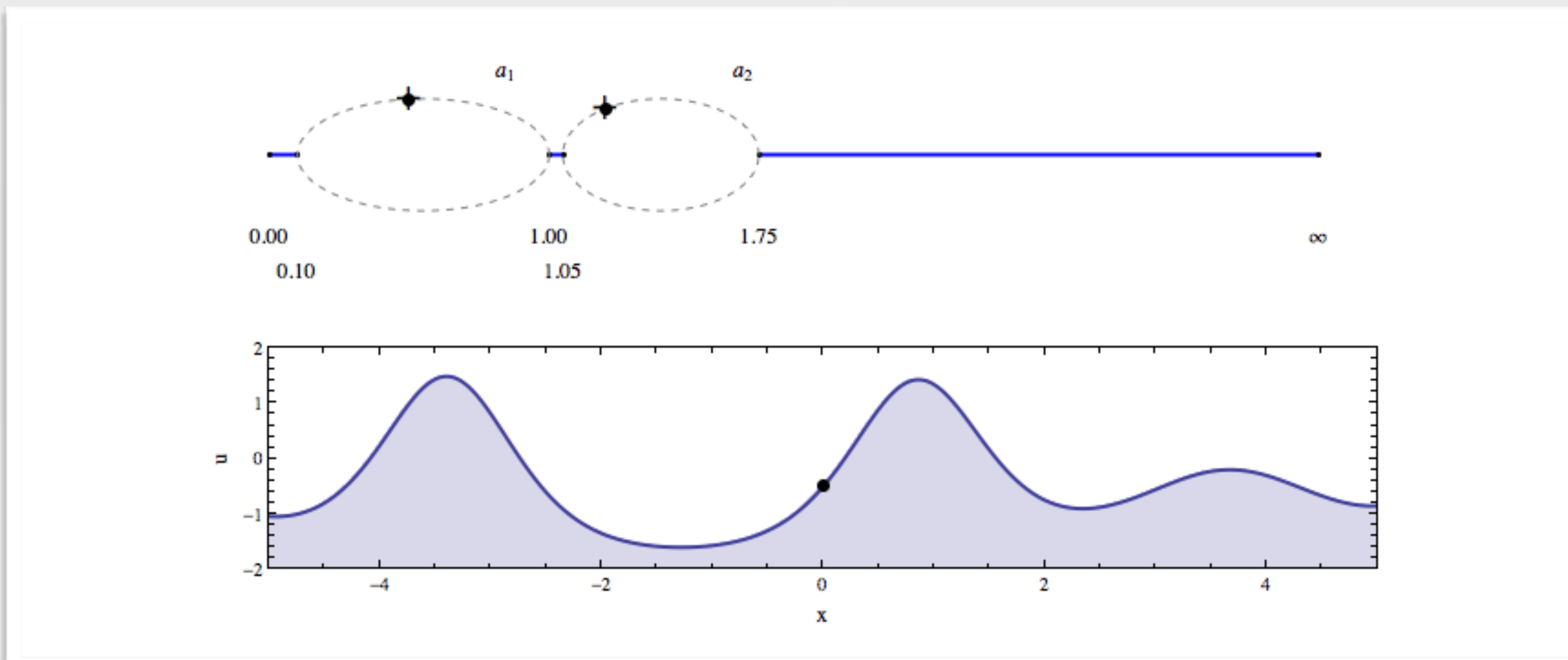


$t\ = 3$

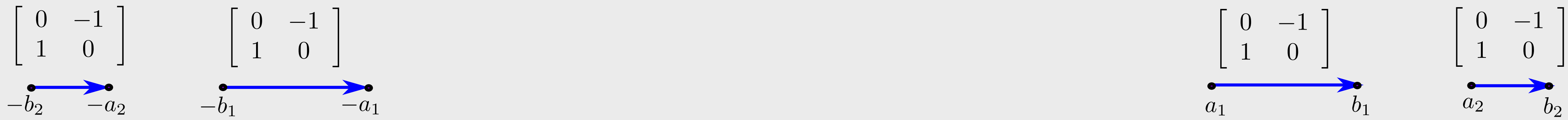


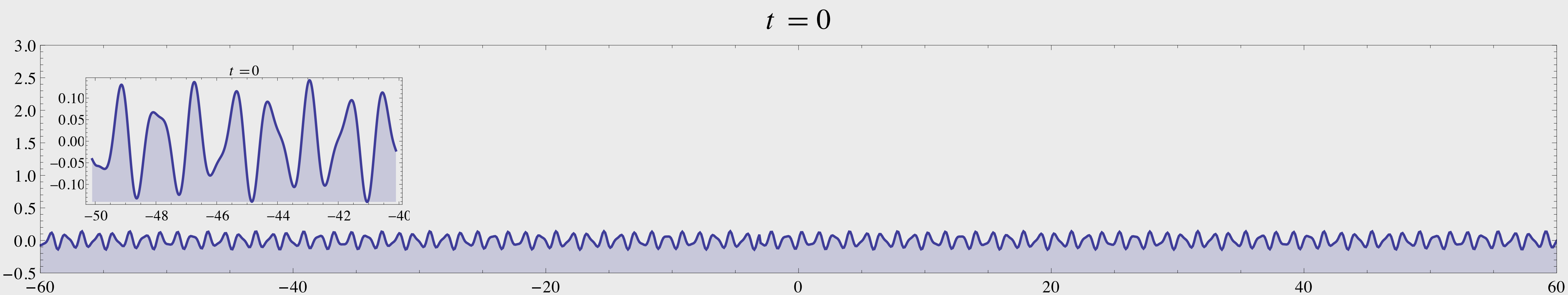


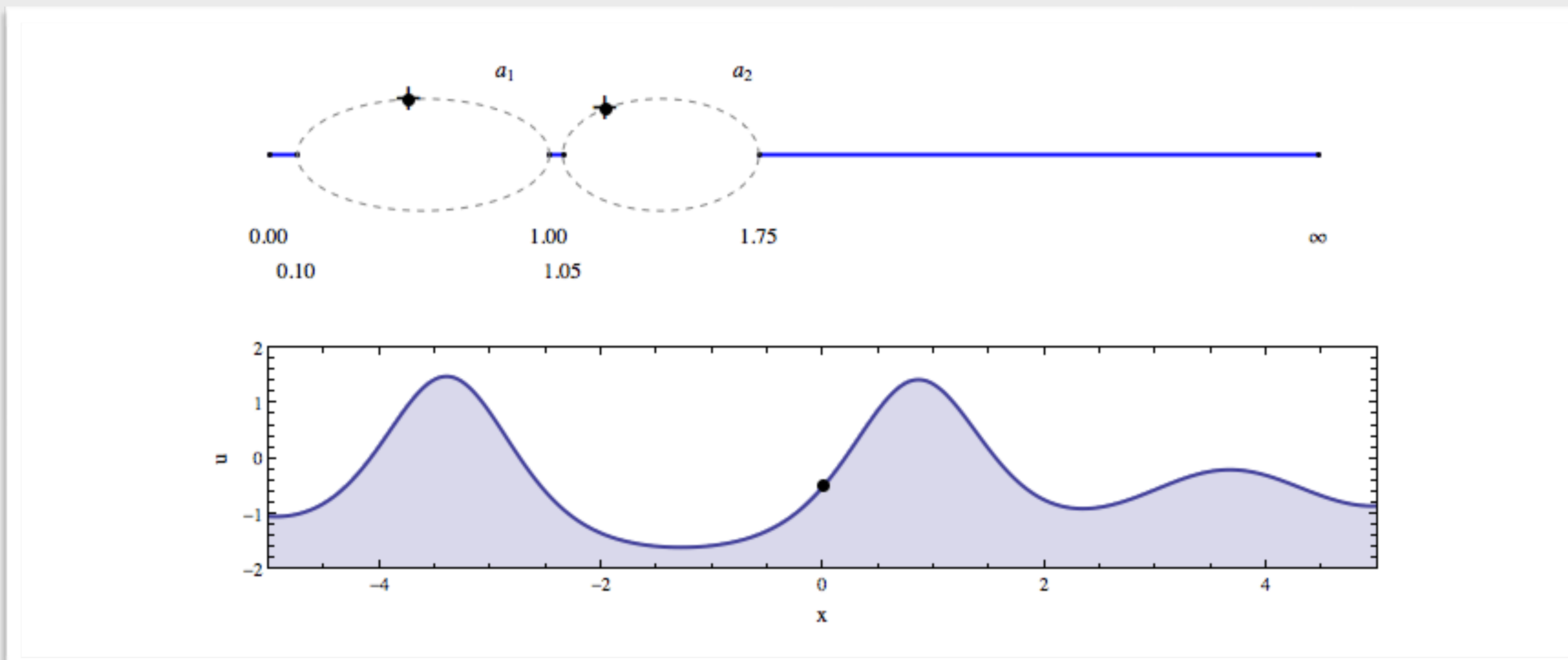
0 −1
1 0
−b2 −a2
−b1 −a1
a1 b1
a2 b2
t = 0
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
−60
−40
−20
0
20
40
60
0.10
0.05
0.00
−0.05
−0.10
−50
−48
−46
−44
−42

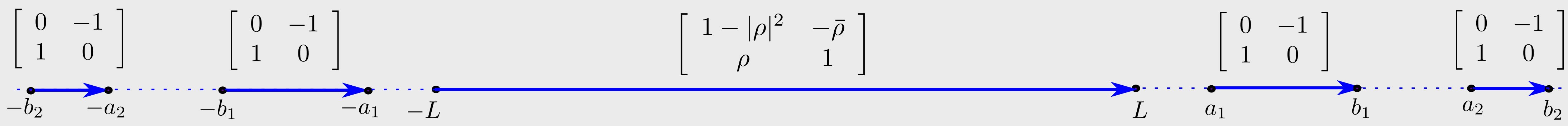


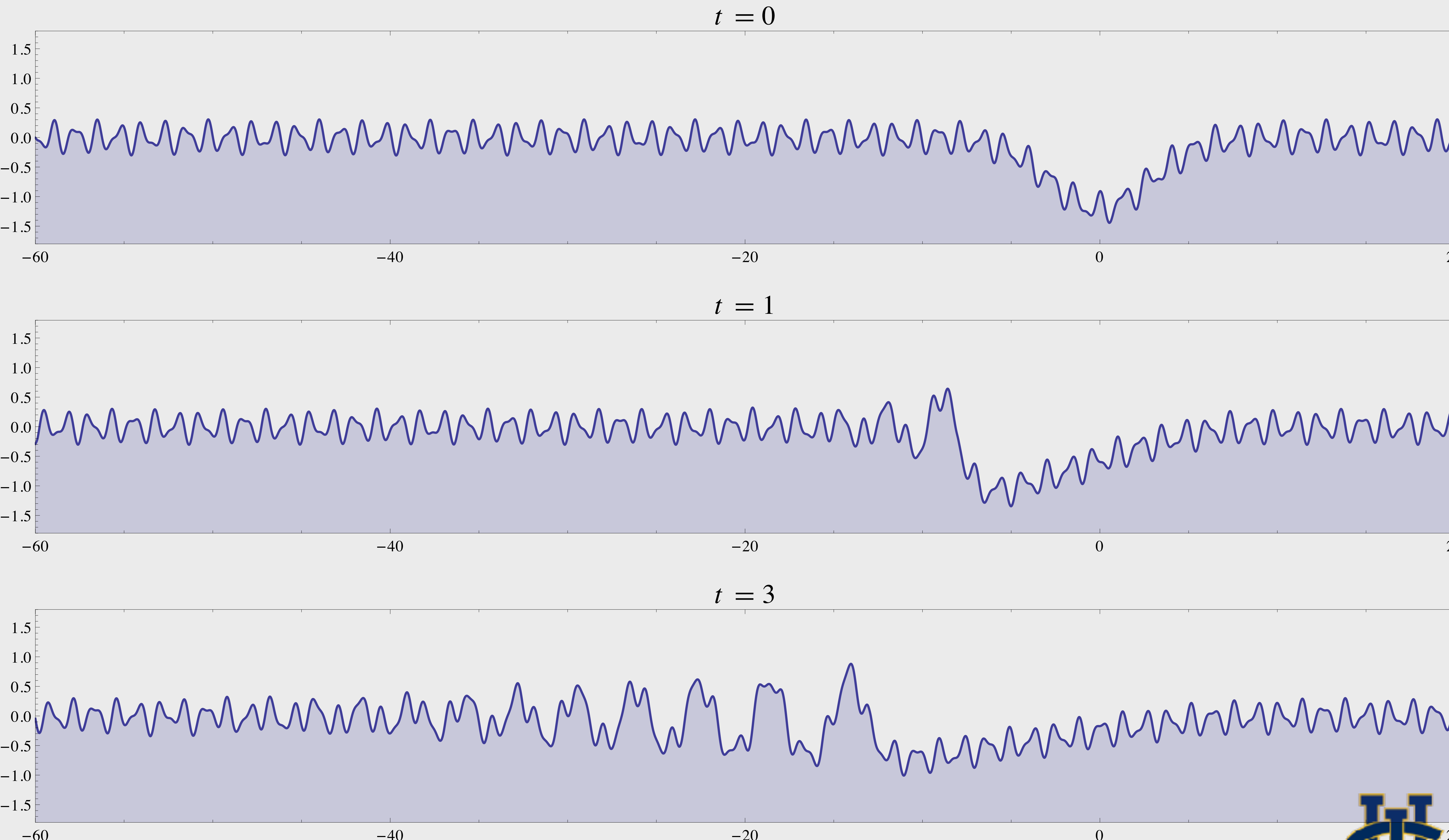
a1
a2
0.00
0.10
1.00
1.05
1.75
∞
u
x
2
1
0
−1
−2
−4
−2
0
2
4


$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$-b_2$ $-a_2$
$-b_1$ $-a_1$
a_1 b_1
a_2 b_2

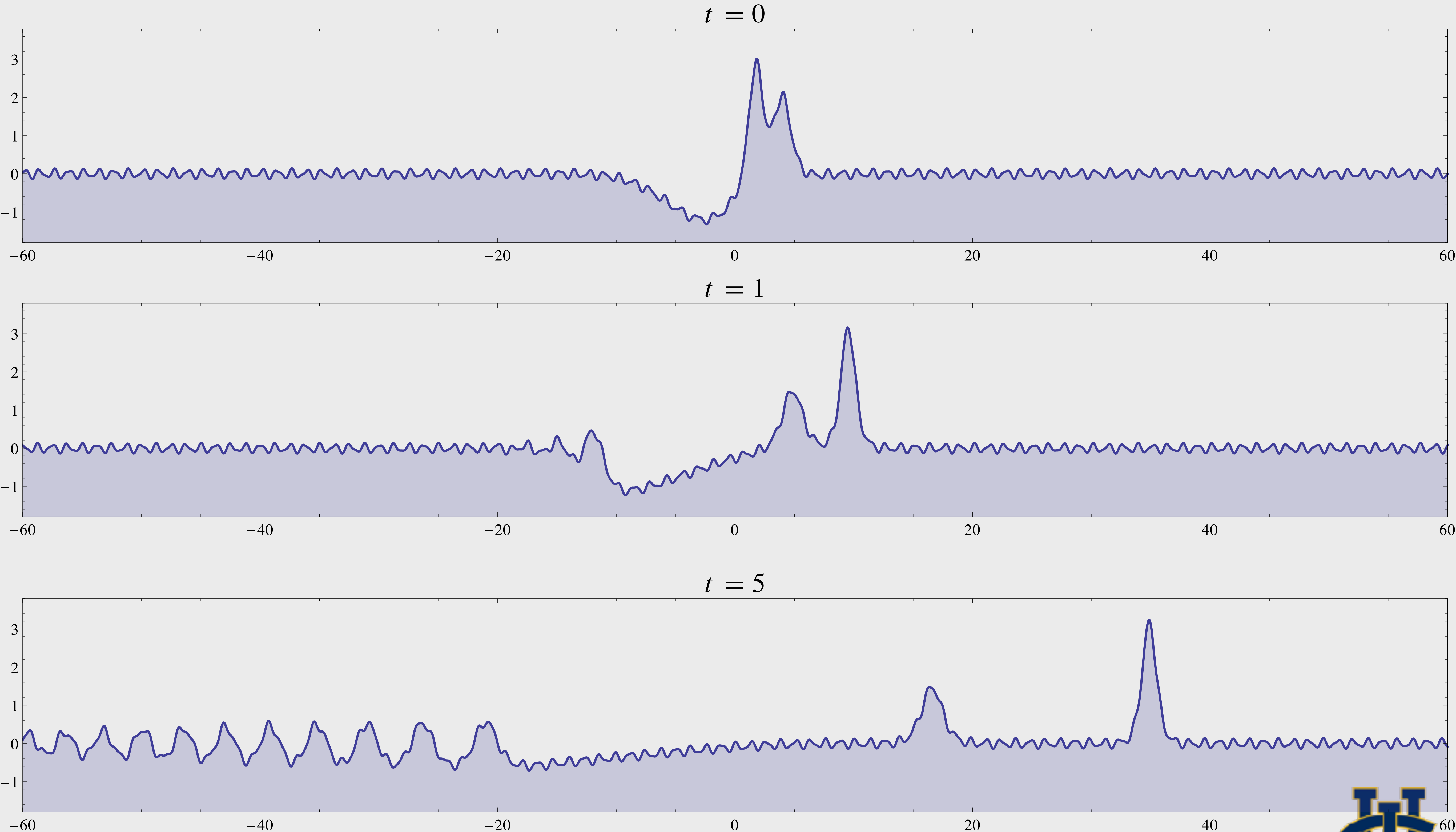

$t = 0$
3.0 2.5 2.0 1.5 1.0 0.5 0.0 −0.5
−60 −40 −20 0 20 40 60
$t = 0$
0.10 0.05 0.00 −0.05 −0.10
−50 −48 −46 −44 −42 −40


a_1
a_2
0.00
0.10
1.00
1.05
1.75
∞
u
x
2 1 0 −1 −2
−4 −2 0 2 4


$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 1-|\rho|^2 & -\bar{\rho} \\ \rho & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$-b_2$ $-a_2$ $-b_1$ $-a_1$ $-L$ L a_1 b_1 a_2 b_2


$t = 0$
$t = 1$
$t = 3$
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-60
-40
-20
0
20

With some solitons



Dispersive shock wave solutions of the KdV equation

w/ Deniz Bilman

The KdV equation

Consider solving the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty),$$

with initial data

$$q(x,0) = q_0(x) = u_0(x) - c^2 H(x), \quad H(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Assume that $u_0(x)$ has its only discontinuity at $x = 0$ and decays rapidly as $|x| \to \infty$.

For $u_0 = 0$, we have the so-called dispersive Riemann problem for the KdV equation.

General data can be treated using the Galilean boost.

Other work

- Schrödinger scattering theory for such potentials:

  A Cohen and T Kappeler. Scattering and inverse scattering for steplike potentials in the Schrodinger equation. *Indiana Univ. Math. J.*, 34:127–180, 1985

- Existence of solutions using GLM:

  A Cohen. Solutions of the Korteweg–de Vries equation with steplike initial profile. *Commun. Partial Differ. Equations*, 9(8):751–806, jan 1984

  T Kappeler. Solutions of the Korteweg-deVries equation with steplike initial data. *J. Differ. Equ.*, 63(3):306–331, jul 1986

- Riemann–Hilbert approach to rarefaction:

  K Andreiev, I Egorova, T L Lange, and G Teschl. Rarefaction waves of the Korteweg–de Vries equation via nonlinear steepest descent. *J. Differ. Equ.*, 261(10):5371–5410, 2016

  K. Andreiev and I. Egorova. On the Long-Time Asymptotics for the Korteweg-de Vries Equation with Steplike Initial Data Associated with Rarefaction Waves. *J. Math. Physics, Anal. Geom.*, 13(4):325–343, dec 2017

- DSWs and long-time asymptotics:

  M J Ablowitz and D E Baldwin. Dispersive shock wave interactions and asymptotics. *Phys. Rev. E*, 87(2):022906, feb 2013

- Riemann–Hilbert approach to dispersive shock waves:

  I Egorova, Z Gladka, V Kotlyarov, and G Teschl. Long-time asymptotics for the Korteweg–de Vries equation with step-like initial data. *Nonlinearity*, 26(7):1839–1864, jul 2013

Riemann–Hilbert Problem 1. *The function* $\Phi_1 : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{1\times 2}$, $\Phi_1(z) = \Phi_1(z; x, t)$, *satisfies*

$$
\begin{aligned}
\Phi_1^+(s) &= \Phi_1^-(s)\begin{bmatrix} 1-|R_1(s)|^2 & -\overline{R_1}(s)\mathrm{e}^{2\mathrm{i}sx+8\mathrm{i}s^3t} \\ R_1(s)\mathrm{e}^{-2\mathrm{i}sx-8\mathrm{i}s^3t} & 1 \end{bmatrix}, \quad s \in \mathbb{R}, \\
\Phi_1^+(s) &= \Phi_1^-(s)\begin{bmatrix} 1 & 0 \\ -\frac{c(z_j)}{s-z_j}\mathrm{e}^{-2\mathrm{i}z_jx-8\mathrm{i}z_j^3t} & 1 \end{bmatrix}, \quad s \in \Sigma_j, \\
\Phi_1^+(s) &= \Phi_1^-(s)\begin{bmatrix} 1 & -\frac{c(z_j)}{s+z_j}\mathrm{e}^{-2\mathrm{i}z_jx-8\mathrm{i}z_j^3t} \\ 0 & 1 \end{bmatrix}, \quad s \in -\Sigma_j, \\
\Phi_1(z) &= \begin{bmatrix} 1 & 1 \end{bmatrix} + O(z^{-1}), \quad z \in \mathbb{C}\setminus\mathbb{R},
\end{aligned}
$$

with the symmetry condition

$$
\Phi_1(-z) = \Phi_1(z)\sigma_1, \quad z \in \mathbb{C}\setminus\Gamma, \quad \Gamma = \mathbb{R} \cup \bigcup_j (\Sigma_j \cup -\Sigma_j).
$$

Riemann–Hilbert Problem 2. *The function* $\Phi_2 : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}^{1\times 2}$, $\Phi_2(z) = \Phi_2(z; x, t)$ *satisfies*

$$\Phi_2^+(s) = \Phi_2^-(s)\begin{bmatrix} 1-|R_{\mathrm{r}}(s)|^2 & -R_{\mathrm{r}}(-s)e^{-2i\lambda(s)x-8i\varphi(s)t} \\ R_{\mathrm{r}}(s)e^{2i\lambda(s)x+8i\varphi(s)t} & 1 \end{bmatrix}, \quad s^2 > c^2,$$

$$\Phi_2^+(s) = \Phi_2^-(s)\begin{bmatrix} 1 & -R_{\mathrm{r}}(-s)e^{-2i\lambda^-(s)x-8i\varphi^-(s)t} \\ 0 & 1 \end{bmatrix}\sigma_1\begin{bmatrix} 1 & 0 \\ R_{\mathrm{r}}(s)e^{2i\lambda^+(s)x+8i\varphi^+(s)t} & 1 \end{bmatrix}, \quad -c \le s \le c,$$

$$\Phi_2^+(s) = \Phi_2^-(s)\begin{bmatrix} 1 & 0 \\ -\frac{C(z_j)}{s-z_j}e^{2i\lambda(z_j)x+8i\varphi(z_j)t} & 1 \end{bmatrix}, \quad s \in \Sigma_j,$$

$$\Phi_2^+(s) = \Phi_2^-(s)\begin{bmatrix} 1 & -\frac{C(z_j)}{s+z_j}e^{2i\lambda(z_j)x+8i\varphi(z_j)t} \\ 0 & 1 \end{bmatrix}, \quad s \in -\Sigma_j,$$

$$\Phi_2(z) = \begin{bmatrix} 1 & 1 \end{bmatrix} + O(z^{-1}), \;\; z \to \infty, \quad \varphi(s) = \lambda^3(s) + 3/2c^2\lambda(s),$$

with the symmetry condition

$$\Phi_2(-z) = \Phi_2(z)\sigma_1, \quad z \in \mathbb{C}\setminus\mathbb{R}.$$

Riemann–Hilbert Problem 2. *The function* $\Phi_2 : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}^{1\times 2}$, $\Phi_2(z) = \Phi_2(z;x,t)$ *satisfies*

$$\Phi_2^+(s) = \Phi_2^-(s)\begin{bmatrix} 1-|R_{\mathrm{r}}(s)|^2 & -R_{\mathrm{r}}(-s)\mathrm{e}^{-2\mathrm{i}\lambda(s)x-8\mathrm{i}\varphi(s)t} \\ R_{\mathrm{r}}(s)\mathrm{e}^{2\mathrm{i}\lambda(s)x+8\mathrm{i}\varphi(s)t} & 1 \end{bmatrix}, \quad s^2 > c^2,$$

$$\Phi_2^+(s) = \Phi_2^-(s)\begin{bmatrix} 1 & -R_{\mathrm{r}}(-s)\mathrm{e}^{-2\mathrm{i}\lambda^-(s)x-8\mathrm{i}\varphi^-(s)t} \\ 0 & 1 \end{bmatrix}\sigma_1\begin{bmatrix} 1 & 0 \\ R_{\mathrm{r}}(s)\mathrm{e}^{2\mathrm{i}\lambda^+(s)x+8\mathrm{i}\varphi^+(s)t} & 1 \end{bmatrix}, \quad -c \leq s \leq c,$$

$$\Phi_2^+(s) = \Phi_2^-(s)\begin{bmatrix} 1 & 0 \\ -\frac{C(z_j)}{s-z_j}\mathrm{e}^{2\mathrm{i}\lambda(z_j)x+8\mathrm{i}\varphi(z_j)t} & 1 \end{bmatrix}, \quad s \in \Sigma_j,$$

$$\Phi_2^+(s) = \Phi_2^-(s)\begin{bmatrix} 1 & -\frac{C(z_j)}{s+z_j}\mathrm{e}^{2\mathrm{i}\lambda(z_j)x+8\mathrm{i}\varphi(z_j)t} \\ 0 & 1 \end{bmatrix}, \quad s \in -\Sigma_j,$$

$$\Phi_2(z) = \begin{bmatrix} 1 & 1 \end{bmatrix} + O(z^{-1}), \quad z \to \infty, \quad \varphi(s) = \lambda^3(s) + 3/2c^2\lambda(s),$$

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with the symmetry condition

$$\Phi_2(-z) = \Phi_2(z)\sigma_1, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Deformations

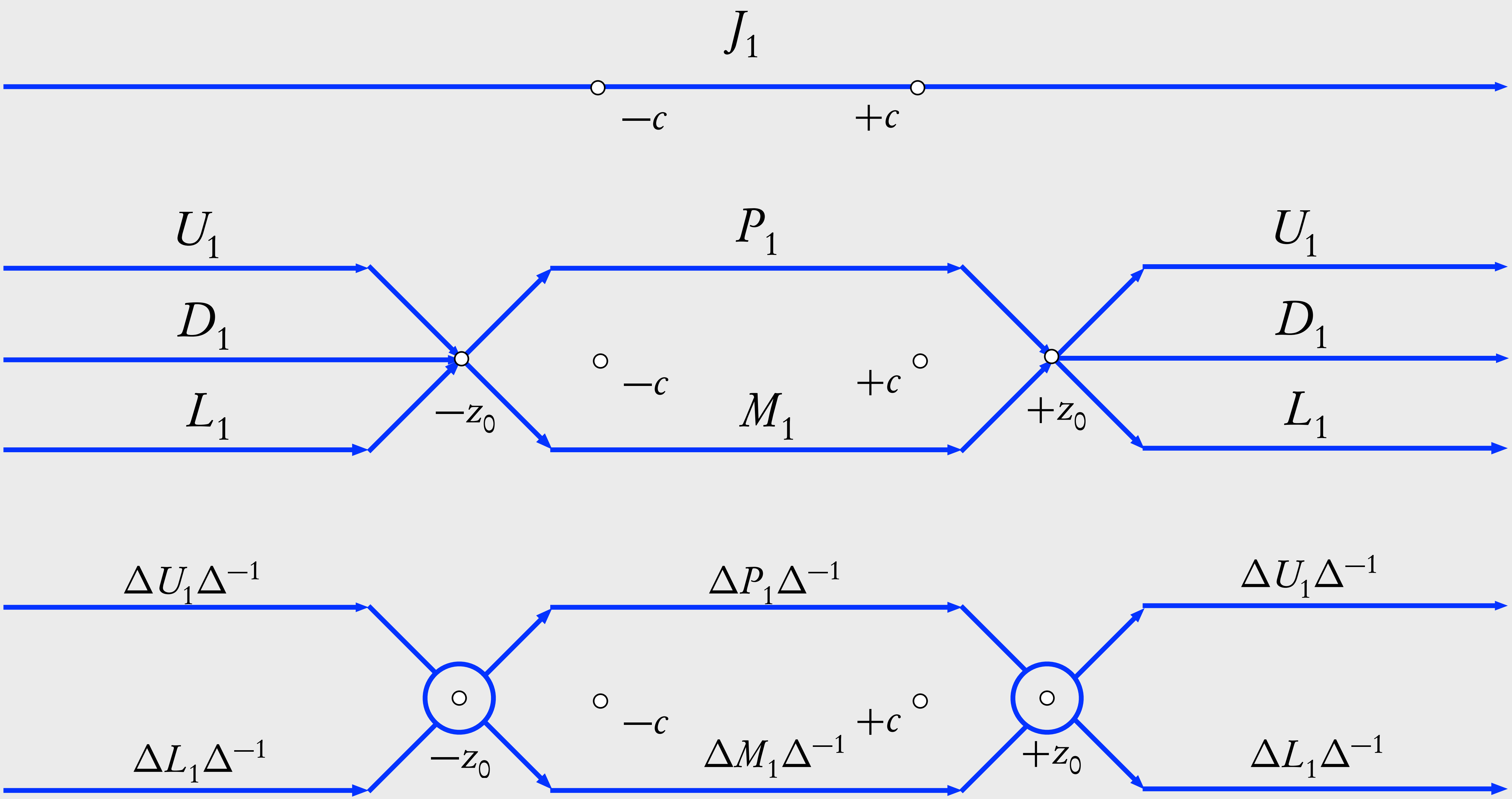
We have not implemented all the deformations that are required to compute $q(x,t)$ for all (x,t).

Two deformations allow us to compute the solution for all x and small time, $0 < t < 2$.

More to come, including $t \to \infty$!

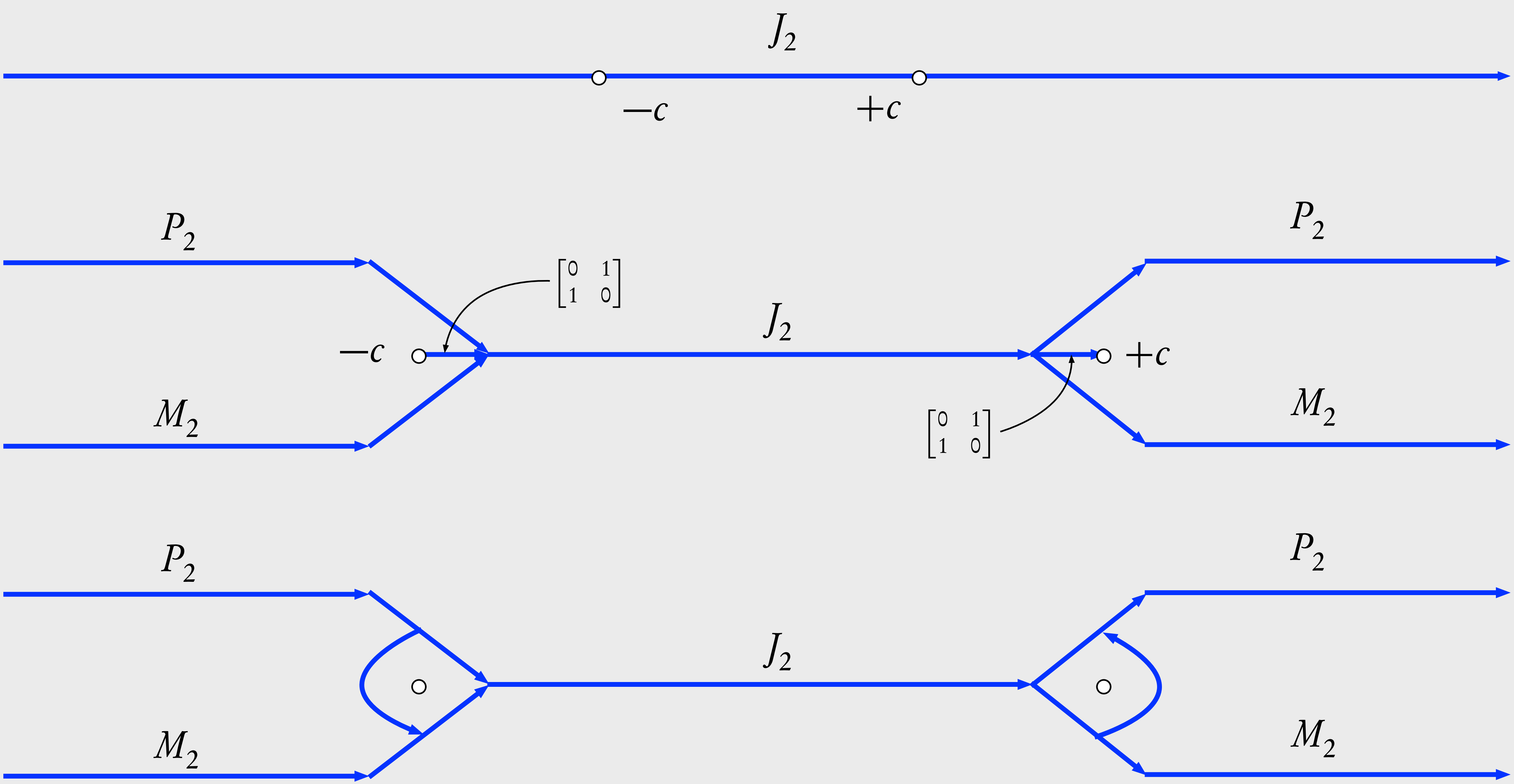
All deformations are performed assuming $u_0 \in L^1(\mathbb{R}, e^{\delta|x|}dx)$ for some $\delta > 0$.

$$x < 0 \qquad -\sqrt{\frac{-x}{12t}} < -c - \epsilon \qquad \epsilon > 0$$



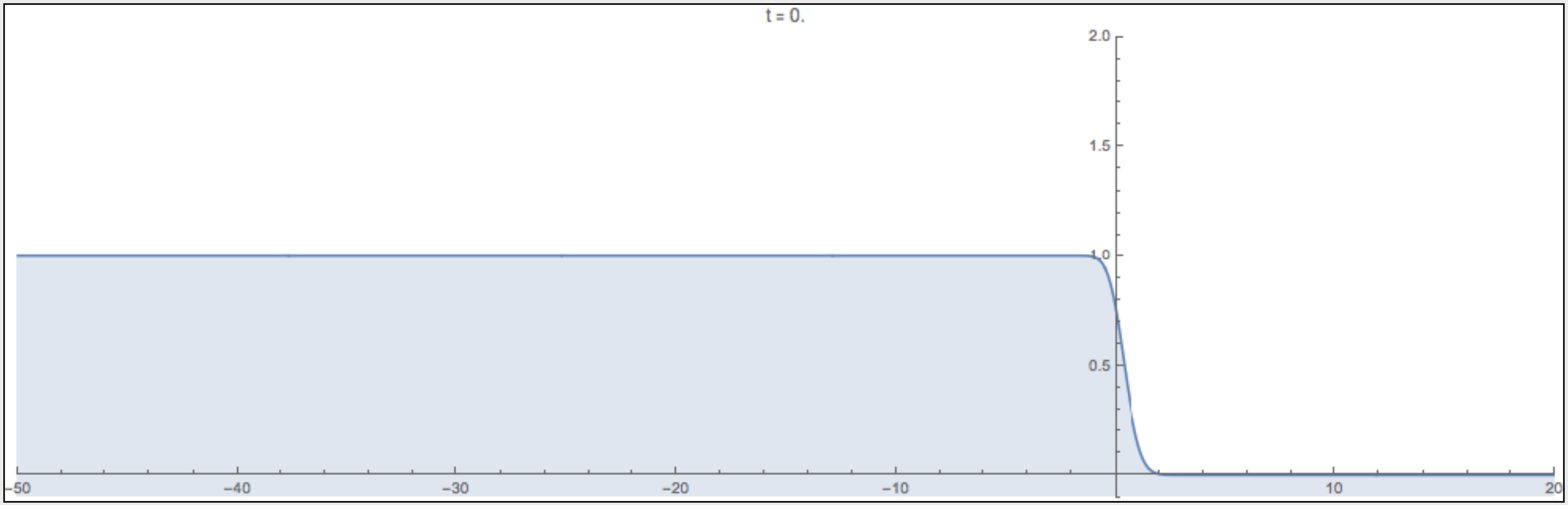
The LDU-factorization fails on $[-c, c]$.

$$x \geq -2c^2 t$$

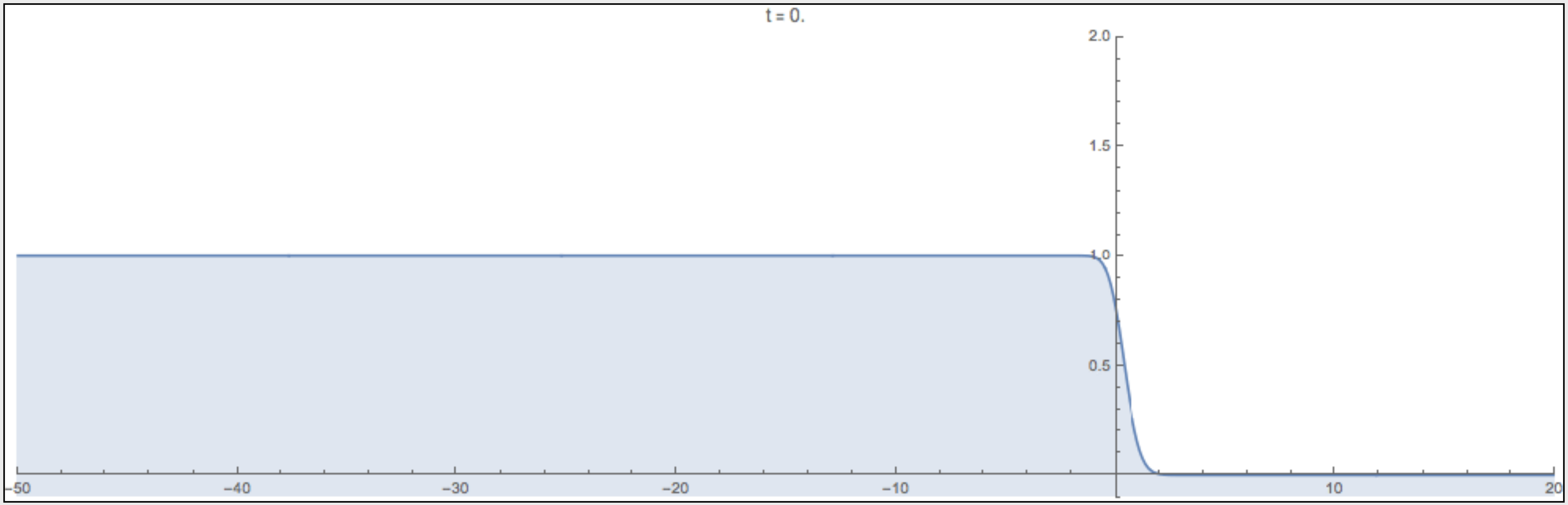


J_2
$-c$
$+c$
P_2
P_2
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
J_2
$-c$
$+c$
M_2
M_2
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
P_2
P_2
J_2
M_2
M_2

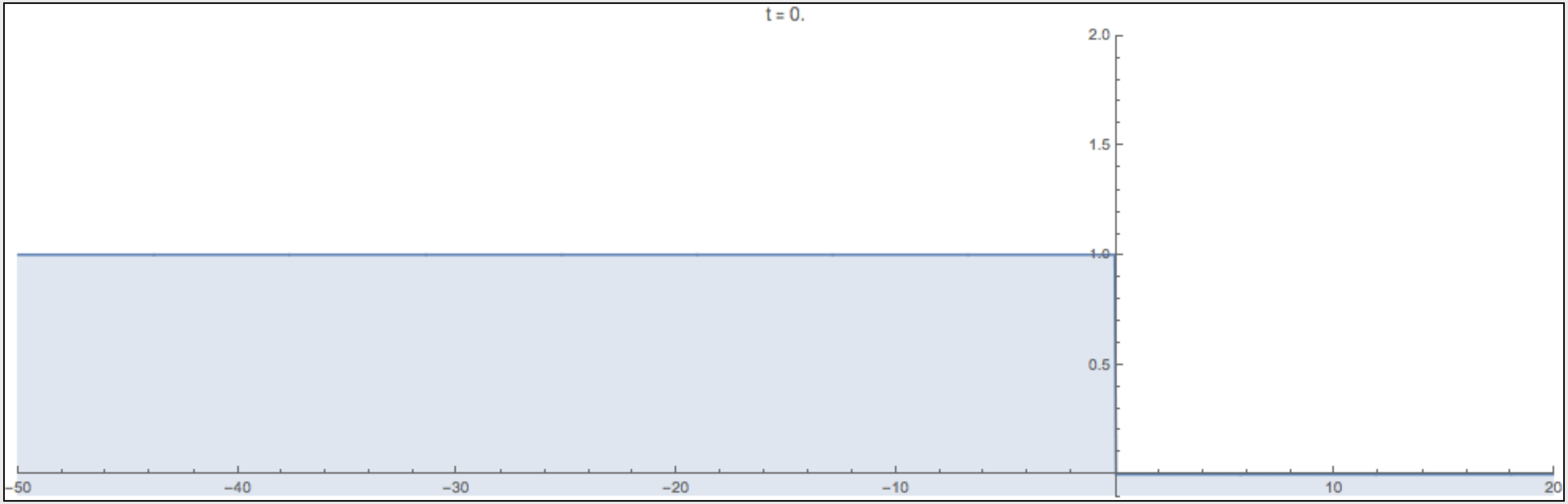
$$q_0(x) = \frac{1}{4}(1+\mathrm{erf}(x))^2 \qquad c = 1$$



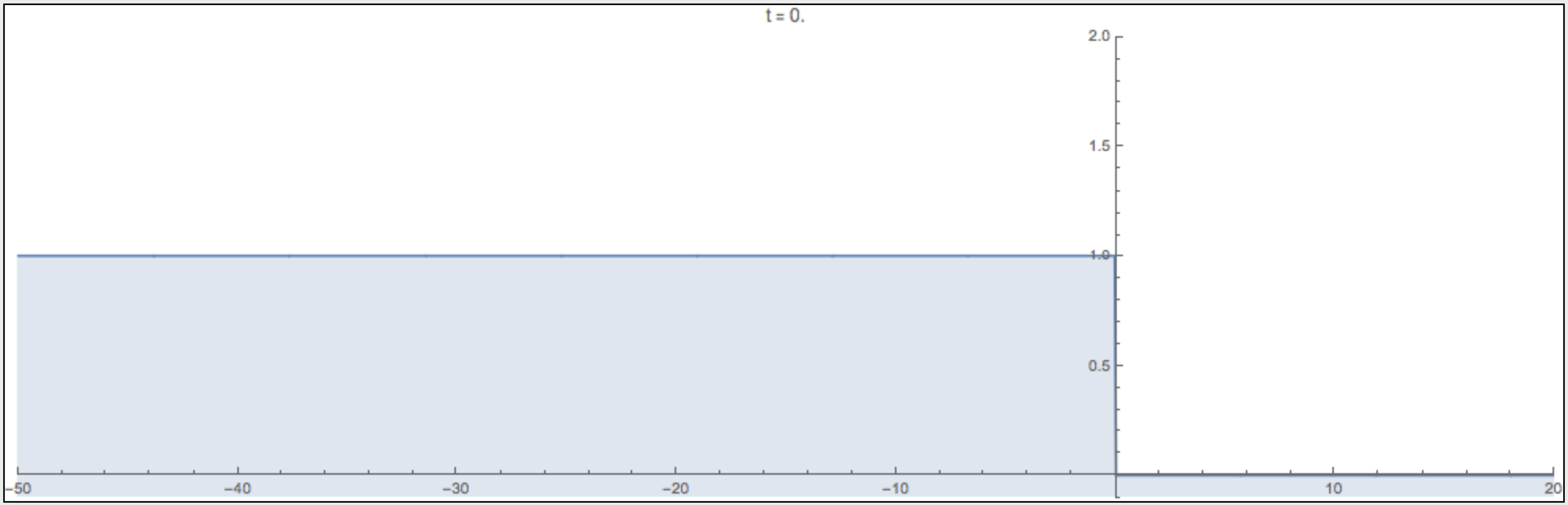
$$q_0(x) = \frac{1}{4}(1 + \mathrm{erf}(x))^2 \qquad c = 1$$



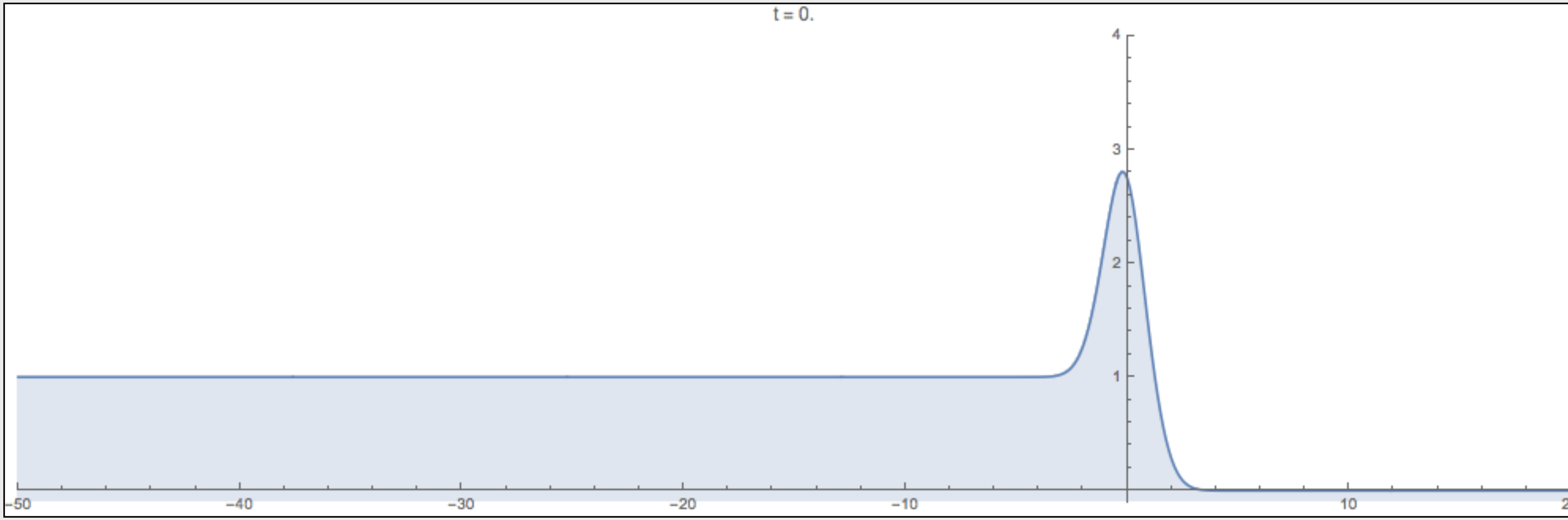
$$q_0(x) = c^2(1 - H(x)) \qquad c = 1$$



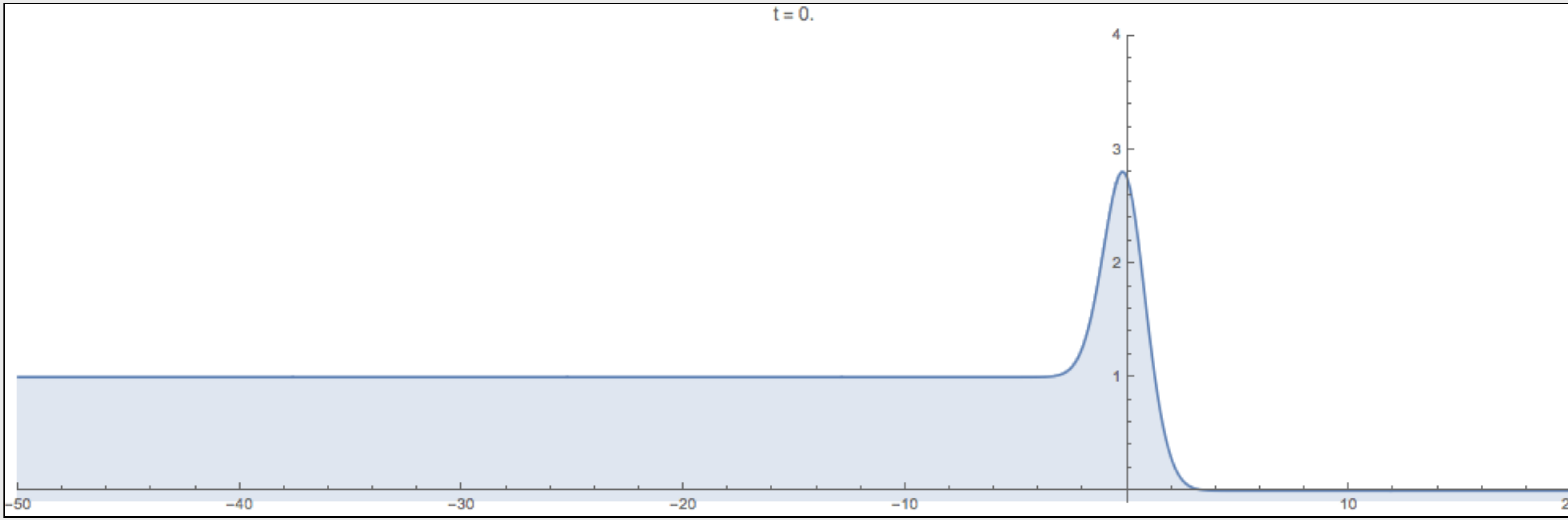
$$q_0(x) = c^2(1 - H(x)) \qquad c = 1$$



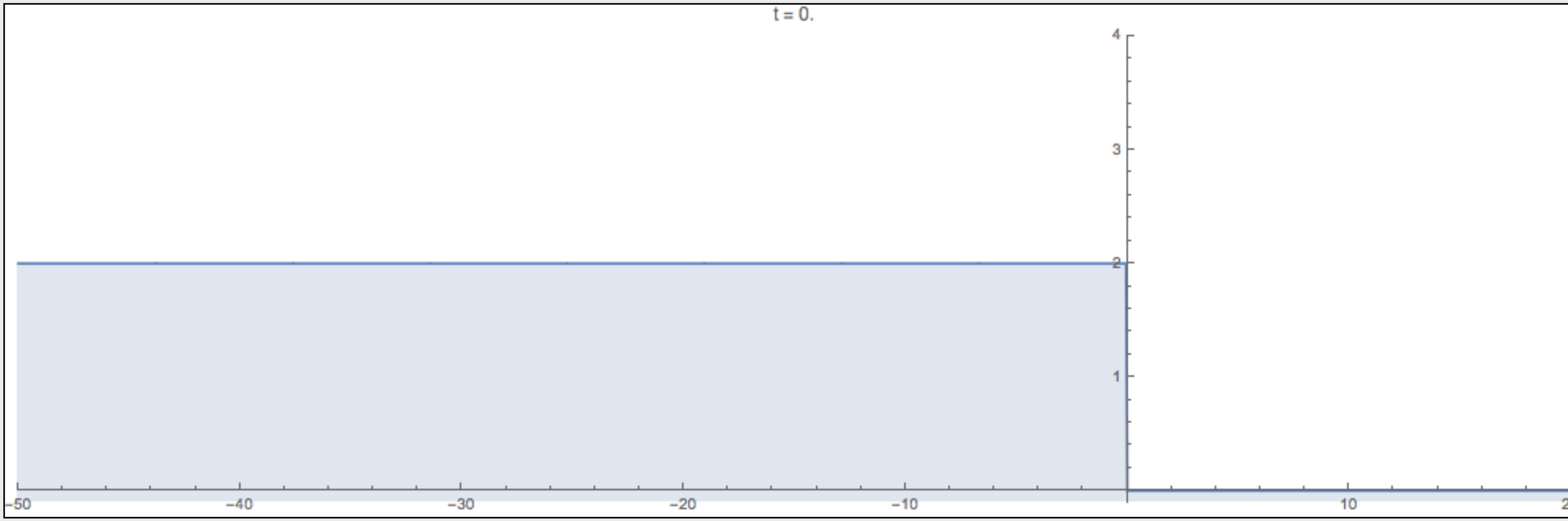
$$q_0(x) = \frac{1}{4}(1 + \mathrm{erf}(x))^2 + 2\mathrm{e}^{-x^2/2} \qquad c = 1$$



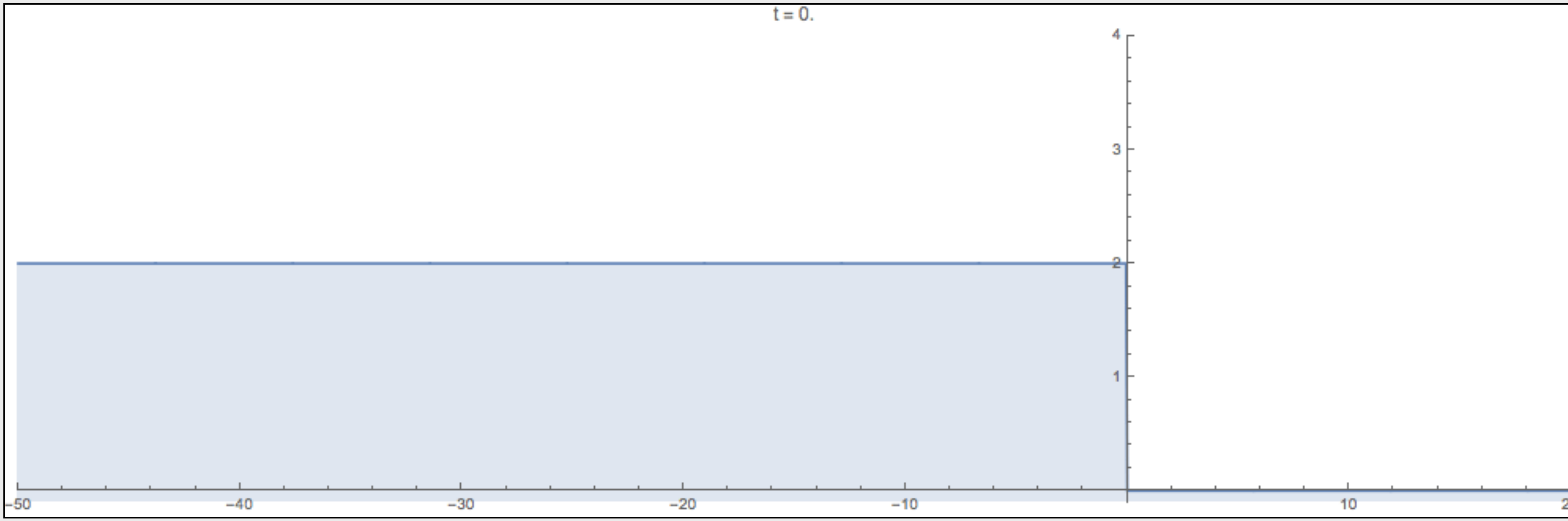
$$q_0(x) = \frac{1}{4}(1 + \mathrm{erf}(x))^2 + 2\mathrm{e}^{-x^2/2} \qquad c = 1$$

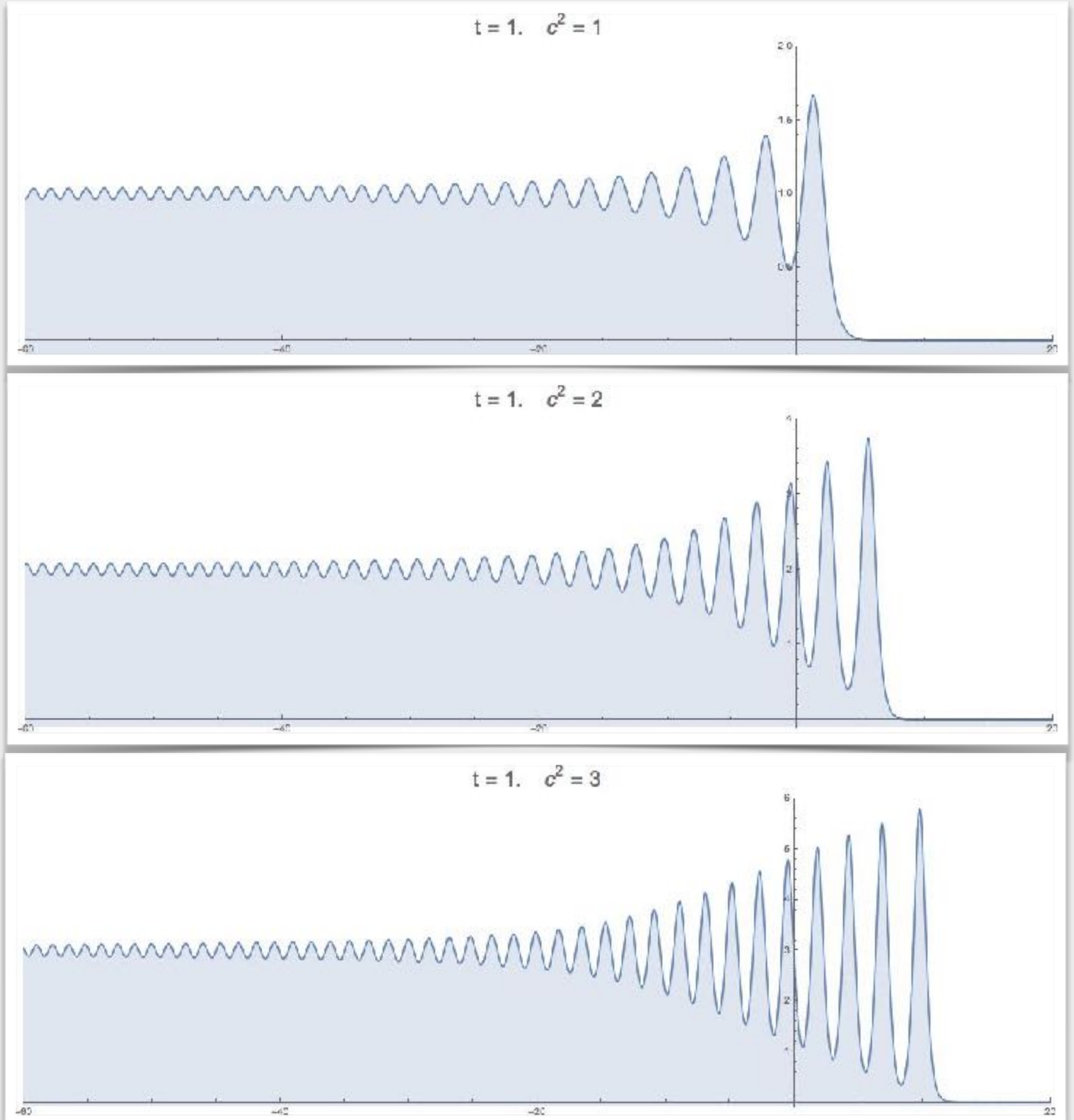


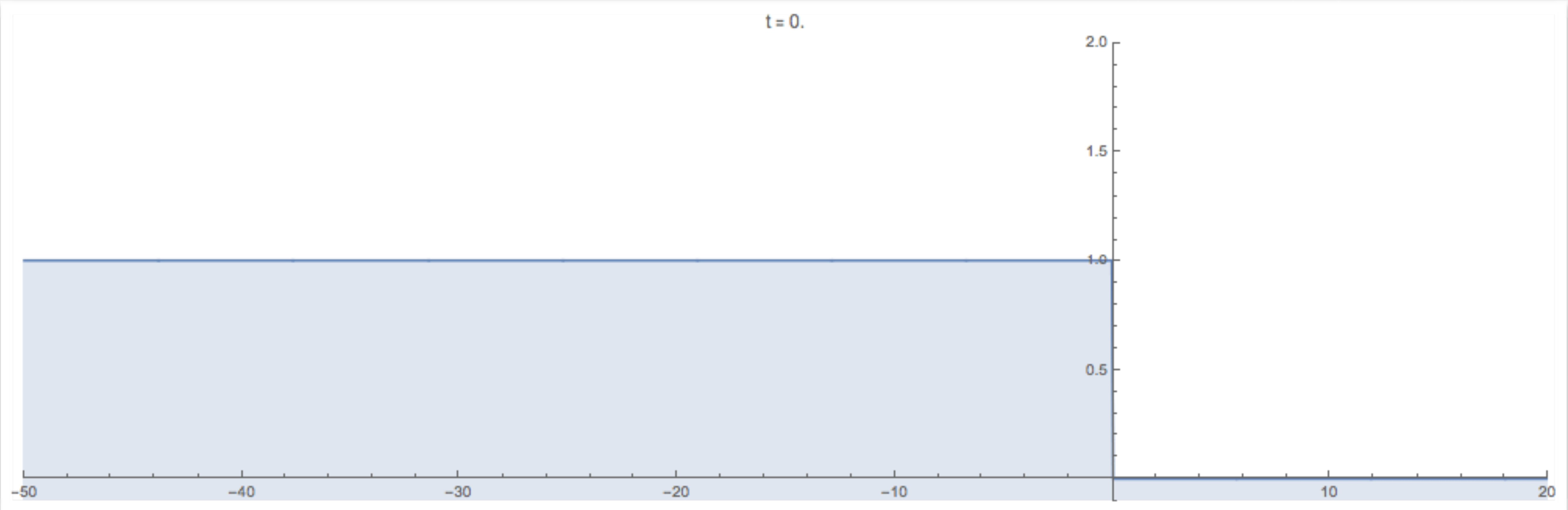
$$q_0(x) = c^2(1 - H(x)) \qquad c = \sqrt{2}$$

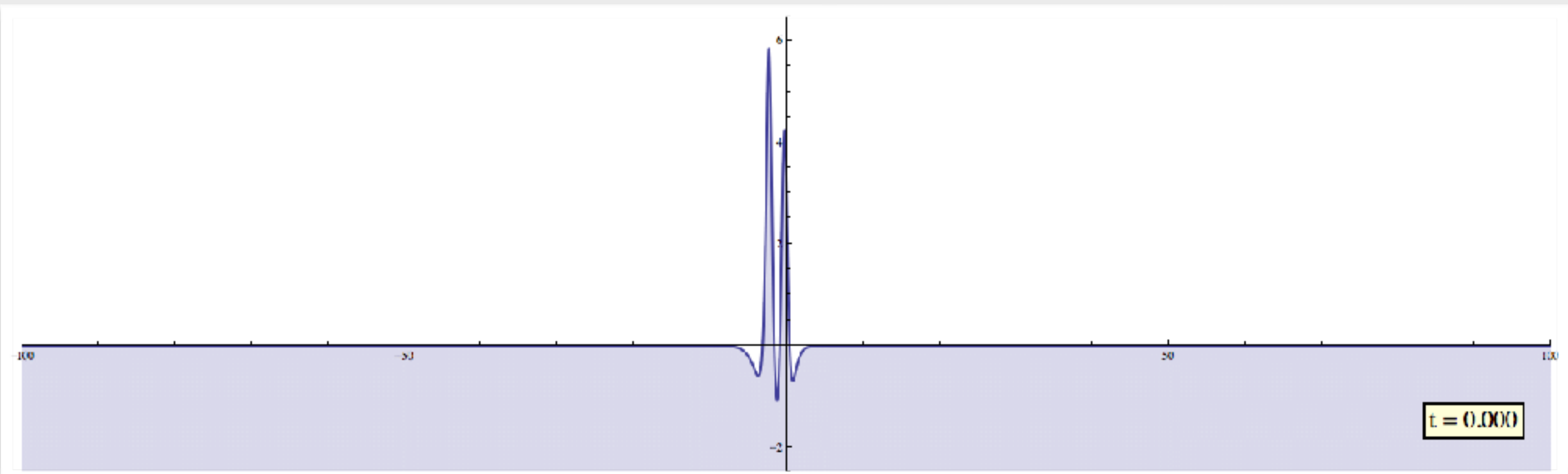


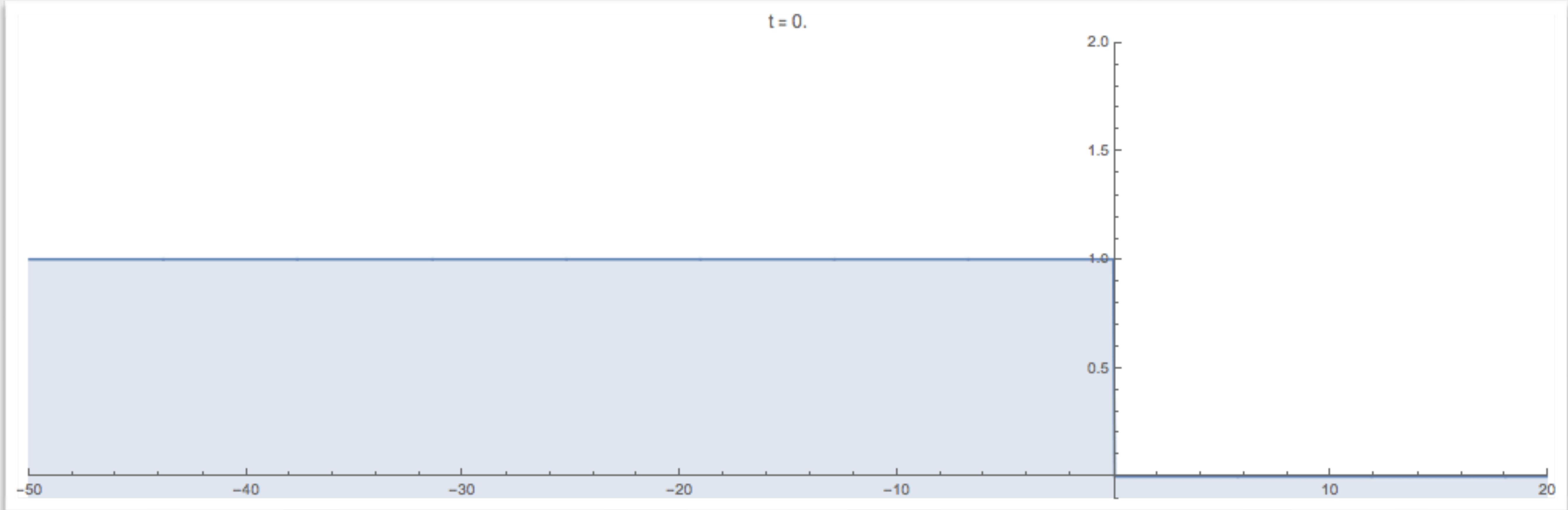
$$q_0(x) = c^2(1 - H(x)) \qquad c = \sqrt{2}$$

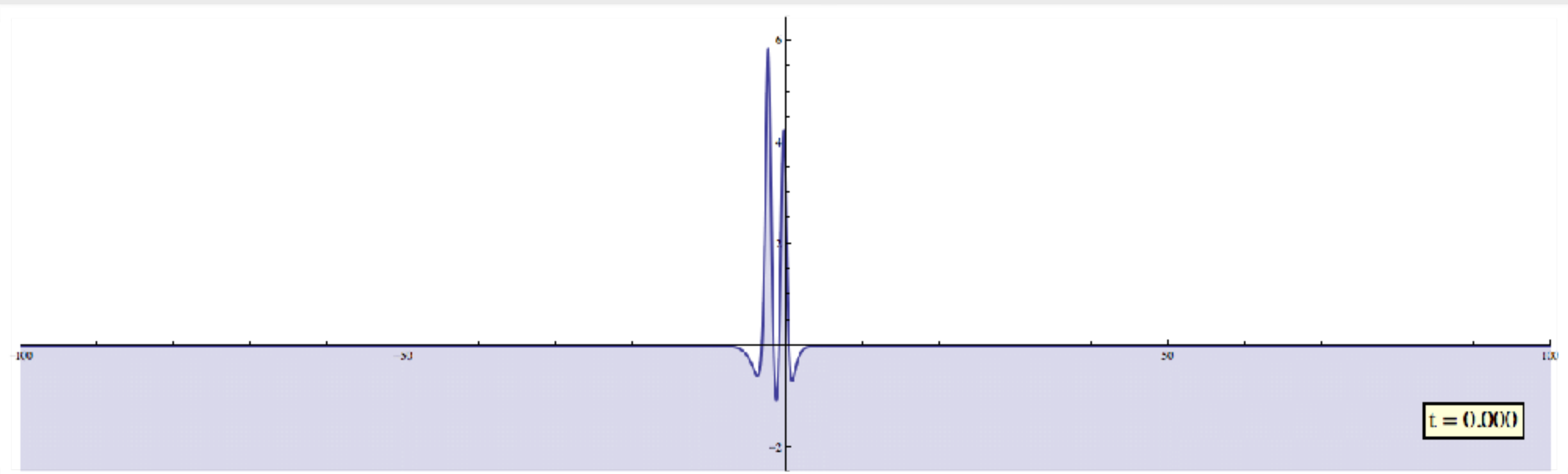


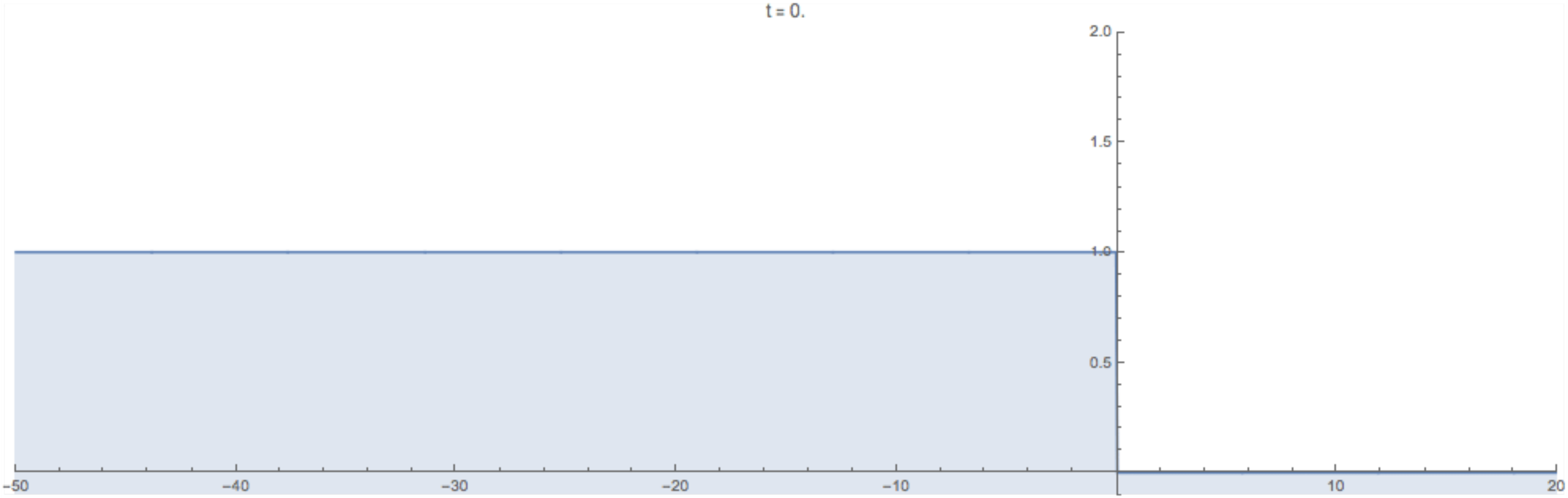

t = 1.   c² = 1
t = 1.   c² = 2
t = 1.   c² = 3

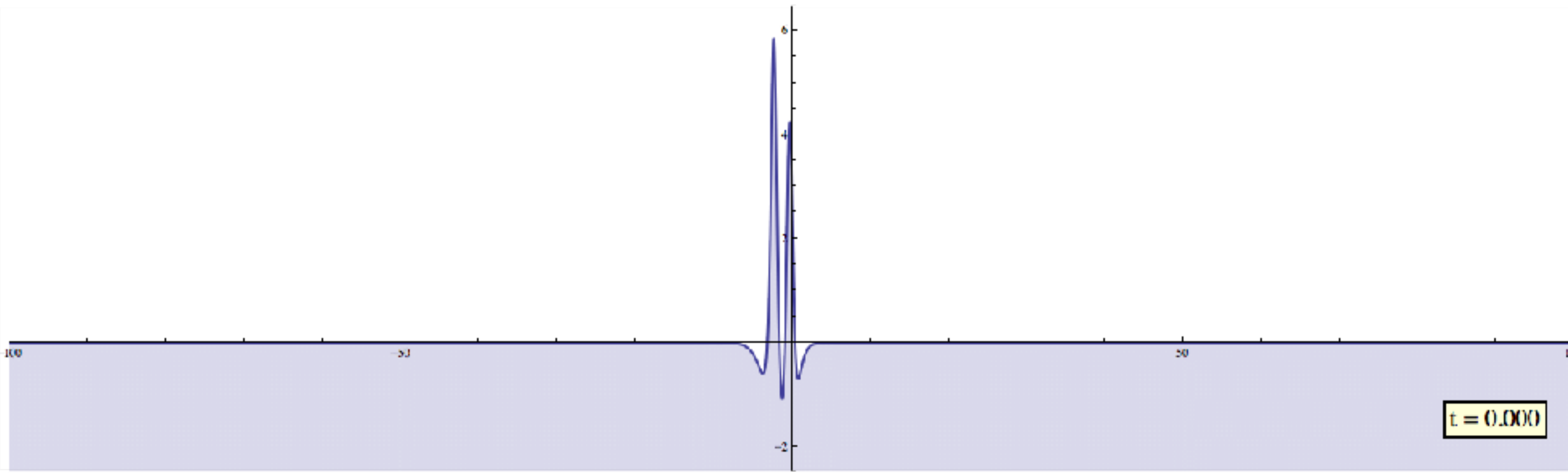

t = 0.
2.0
1.5
1.0
0.5
−50
−40
−30
−20
−10
10
20


t = 0.000


t = 0.
2.0
1.5
1.0
0.5
−50
−40
−30
−20
−10
10
20


t = 0.000


t = 0.
2.0
1.5
1.0
0.5
−50
−40
−30
−20
−10
10
20


6
4
−2
−100
−50
50
100
t = 0.000