Control of Linear Viscoelastic Models

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Integrable Systems in Mathematics,
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Contents

Control problems

Question : How to steer the solution trajectory $x(t)$ of a differential equation towards a desired profile using a control $u(t)$?

$$\dot{x} = Ax + Bu(t), \quad x(0) = x_0,$$

Either an ODE evolving in $\mathbb{R}^N$, or a PDE evolving in some function space.

The control operator B may act only on some of the equations;

Or for PDEs, it may act in an open subset of the domain,
or on parts of the boundary.

Aim : Choose a control $u(t)$ so that $x(T)$ reaches a desired state.

General Framework : A is a C^0 Semigroup defined on a Hilbert space H.

Control $u(t)$ takes values on a subspace U of H;

B is a bounded linear operator from U to H.

Controllability

System (A, B) is controllable in time $T > 0$:
if there is an admissible control u to steer the system from any x_0 to x_1.

The solution to the system is

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)\,ds.$$

This motivates the study of operator $F_T \; : \; L^2(0, T; U) \longmapsto H$

$$F_T(u) = \int_0^T e^{(T-s)A}Bu(s)\,ds.$$

The system (A, B) is controllable $\Leftrightarrow$ $\operatorname{Im} F_T = H$.

F_T is surjective $\Leftrightarrow$ For its adjoint operator $F_T^* \in \mathcal{L}(H^*, L^2(0, T; U^*))$,
norm of $F_T^* y_T$ is bounded below by norm of y_T.

Adjoint system

The adjoint operator F_T^* involves the solution of adjoint system, with terminal condition y_T:

$$-\dot{y}(t) = A^* y(t), \quad t > 0, \quad y(T) = y_T.$$

Then

$$F_T^* y_T(\cdot) = B^* e^{(T-\cdot)A^*} y_T = B^* y(\cdot).$$

Seek conditions on F_T^* to deduce controllability results for the original system.

Comparing the two systems, deduce from

$$\int_0^T (\dot{x}, y(t))dt = \int_0^T (Ax + Bu(t), y(t))dt,$$

after using the adjoint equation, the basic identity

$$(x(T), y_T) - (x_0, y(0)) = \int_0^T (Bu(t), y(t))dt.$$

Exact and Null Controllability

Lemma : If the solutions of the adjoint system satisfy

$$\|y_T\| \leq C\|B^* y\|_{L^2(0,T;U^*)}$$

then the original system is exactly controllable : There exists a control u driving the system to any point x_1 in H.

Proof

Let L be the linear mapping from $L^2(0,T;U^*)$ to H^*,

$$L(F_T^* y_T) = L(B^* y(t)) = y_T$$

By assumption, it is bounded.

Fix $x(0) = 0$; choose $u = L^*(x_1)$. Then $x(T) = x_1$.

Similar condition for Null controllability: There exists a control u driving the system from any point in H to zero.

Approximate Controllability

Approximately controllable : The reachable states form a dense subset in H.

$$\overline{\text{Image }(F_T)} = \text{Kernel } F_T^*,$$

$\Leftrightarrow$ Null space of F_T^* is zero.

Lemma : If for the solutions of the adjoint system, $B^*y(t) \equiv 0 \Rightarrow y(t) \equiv 0$, then the original system is approximately controllable.

Proof

Suppose not. Then can pick a nonzero y_T orthogonal to every reachable state x_T.

Set $x(0)$ be zero. Then from the earlier identity

$$0 = (x(T), y_T) = \int_0^T (Bu(t), y(t))dt = \int_0^T (u(t), B^*y(t))dt$$

for every choice of control u. Thus $B^*y(t) \equiv 0$.

By assumption $y(t) \equiv 0$ and thus $y_T = 0$ too.

Case of ODE

For finite dimensional linear operators surjectivity $\Leftrightarrow$ injectivity.
Kalman Rank condition gives controllability.

All three Controllability notions are equivalent!

ODE system is reversible $\Rightarrow$ Exact Controllability is equivalent to Null Controllability.

The subspace of reachable states is dense, then it has to be the whole space. So Approximate Controllability equivalent to Exact Controllability!

Assuming simple eigenvalues $\{\lambda_k\}_{k=1}^n$ for the matrix A^*, we can write

$$y_T = \sum_{k=1}^{n} \alpha_k v_k \;\Rightarrow\; y(t) = \sum_{k=1}^{n} \alpha_k v_k e^{\lambda_k t}$$

If there is a u_0 not orthogonal to all $B^* v_k$, then system is approximately controllable.

Case of ODE

Suppose that $B^* y(t) = 0$. Then for any u in U,

$$(u, B^* y(t)) = 0, \Rightarrow \sum_{k=1}^{n} \alpha_k \,(u, B^* v_k)\, e^{\lambda_k t} = 0.$$

From this finite sum of exponentials, how to conclude each coefficient is zero?

Useful tool in such a situation is Laplace transform :

$$\mathcal{L}(B^* y(t))(\mu) = \int_0^\infty \sum_{k=1}^{n} \alpha_k (u_0, B^* v_k) e^{(\lambda_k - \mu)t} dt = \sum_{k=1}^{n} \frac{\alpha_k}{\lambda_k - \mu} (u_0, B^* v_k).$$

This is a meromorphic function with poles at $\{\lambda_k\}_{k=1}^{n}$.
Choose a contour Γ enclosing just one λ_k. By Cauchy's residue theorem

$$\int_\Gamma \mathcal{L}(B^* y(t))(\mu) = 2\pi i \; \alpha_k \; (u_0, B^* v_k).$$

As $B^* y(t) \equiv 0$, its Laplace transform and its integral vanish. Thus $\alpha_k = 0$.

Case of PDE

- View PDE system as ODE evolving in an infinite dimensional space.
- All three controllability notions are now distinct!
- In general, hyperbolic PDE is Exactly Controllable for large T.
- Parabolic PDE Null Controllable for $T > 0$

If the spatial operator has discrete spectrum and eigenfunction expansion holds, then solution can be expressed as an infinite sum.

Can we extend the ODE case arguments to PDE case for approximate controllability?

Case of PDE

If B is an interior localization operator χ_ω, then for approximate controllability, the condition is :

If $y(x,t)|_{\omega\times(0,T)} = 0$, then does y vanish everywhere in $\Omega \times (0,T)$?

Such unique continuation property is true for eigenfunctions of Laplacian and also Stokes operator in time independent case.

For the parabolic part of a PDE system, one can expect such a result and also to use similar arguments as in the ODE case.

Challenges :

- Is the solution defined as an infinite sum, extended for all $t \geq 0$?
- Can we take the Laplace transform for this infinite sum?
- Can we use the residue theorem to isolate the coefficient?

Visco-elastic flows

Fluid flow with velocity u , constant density $\rho > 0$, pressure p and stress tensor τ .

Conservation of mass and momentum:

$$\nabla . u = 0, \quad \rho u_t \;=\; \nabla . \tau - \nabla p.$$

Constitutive Law: Relates the symmetric stress tensor to the motion.

Linear viscoelasticity : For a function G - stress relaxation modulus,

$$\tau(x,t) = \int_0^{\infty} G(s)\Big(\nabla u(t-s) + (\nabla u)^T(t-s)\Big) ds.$$

Newtonian Fluid : $G(s) = \delta(s)$, $\tau_{New} = \eta(\nabla u + (\nabla u)^T), \quad \eta > 0,$

Maxwell's theory of linear elasticity :

$$G(s) = \kappa exp(-\lambda s), \; \lambda > 0, \; \kappa > 0, \; 1/\lambda = \text{ Stress relaxation time.}$$

Then stress tensor satisfies :

$$\tau_t + \lambda \tau = \kappa\Big(\nabla u + (\nabla u)^T\Big), \quad ,$$

Jeffreys and Maxwell Models

Jeffreys model: Stress is a linear combination of Newtonian and Maxwell stress:

$$G(s) = \eta\delta(s) + \kappa exp(-\lambda s); \quad \text{Total Stress} = \tau_{New} + \tau.$$

Maxwell model: $\tau_t + \lambda\tau = \kappa\Big(\nabla u + (\nabla u)^T\Big).$

System in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$

$$\begin{aligned} \rho u_t &= \eta\triangle u + \nabla.\tau - \nabla p + f\chi_{\mathcal{O}} \quad \text{in } \Omega \times (0,T), \\ \nabla.u = 0 &\quad \text{in } \Omega \times (0,T), \quad u = 0 \quad \text{in } \partial\Omega \times (0,T), \\ \tau_t + \lambda\tau &= \kappa(\nabla u + (\nabla u)^T) \quad \text{in } \Omega \times (0,T), \\ u(\cdot,0) &= u_0, \quad \tau(\cdot,0) = \tau_0 \quad \text{in } \Omega, \end{aligned}$$

f is a control localized in $\mathcal{O}$, an open subset of Ω.

- For $\eta > 0$, Jeffreys system,
- For $\eta = 0$, Maxwell system.

Visco-elastic flows

Experimental data for viscoelastic fluids indicate :
several relaxation modes are needed to fit the data.

- Single relaxation mode:

$$\tau_t + \lambda\tau = \kappa\Big(\nabla u + (\nabla u)^T\Big), \quad \lambda > 0, \quad \kappa > 0,$$

- Several relaxation mode: $\tau = \sum_{i=1}^{N} \tau_i,$, for $i = 1, \cdots, N$,

$$(\tau_i)_t + \lambda_i\tau_i = \kappa_i\Big(\nabla u + (\nabla u)^T\Big), \quad \lambda_i > 0, \quad \kappa_i > 0,$$

- Infinite relaxation mode: If $\tau = \sum_{i=1}^{\infty} \tau_i,$, each τ_i satisfying the above ODE.

Known results

- **M. Renardy, (Syst. Control Lett. 54, 2005):**

  1-dimensional multimode linear Jeffreys and Maxwell systems with localized interior control :

  Exact controllability for single-mode Maxwell fluids and

  approximate controllability of multimode Maxwell and Jeffreys fluids.

- **A. Doubova, E. Fernández-Cara, (Syst. Control Lett. 61, 2012):**

  Higher dimensional single mode Jeffreys system :

  approximate controllability for velocity.

  Higher dimensional single mode Maxwell system :

  exact controllability with an underdamped assumption on the system.

- **M. Renardy, (Syst. Control Lett. 58, 2009):** No observability estimate possible in any Sobolev norm for Jeffreys and multimode Maxwell models.

Background for our work

The story so far

- Jeffreys system and multimode Maxwell system are expected to be approximately controllable only.
- This is the best result possible expected for these systems!
- approximate controllability for both velocity and stress variables are expected
- single mode Maxwell system is expected to be exactly controllable without any extra assumptions.
- The integral term for τ, used in earlier work is probably too restrictive.

Scope of our work

Our approach

- Rewrite the system in a more tractable way.
- Check if the operator or at least its restriction to a subspace generates an analytic semigroup.
- Analyze the spectrum.
- Check for eigenfunction expansion.
- Use Laplace transform and Residue theorem to prove a unique continuation type result for the transformed system.
- Conclude for the original system.

A new system

The stress tensor is

$$\tau(x,t) = \tau_0 + \int_0^\infty e^{-\lambda(t-s)}\Big(\nabla u + (\nabla u)^T(x,s)\Big)\,ds.$$

The integral term always lies in the space of tensors which are symmetric part of the gradient of divergence-free vectors vanishing on the boundary.

So the controllability result for the stress tensor τ, is expected only in this subspace. Hence seek

$$\tau = (\nabla v + (\nabla v)^T),$$

for divergence free v vanishing on the boundary.

Thus the new system for divergence free (u, v) vanishing on boundary,

$$\rho u_t \;=\; \eta\triangle u + \triangle v - \nabla p + f\chi_{\mathcal{O}} \quad \text{in } \Omega\times(0,T),$$

$$v_t + \lambda v = \kappa u \quad \text{in } \Omega\times(0,T),$$

$$u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 \quad \text{in } \Omega.$$

where $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$, an open subset of $\mathcal{O}$.

Jeffreys system: $\eta > 0$

Divergence free function Spaces:

$$V_n^0(\Omega) = \left\{ u \in (L^2(\Omega))^d \mid \nabla.u = 0 \quad \text{in } \Omega, \quad u.n = 0 \quad \text{in } \partial\Omega \right\},$$

$$V_0^1(\Omega) = \left\{ v \in (H_0^1(\Omega))^d \mid \nabla.v = 0 \quad \text{in } \Omega \right\}.$$

The Leray projector is $P : L^2(\Omega) \to V_n^0(\Omega)$ and the Stokes operator is

$$A_0 = -P\triangle, \quad D(A_0) = V^2(\Omega) \cap V_0^1(\Omega). \tag{3.1}$$

For Jeffreys system, the linear operator $(A, D(A))$ on $V_n^0 \times V_0^1$

$$\begin{aligned} A &= \begin{pmatrix} -\frac{\eta}{\rho} A_0 & -A_0 \\ \frac{\kappa}{\rho} I & -\lambda I \end{pmatrix}, \\ D(A) &= \{(\tilde{u}, v) \in (V_0^1(\Omega))^2 \,|\, \eta\tilde{u} + \rho v \in D(A_0)\}. \end{aligned} \tag{3.2}$$

- $(A, D(A))$ generates an analytic C_0-semigroup on $V_n^0 \times V_0^1$. The same result holds for its adjoint.

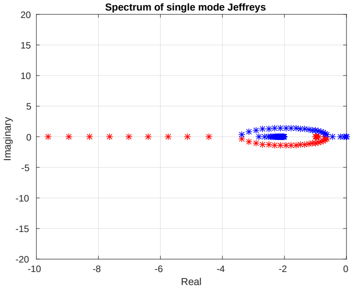
Spectral analysis of A^*

- Look for eigenvalues of A^* using eigenvalues of Stokes opertor :

$$-\Delta\sigma_k + \nabla q_k = \Lambda_k \sigma_k.$$

- For each k, eigenvalues μ_k are roots of a quadratic equation involving Λ_k.
- All eigenvalues are real except finitely many complex ones.
- All eigenvalues are in the left side of the complex plane inside a sector.
- Two classes of eigenvalues:
  - One convergent : $\mu_{k,1} \to -(\kappa/\eta + \lambda), \quad \text{as} \quad k \to \infty,$
  - Other one divergent : $\mu_{k,2}$ goes to $-\infty$ like $-\eta\Lambda_k/\rho$ as $k \to \infty$,
- Eigenfunctions of A^* form a Riesz Basis : Image of an orthonormal basis under a linear transformation.
- Eigenfunction expansion available for solutions of adjoint problem.

Single mode Jeffreys system



Approximate Controllability for Jeffreys system

Let (σ, ϕ) be the solution of adjoint problem with terminal condition and let $\sigma|_{\mathcal{O}\times(0,T)} = 0$, then to prove $(\sigma_T, \phi_T) = (0, 0)$:

- The solution $e^{(T-t)A^*}(\sigma_T, \phi_T)$, is analytic in t in a sector containing positive t axis, as A^* generates an analytic semigroup.
- By analytic continuation, $\sigma|_{\mathcal{O}\times(0,\infty)} = 0$.
- The solution has the expansion in terms of eigenfunctions :

$$\sigma(\cdot, T-t) = \sum_{k\in\mathbb{N}} \alpha_{k,1} e^{t\mu_{k,1}} \sigma_{k,1} + \alpha_{k,2} e^{t\mu_{k,2}} \sigma_{k,2}.$$

- Taking Laplace transform of $\sigma|_{\mathcal{O}\times(0,\infty)}$,

$$0 = \sum_{k\in\mathbb{N}} \frac{\alpha_{k,1}}{\mu-\mu_{k,1}} \sigma_{k,1}|_{\mathcal{O}} + \frac{\alpha_{k,2}}{\mu-\mu_{k,1}} \sigma_{k,2}|_{\mathcal{O}},$$

  $\sigma_{k,l}$, $l = 1, 2$ restricted on $\mathcal{O}$ is nonzero.
- Using Residue theorem, show $\alpha_{k,l} = 0$ for all k, $l = 1, 2$.

Result for Jeffreys system

- Define the subspace $L^2(\Omega;\mathcal{L}_s(\mathbb{R}^d))$, where $\mathcal{L}_s(\mathbb{R}^d)$ is the space of symmetric real $d \times d$ matrix.
- Define : $R_0 = \{Dv \mid v \in V_0^1(\Omega)\}, \quad \text{and} \quad R_{\lambda,T}(\tau_0) = e^{-\lambda T}\tau_0 + R_0.$

Theorem

For any u_0, u_r in V_n^0 and $\tau_0 \in L^2(\Omega;\mathcal{L}_s(\mathbb{R}^d))$, $\tau_r \in R_{\lambda,T}(\tau_0)$ and any $\epsilon > 0$, there exists a localized interior control $f \in L^2(0,T;L^2(\mathcal{O}))$ such that (u,τ), the solution of Jeffreys system satisfies

$$\|(u(\cdot,T),\tau(\cdot,T)) - (u_r,\tau_r)\|_{V_n^0 \times L^2(\Omega;\mathcal{L}_s(\mathbb{R}^d))} < \epsilon.$$

- Several relaxation mode: the same result holds.
- Infinite relaxation mode: (with D. Mitra, M. Renardy, J. Differential Equns.). Approximate Controllability under some assumptions on $\{\kappa_i\}_{i=1}^{\infty}$ and $\{\lambda_i\}_{i=1}^{\infty}$.

Single Mode Maxwell system : $\eta = 0$

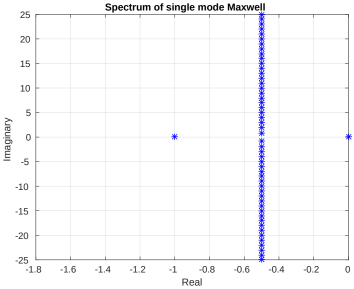
Transformed system for divergence free (u, v) vanishing on boundary,

$$\rho u_t = \triangle v - \nabla p + f\chi_{\mathcal{O}} \quad \text{in } \Omega \times (0, T),$$
$$v_t + \lambda v = \kappa u \quad \text{in } \Omega \times (0, T),$$
$$u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0 \quad \text{in } \Omega.$$

- Spectral analysis as before;
- Eigenvalues are roots of a quadratic equation involving Λ_k, a pair of complex numbers with
  - convergent real part and
  - divergent imaginary part.
- System can be reduced to a damped wave equation by eliminating u:

$$\rho(v_{tt} + \lambda v_t) = \kappa(\Delta v - \nabla p + f\chi_O).$$

Single mode Maxwell System



Single Mode Maxwell system

- A geometric condition on the controlled domain $\mathcal{O}$: Pick a point x_0 outside Ω and define $\Gamma = \{x \in \partial\Omega \,|\, (x - x_0) \cdot n \geq 0\}$, Assume that $\mathcal{O}$ contains all points in Ω which are sufficiently close to Γ.
- Set $R(x_0) = \max_{x \in \overline{\Omega}} |x - x_0|, \qquad T(x_0) = 2R(x_0)\sqrt{\rho/\kappa}.$
- Exact controllability persists under compact perturbations of the infinitesimal generator provided that approximate controllability holds.
- Show approximate controllability using geometric condition and Holmgren Theorem.

Theorem

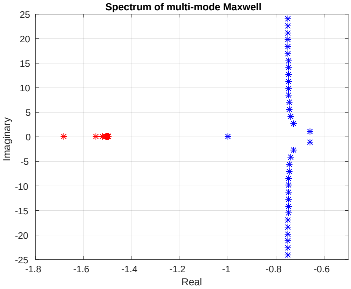
(u, v)-system is exactly controllable it $V_n^0 \times V_0^1$ at time $T > T(x_0)$ using a L^2-control localized in $\mathcal{O}$.

- (u, τ) is exact controllable under the restriction on stress tensor.

Multi Mode Maxwell system

- Several relaxation mode: situation changes completely!
- Spectral analysis gives only implicit form of $(N+1)$ eigenvalues $\{\mu_k^j\}_{j=1}^{N+1}$ for each k.
- Spectrum contains :
  - $(N-1)$ sequences of convergent real eigenvalues and
  - one sequence of a pair of complex eigenvalues as in single mode case with
    - convergent real part and
    - divergent imaginary part.

Multimode Maxwell System



Multimode Maxwell system

- Mixed system of hyperbolic and parabolic types.
- Solution of the adjoint system can be decoupled into two parts-
  one part coming from an analytic semigroup and
  other part associated with damped wave equation.
- Extend the solution on the whole positive t axis :
  for the first part using analyticity and
  for the latter, using observability inequality, as an approximation of single mode case.
- Then use Laplace transform as before.
- Approximate controllability follows as before.

Summary of our results

S. Chowdhury, D. Mitra, M. Ramaswamy, M. Renardy, *J. Math. Fluid Mech.* , 2016.
Higher-dimensional Jeffreys and Maxwell systems with a distributed control acting only in velocity equation.

- single mode and multimode Jeffreys system:
  approximate controllability for both velocity and stress variables.
- single mode Maxwell system:
  exact controllability for both variables without the underdamped condition.
- multimode Maxwell system :
  only approximate controllability result for both variables .

D. Mitra, M. Ramaswamy, M. Renardy, *J. Differential Equns.* , 2017.

- Similar results for infinite mode fluid models.

Conclusions and perspectives

- For the several mode case, can we settle null controllability question for the viscoelastic flows?
- Jeffreys system has accumulation point in the spectrum, localized control cannot give null controllability but moving control may give. Under investigation.
- Stabilization of the fluid numerically and theoretically?
- What can be obtained if the stress tensor is in the integral form instead of discrete sum? - Our spectral analysis and Laplace transform argument may not work.
- What can be said about linear models but with variable coefficients?
- How to tackle nonlinear models?
- Can we study the controllability or stabilizability of the systems around a variable trajectory?