Integrability in the Laplacian Growth Problem

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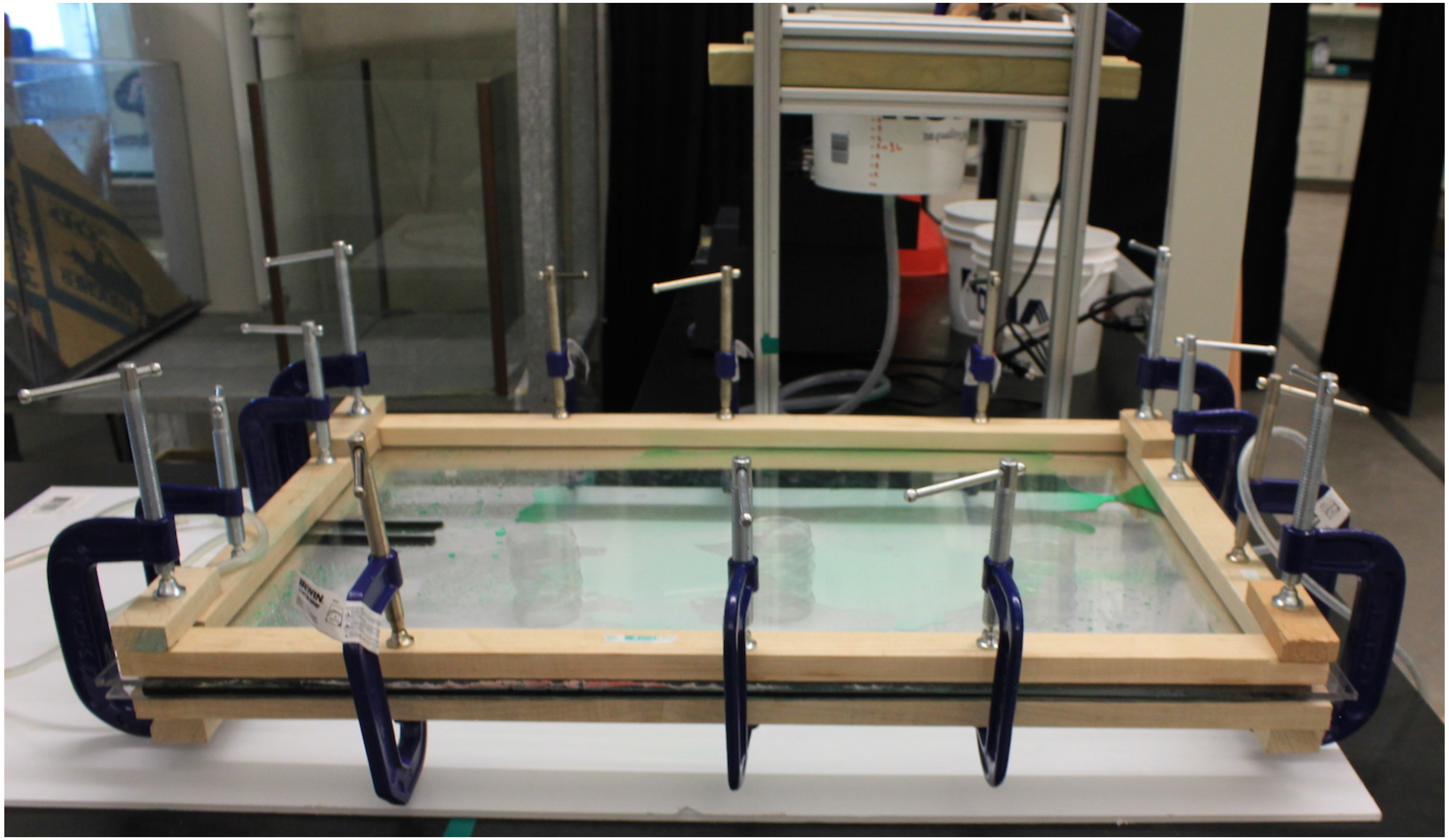
ICTS, Bangalore
Integrable systems in M, CM, and SM.

Physics of Laplacian Growth

The Physical Setup



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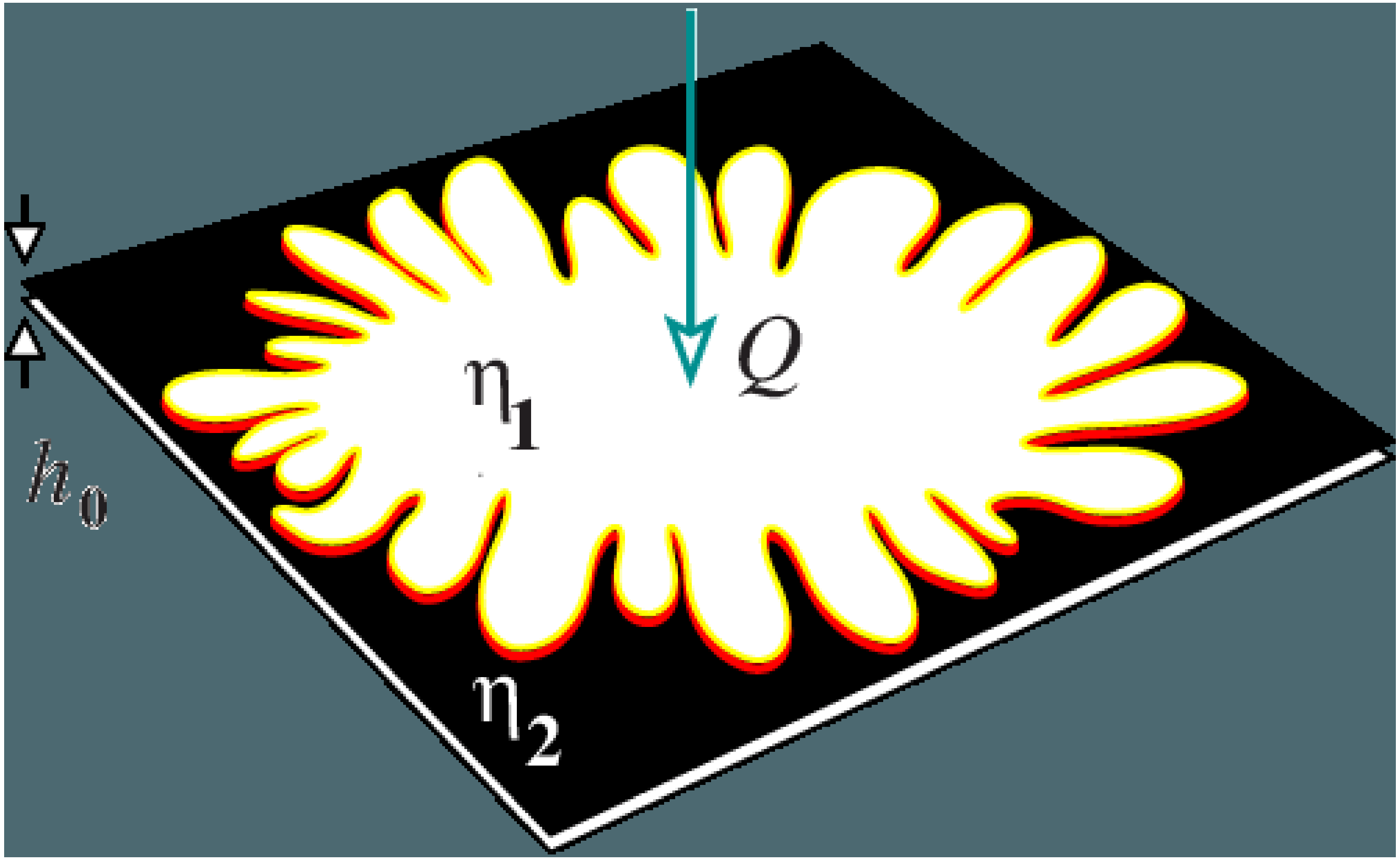

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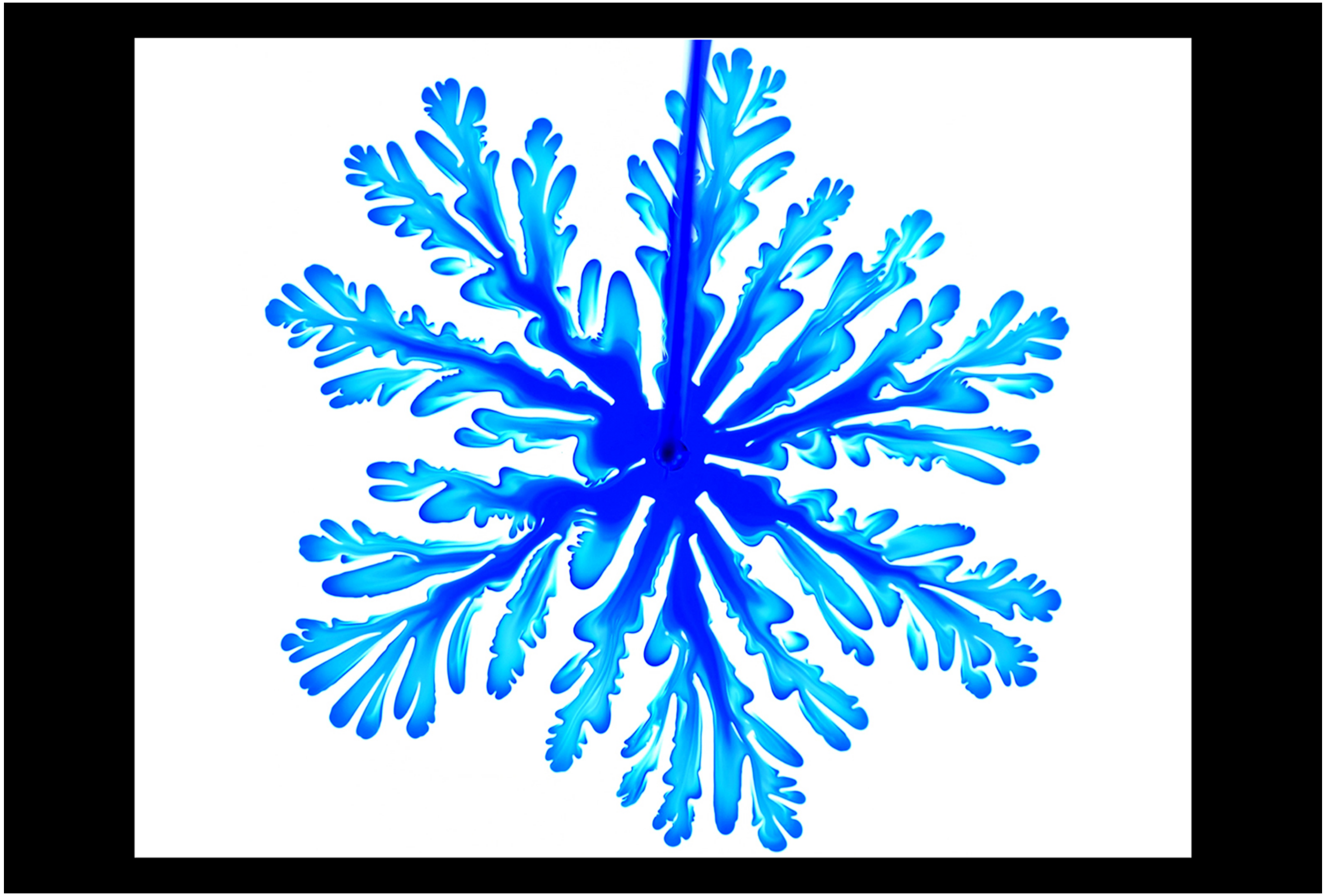

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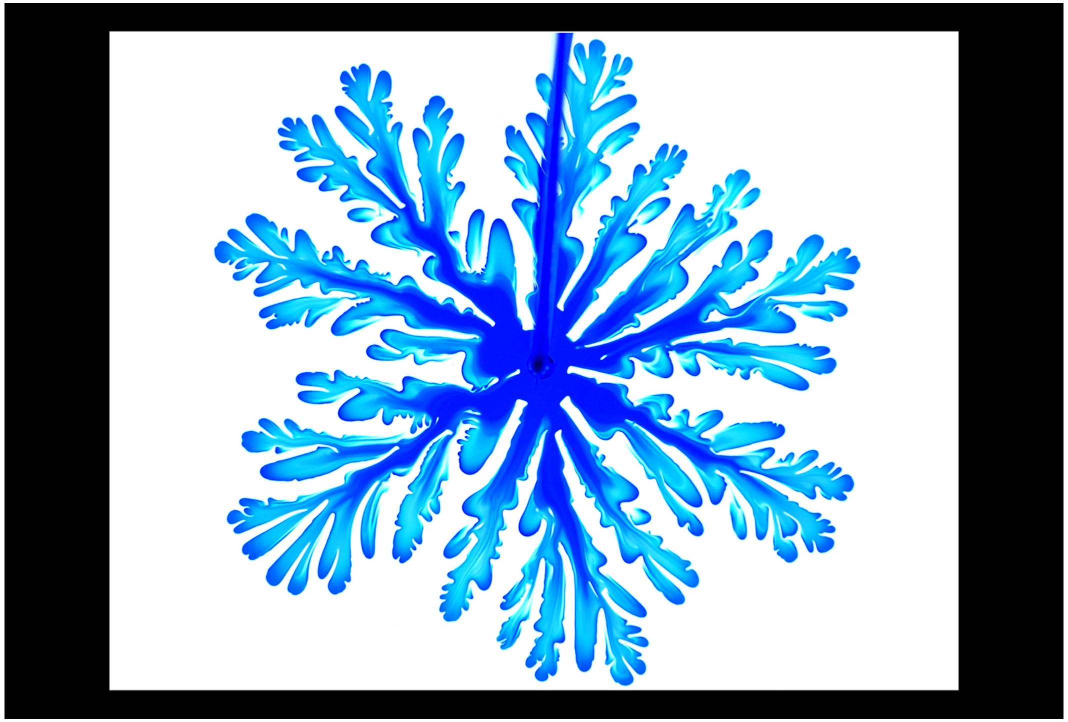

Box counting (Hausdorff) dimension: D=1.71...

Mathematics of Laplacian Growth

D'Arcy/Pouiseuille law

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$$b^2\vec{v} = -\frac{b^4}{12\eta}\vec{\nabla}P$$

- This is called D'Arcy's law
- The low viscosity of the fliuid in the droplet does not support pressure gradients $\Rightarrow$ Pressure is constant inside the droplet.

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- **This Defines Laplacian Growth.**

Fundamental Properties

$$\nabla^2 P = 0$$

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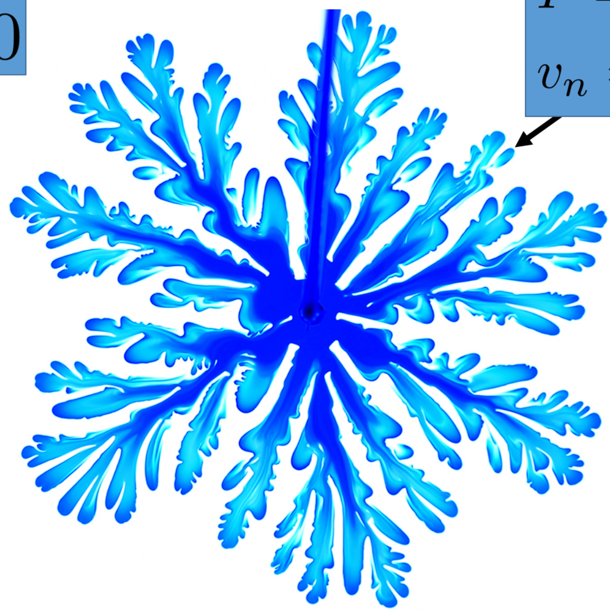
$$v_n = \partial_n P$$



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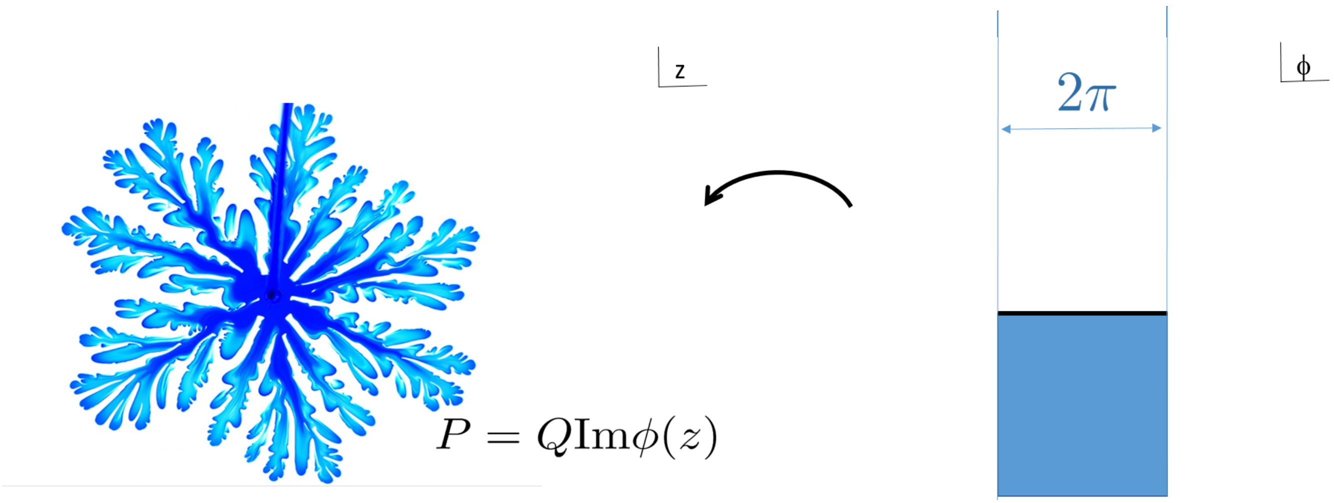

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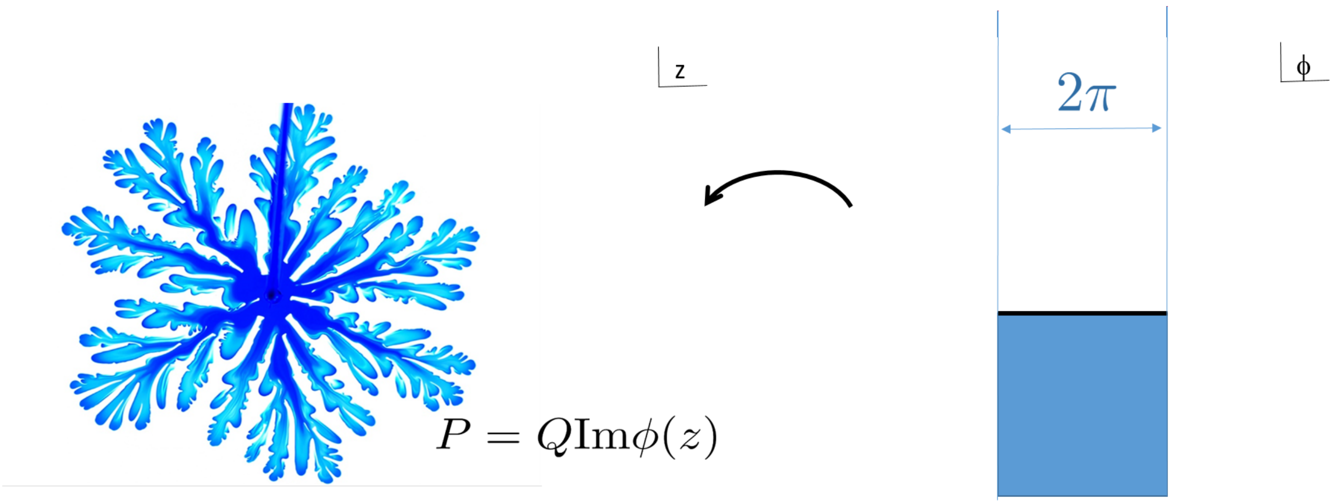

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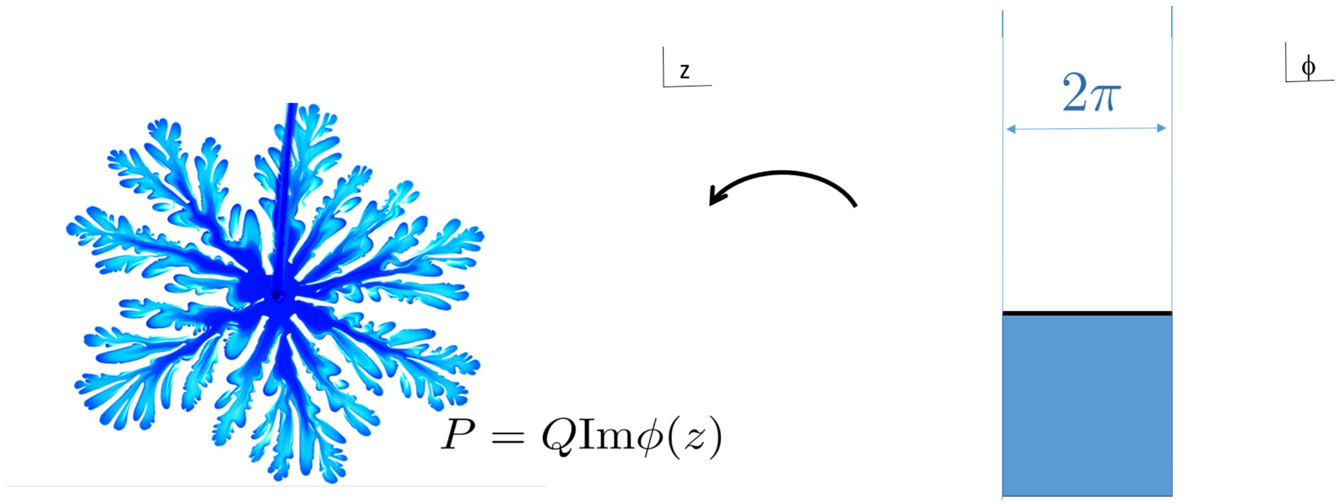

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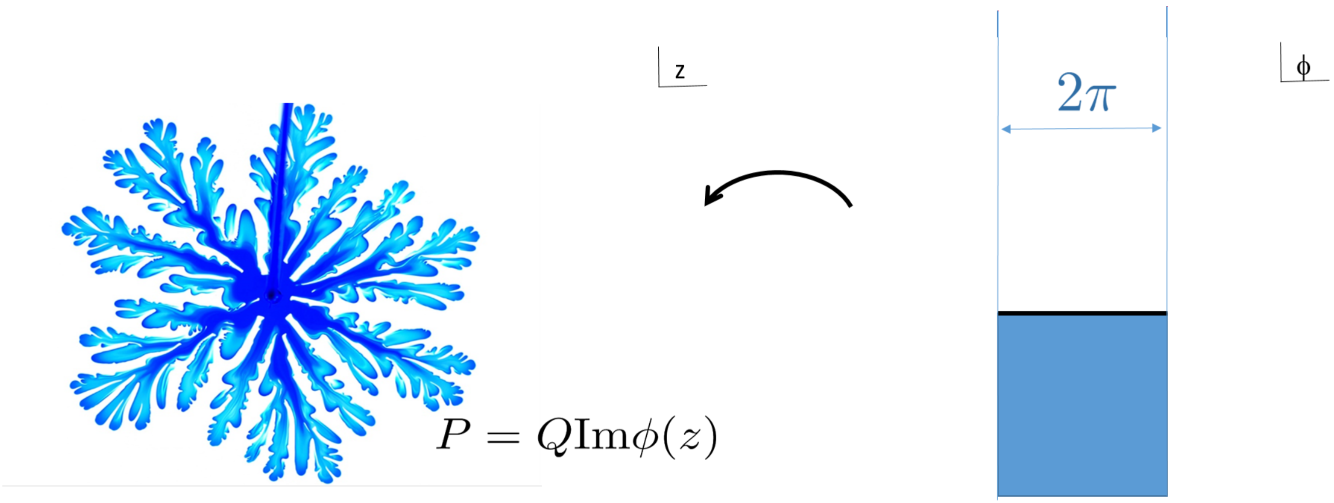

$P = Q\text{Im}\phi(z)$

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- An infinite number of conserved Harmonic moments [Richardson]

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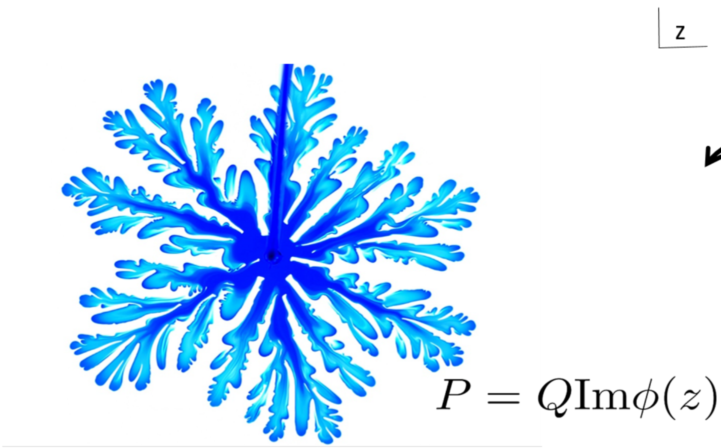

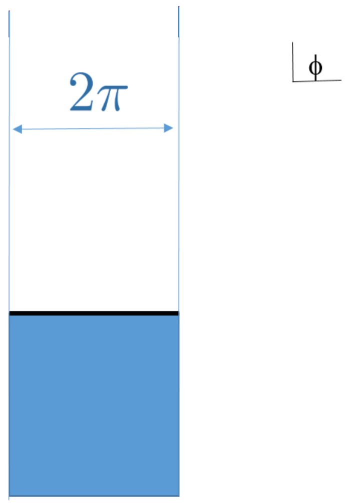

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$t_n = \iint_{\text{exterior}} z^{-k} d^2 z$, and c.c.,

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$$z_+^k(\phi, t) = e^{\imath k\phi} a_0^{(k)}(\boldsymbol{t}) + e^{\imath(k-1)\phi} a_1^{(k)}(\boldsymbol{t}) + + \cdots + a_k^{(k)}(\boldsymbol{t}) + \cancel{a_{k+1}^{(k)}(\boldsymbol{t}) e^{-\imath k\phi}} + \cancel{\ldots}$$

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- Main Eq. is a constraint: $\{z, \bar{z}\} = 1$

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- The integrable structure is a dispersionless limit of another integrable system
- We are interested in the statistics of solutions – initial value problem arguably less interesting.
- Hence we may is search for a quantum integrable problem for which Laplacian growth is a semiclassical approximation

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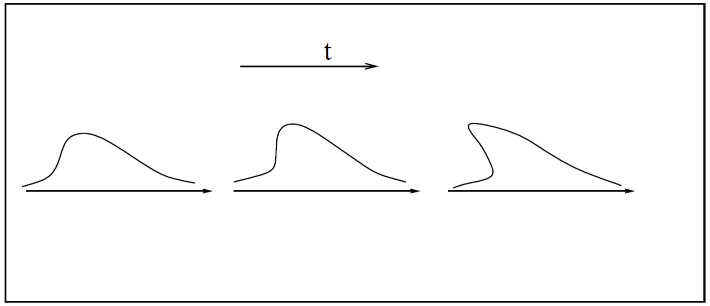

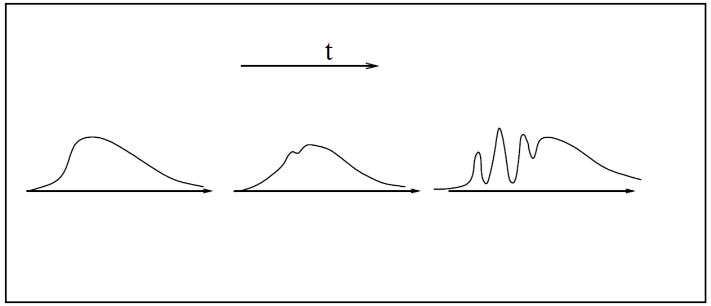

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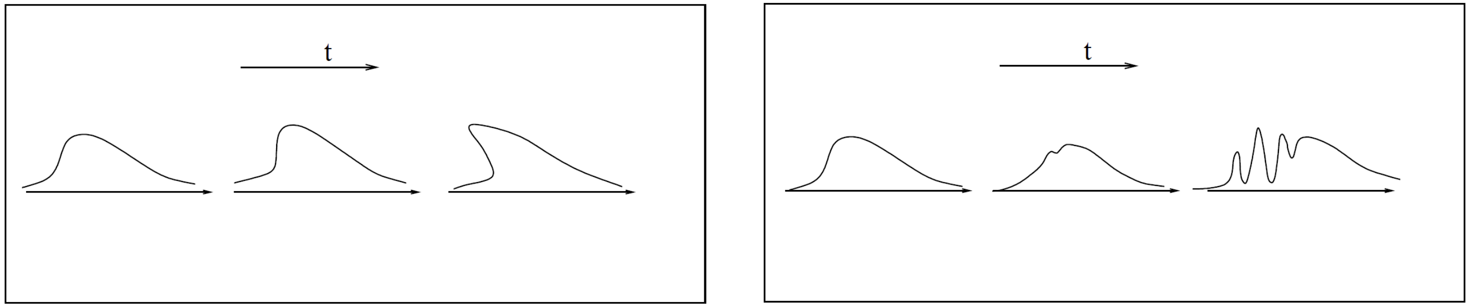

- The limit $\varepsilon \to 0$ is well defined, in the sense that we may keep track of the *envelope* of oscillation

Whitham Equations

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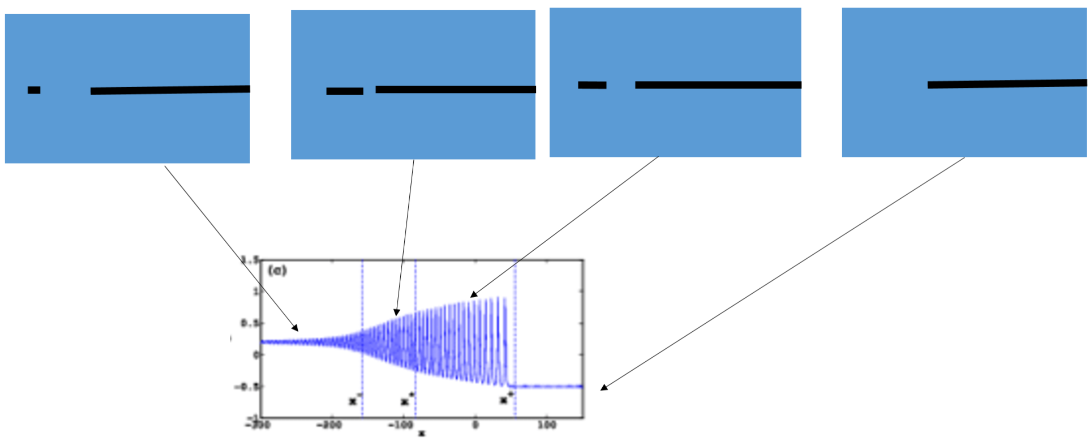
- Given a (multi-)periodic solution, $u(x,t)$, of the KdV we may
- Associate with it the spectrum of the Lax operator

$$H(t) = -\partial_x^2 + u(x,t)$$

- Which happens to be multi-gapped



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$$\hat{L} = a_1(t)e^{\partial_t} + a_0(t) + a_{-1}(t)e^{-t} + \ldots$$

$$\hat{L} = \begin{pmatrix} a_0(1) & a_1(1) & 0 & \ldots & \ldots & \ldots \\ a_{-1}(2) & a_0(2) & a_1(2) & 0 & \ldots & \ldots \\ a_{-2}(3) & a_{-1}(3) & a_0(3) & a_1(2) & 0 & \ldots \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}$$

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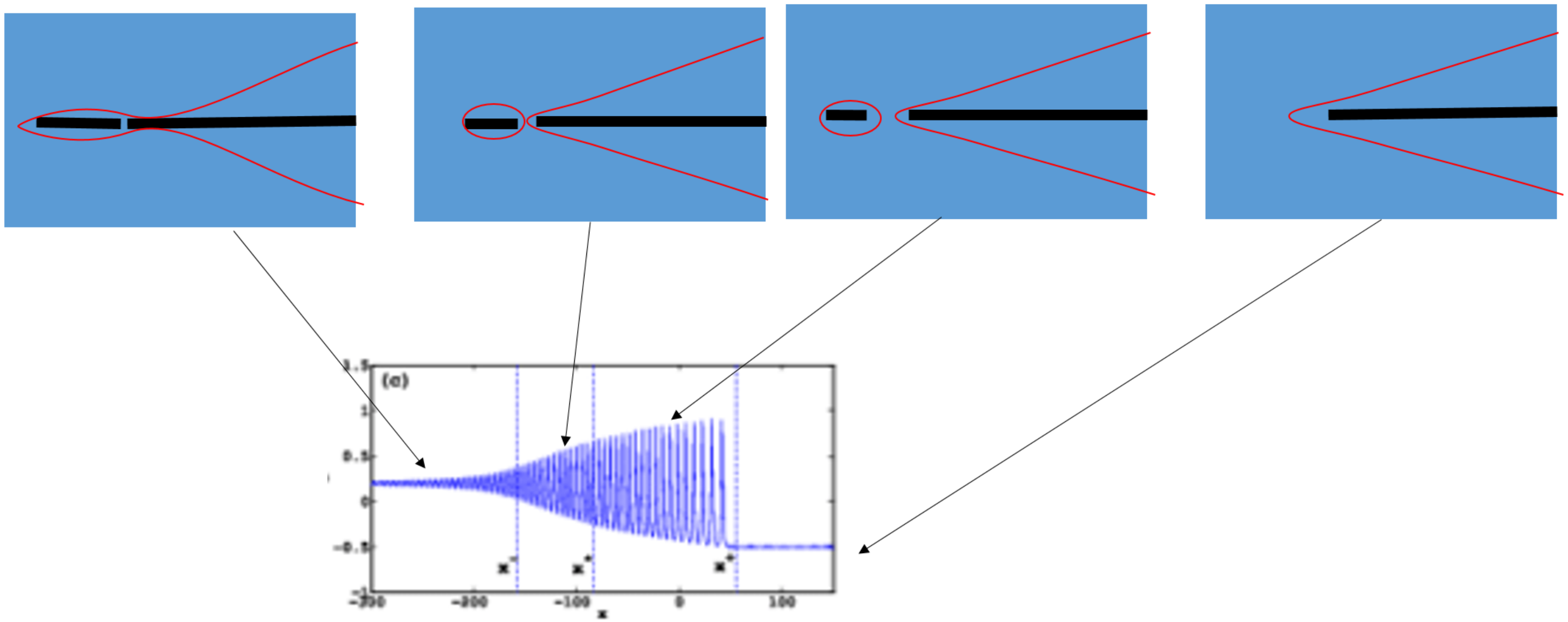
2DTL	**LG**
$\hat{L} = a_1 e^{\partial_t} + a_0 + \ldots$	$z = a_1 e^{\imath\phi} + a_0 + \ldots$
$[\partial_t, t] = 1$	$\{\varphi, t\} = 1$
$\hat{H}_k = \hat{L}^k_+$	$H_k = z^k_+$
$[\hat{L}, \hat{L}^\dagger] = 1$	$\{z, \bar{z}\} = 1$

Dispersive Regularization of Laplacian Growth



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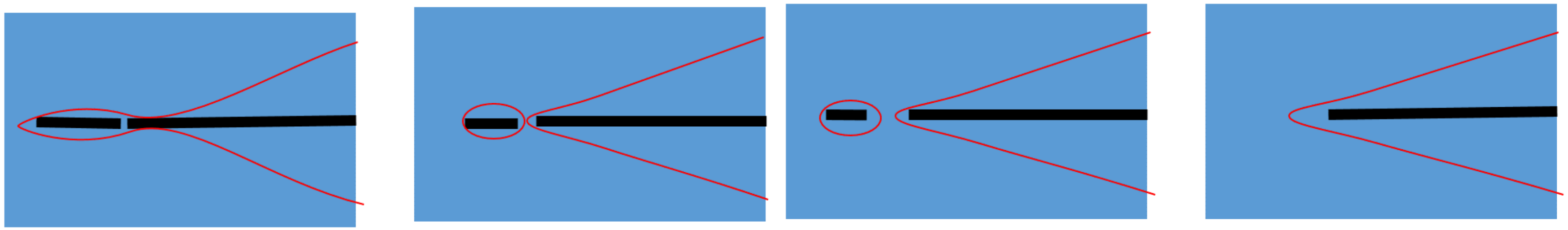

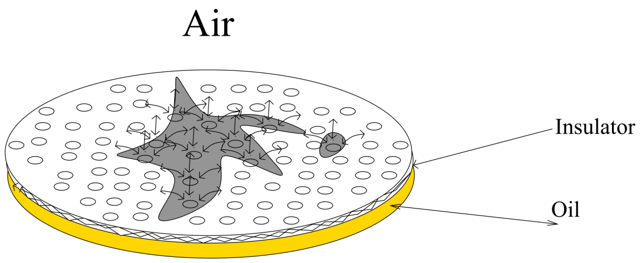

Remaning Issues

Some Open Problems and Questions

We are interested in the statistics of solutions which have a fractal property:

$$A \sim t \sim R^D$$

- Does conformal invariance help? Representation theory of (centrally extended) conformal transformations seems to be only tool to obtain discrete spectrum of scaling dimensions.
- Does integrability help?
- Should we quantize the problem again to obtain statistical mechanics?

quanum Koreteweg de Vries (qKdV)

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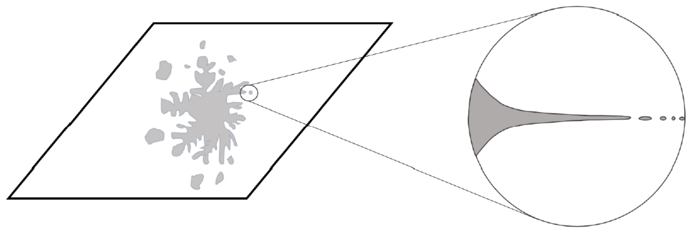

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- Seems to obey all requirements of an ensemble of growing self-similar cluster

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- The $\hat{T}$ has the following algebra: $[\hat{T}(x), \hat{T}(y)] =$ $2(\hat{T}(x) + \hat{T}(y))\delta'(x-y) + \frac{\pi c}{6}\delta'''(x-y)$

Quantum KdV and the Virasoro Algebra II

- A set of commuting Hamiltonians can be defined which are quantization of the KdV Hamiltonians

$$\hat{H}^1 = \oint \frac{dx}{2\pi} \hat{T}$$

$$\hat{H}^2 = \oint \frac{dx}{2\pi} : \hat{T}^2 :$$

$$\hat{H}^3 = \oint \frac{dx}{2\pi} : \hat{T}^3 : + \frac{c+2}{12} : \frac{\hat{T}'^2}{2} :$$

$$\ldots ,$$

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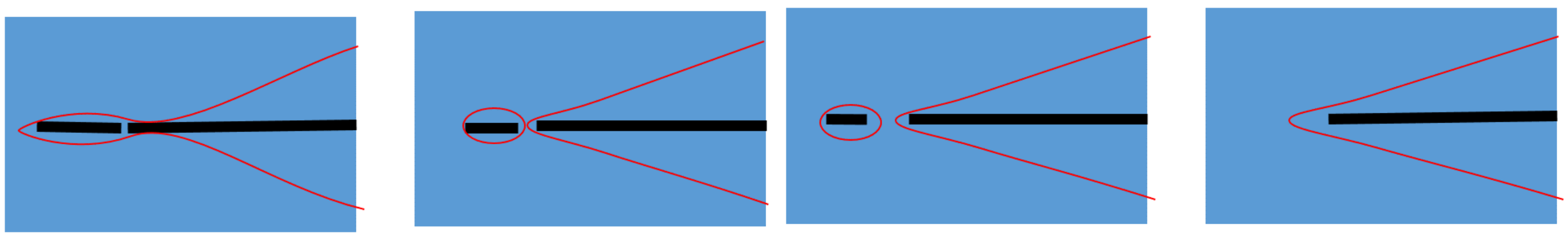

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- Namely, this probabilty distribtuion describes an ensemble of growing fracrtal clusters with fractal dimension with fractal dimension D.

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- In a certain limit that is the KdV equation
- The KdV equation may be quantized thus leading putatively to
- A statistical mechanics description of Laplcian growth with a discrete spectrum of fractal dimensions.