Intersection Theory on Moduli Space of Curves and their connection to Integrable Systems

ICTS Bangalore

Chitrabhanu Chaudhuri

IISER Pune

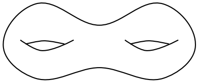
17 July, 2018

Moduli of Riemann Surfaces

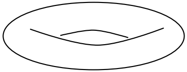
A Riemann surface is a 1 dimensional complex manifold. We are interested in compact Riemann surfaces. Any such surface is topologically, a sphere with g handles attached.

Moduli of Riemann Surfaces

A Riemann surface is a 1 dimensional complex manifold. We are interested in compact Riemann surfaces. Any such surface is topologically, a sphere with g handles attached.



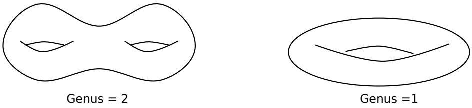
Genus = 2



Genus =1

Moduli of Riemann Surfaces

A Riemann surface is a 1 dimensional complex manifold. We are interested in compact Riemann surfaces. Any such surface is topologically, a sphere with g handles attached.



The genus of the surface is the number of handles that we attach to the sphere. For example the 2-sphere has genus 0, where as the torus has genus 1.

The moduli space, M_g, is a space parametrizing isomorphism classes of compact Riemann Surfaces of genus g.

The moduli space, M_g, is a space parametrizing isomorphism classes of compact Riemann Surfaces of genus g.

It can be realized as a quotient of a complex manifold of dimension $3g - 3$ by the action of a discrete group. Such a space is called a complex orbifold. This space is not compact.

The moduli space, M_g, is a space parametrizing isomorphism classes of compact Riemann Surfaces of genus g.

It can be realized as a quotient of a complex manifold of dimension $3g-3$ by the action of a discrete group. Such a space is called a complex orbifold. This space is not compact.

We shall also consider the moduli spaces of genus g, compact, Riemann surfaces with n marked points, $(C; x_1, \ldots, x_n)$. This space is denoted by $M_{g,n}$. The points $x_1, \ldots, x_n$ are all distinct.

There is a compactification of the moduli space M_g which we denote by $\overline{M}_g$. This space parametrizes “pinched” Riemann surfaces. The pinchings are called singularities.

There is a compactification of the moduli space M_g which we denote by $\overline{M}_g$. This space parametrizes "pinched" Riemann surfaces. The pinchings are called singularities.

Similarly there is a compactification of $M_{g,n}$ denoted by $\overline{M}_{g,n}$. This space parametrizes pinched and marked Riemann surfaces which are degenerations of Riemann surfaces of genus g with n marked points.

There is a compactification of the moduli space M_g which we denote by $\overline{M}_g$. This space parametrizes "pinched" Riemann surfaces. The pinchings are called singularities.

Similarly there is a compactification of $M_{g,n}$ denoted by $\overline{M}_{g,n}$. This space parametrizes pinched and marked Riemann surfaces which are degenerations of Riemann surfaces of genus g with n marked points.

Such a surface is called stable, in particular

$$\chi(C - \{x_1, \ldots, x_n\} - \{\text{singular points}\}) < 0. \tag{1}$$

There is a compactification of the moduli space M_g which we denote by $\overline{M}_g$. This space parametrizes "pinched" Riemann surfaces. The pinchings are called singularities.

Similarly there is a compactification of $M_{g,n}$ denoted by $\overline{M}_{g,n}$. This space parametrizes pinched and marked Riemann surfaces which are degenerations of Riemann surfaces of genus g with n marked points.

Such a surface is called stable, in particular

$$\chi(C - \{x_1, \ldots, x_n\} - \{\text{singular points}\}) < 0. \tag{1}$$

$\overline{M}_{g,n}$ is a compact complex orbifold of dimension $3g - 3 + n$.

There is a compactification of the moduli space M_g which we denote by $\overline{M}_g$. This space parametrizes "pinched" Riemann surfaces. The pinchings are called singularities.

Similarly there is a compactification of $M_{g,n}$ denoted by $\overline{M}_{g,n}$. This space parametrizes pinched and marked Riemann surfaces which are degenerations of Riemann surfaces of genus g with n marked points.

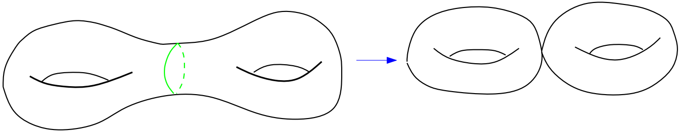
Such a surface is called stable, in particular

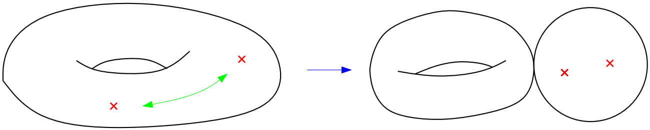
$$\chi(C - \{x_1, \ldots, x_n\} - \{\text{singular points}\}) < 0. \tag{1}$$

$\overline{M}_{g,n}$ is a compact complex orbifold of dimension $3g - 3 + n$.

There is a map $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ obtained by forgetting the last marked point and deleting any unstable component.

Degenerations of marked Riemann Surfaces





Psi classes and the Hodge bundle

There are several line bundles and vector bundles which naturally occur on $\overline{M}_{g,n}$.

Psi classes and the Hodge bundle

There are several line bundles and vector bundles which naturally occur on $\overline{M}_{g,n}$.

Let $\mathcal{L}_i$ be the line bundle on $\overline{M}_{g,n}$ whose fiber at $(C; x_1, \ldots, x_n)$ is the cotangent line at x_i. Define

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{M}_{g,n}, \mathbb{C}). \tag{2}$$

Psi classes and the Hodge bundle

There are several line bundles and vector bundles which naturally occur on $\overline{M}_{g,n}$.

Let $\mathcal{L}_i$ be the line bundle on $\overline{M}_{g,n}$ whose fiber at $(C; x_1, \ldots, x_n)$ is the cotangent line at x_i. Define

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{M}_{g,n}, \mathbb{C}). \tag{2}$$

Let $\mathcal{H}$ be the vector bundle on $\overline{M}_{g,n}$ whose fiber at $(C; x_1, \ldots, x_n)$ is the vector space $H^0(C, \Omega_C)$. This is a rank g vector bundle. Define

$$\lambda_i = c_i(\mathcal{H}) \in H^{2i}(\overline{M}_{g,n}, \mathbb{C}). \tag{3}$$

Intersection numbers

There are certain cohmology classes on $\overline{M}_{g,n}$ which are called tautological classes. It turns out that the intersections of all tautological classes can be obtained just by knowing the intersections of the ψ classes.

Intersection numbers

There are certain cohmology classes on $\overline{M}_{g,n}$ which are called tautological classes. It turns out that the intersections of all tautological classes can be obtained just by knowing the intersections of the ψ classes.

If $d_1 + \ldots + d_n = 3g - 3 + n$, define

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\overline{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

otherwise 0.

Intersection numbers

There are certain cohmology classes on $\overline{M}_{g,n}$ which are called tautological classes. It turns out that the intersections of all tautological classes can be obtained just by knowing the intersections of the ψ classes.

If $d_1 + \ldots + d_n = 3g - 3 + n$, define

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\overline{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

otherwise 0.

For example $\overline{M}_{0,3}$ is a point hence

$$\langle \tau_0 \tau_0 \tau_0 \rangle = \langle \tau_0^3 \rangle = \int_{\overline{M}_{0,3}} 1 = 1.$$

where as if $n \neq 3$

$$\langle \tau_0^n \rangle = 0.$$

Sometimes we write $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$, where, as before $d_1 + \cdots + d_n = 3g - 3 + n$.

Sometimes we write $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$, where, as before $d_1 + \cdots + d_n = 3g - 3 + n$.

Some of these intersection numbers were already known. For example

$$\langle \tau_1 \rangle_{1,1} = \int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}.$$

Sometimes we write $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$, where, as before $d_1 + \cdots + d_n = 3g - 3 + n$.

Some of these intersection numbers were already known. For example

$$\langle \tau_1 \rangle_{1,1} = \int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}.$$

It was also known but harder to show that if $g > 0$

$$\langle \tau_{3g-2} \rangle_{g,1} = \int_{\overline{M}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g g!}$$

Sometimes we write $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$, where, as before $d_1 + \cdots + d_n = 3g - 3 + n$.

Some of these intersection numbers were already known. For example

$$\langle \tau_1 \rangle_{1,1} = \int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}.$$

It was also known but harder to show that if $g > 0$

$$\langle \tau_{3g-2} \rangle_{g,1} = \int_{\overline{M}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g g!}$$

and if $d_1 + \cdots + d_n = n - 3$

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{0,n} = \frac{(n-3)!}{d_1! \cdots d_n!}.$$

We form the generating function

$$F(t_0, t_1, t_2, \ldots) = \sum_{n=1}^{\infty} \sum_{d_1,\ldots,d_n} \frac{1}{n!} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle t_{d_1} \cdots t_{d_n}.$$

We form the generating function

$$F(t_0, t_1, t_2, \ldots) = \sum_{n=1}^{\infty} \sum_{d_1,\ldots,d_n} \frac{1}{n!} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle t_{d_1} \cdots t_{d_n}.$$

The first few terms can be calculated easily and we list a few of them

$$\begin{aligned} F(t_0, t_1, t_2, \ldots) =& \frac{1}{6} t_0^3 + \frac{1}{6} t_0^3 t_1 + \frac{1}{24} t_1 \\ &+ \frac{1}{6} t_0^3 t_1^2 + \frac{1}{24} t_0^4 t_2 + \frac{1}{48} t_1^2 + \frac{1}{24} t_0 t_2 + \ldots \end{aligned}$$

We form the generating function

$$F(t_0, t_1, t_2, \ldots) = \sum_{n=1}^{\infty} \sum_{d_1,\ldots,d_n} \frac{1}{n!} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle t_{d_1} \cdots t_{d_n}.$$

The first few terms can be calculated easily and we list a few of them

$$\begin{aligned} F(t_0, t_1, t_2, \ldots) =& \frac{1}{6} t_0^3 + \frac{1}{6} t_0^3 t_1 + \frac{1}{24} t_1 \\ &+ \frac{1}{6} t_0^3 t_1^2 + \frac{1}{24} t_0^4 t_2 + \frac{1}{48} t_1^2 + \frac{1}{24} t_0 t_2 + \ldots \end{aligned}$$

The Witten conjecture gives an elegant way of recursively finding the coefficients of this generating function and hence for computing all the intersections of the ψ classes.

Witten conjecture

Let

$$U = \frac{\partial^2}{\partial t_0^2} F, \quad \dot{U} = \frac{\partial}{\partial t_0} U, \quad \ddot{U} = \frac{\partial^2}{\partial t_0^2} U \ldots$$

Witten conjecture

Let

$$U = \frac{\partial^2}{\partial t_0^2} F, \quad \dot{U} = \frac{\partial}{\partial t_0} U, \quad \ddot{U} = \frac{\partial^2}{\partial t_0^2} U \ldots$$

The Witten conjecture says that

$$\frac{\partial U}{\partial t_i} = \frac{\partial}{\partial t_0} R_i(U, \dot{U}, \ddot{U}, \ldots);$$

where the polynomials R_i are defined recursively by

$$R_0 = U, \quad \frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1} \left(\dot{U} + 2U \frac{\partial}{\partial t_0} + \frac{1}{4} \frac{\partial^3}{\partial t_0^3} \right) R_n.$$

Witten conjecture

Let

$$U = \frac{\partial^2}{\partial t_0^2} F, \quad \dot{U} = \frac{\partial}{\partial t_0} U, \quad \ddot{U} = \frac{\partial^2}{\partial t_0^2} U \ldots$$

The Witten conjecture says that

$$\frac{\partial U}{\partial t_i} = \frac{\partial}{\partial t_0} R_i(U, \dot{U}, \ddot{U}, \ldots);$$

where the polynomials R_i are defined recursively by

$$R_0 = U, \quad \frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1} \left(\dot{U} + 2U \frac{\partial}{\partial t_0} + \frac{1}{4} \frac{\partial^3}{\partial t_0^3} \right) R_n.$$

This system of equations is called the KdV (Korteweg-de Vries) hierarchy, and the polynomials R_i are called the Gelfand-Dikii polynomials.

The first few Gelfand-Dikii polynomials are

$$\begin{aligned} R_0 =& U, \\ R_1 =& \frac{1}{2}U^2 + \frac{1}{12}\ddot{U}, \\ R_2 =& \frac{5}{6}U^3 + \frac{5}{12}U\ddot{U} + \frac{1}{48}\ddot{\ddot{U}}. \end{aligned}$$

The first few Gelfand-Dikii polynomials are

$$\begin{aligned} R_0 =& U, \\ R_1 =& \frac{1}{2}U^2 + \frac{1}{12}\ddot{U}, \\ R_2 =& \frac{5}{6}U^3 + \frac{5}{12}U\ddot{U} + \frac{1}{48}\ddddot{U}. \end{aligned}$$

The first equation in the KdV hierarchy, that is,

$$\frac{\partial U}{\partial t_1} = \frac{\partial}{\partial t_0} R_1$$

is called the KdV equation.

Witten conjecture appeared first in his 1991 paper "Two-dimensional Gravity and Intersection Theory on Moduli Space".

Witten conjecture appeared first in his 1991 paper "Two-dimensional Gravity and Intersection Theory on Moduli Space".

In this paper Witten also proved that F satisfies the "string equation":

$$\frac{\partial}{\partial t_0} F = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}.$$

Witten conjecture appeared first in his 1991 paper "Two-dimensional Gravity and Intersection Theory on Moduli Space".

In this paper Witten also proved that F satisfies the "string equation":

$$\frac{\partial}{\partial t_0}F = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1}\frac{\partial F}{\partial t_i}.$$

In terms of intersection numbers the string equation reads

$$\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle = \sum_{\{i \mid d_i > 0\}} \langle \tau_{d_1} \cdots \tau_{d_i - 1} \cdots \tau_{d_n} \rangle.$$

Witten conjecture appeared first in his 1991 paper "Two-dimensional Gravity and Intersection Theory on Moduli Space".

In this paper Witten also proved that F satisfies the "string equation":

$$\frac{\partial}{\partial t_0} F = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i}.$$

In terms of intersection numbers the string equation reads

$$\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle = \sum_{\{i \ | \ d_i > 0\}} \langle \tau_{d_1} \cdots \tau_{d_i - 1} \cdots \tau_{d_n} \rangle.$$

It can be shown that the string equation and the KdV equation generate the KdV hierarchy.

Witten conjecture, now a theorem due to Kontsevic and many others, thus can be simply stated as:

Theorem (Witten, Kontsevic)

$$\frac{\partial}{\partial t_1} U = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3} \qquad \textit{(KdV equation).}$$

Witten conjecture, now a theorem due to Kontsevic and many others, thus can be simply stated as:

Theorem (Witten, Kontsevic)

$$\frac{\partial}{\partial t_1} U = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3} \qquad \textit{(KdV equation).}$$

The first proof of this theorem was given By Kontsevic (1992) using a certain cellular decomposition of $M_{g,n}$ and matrix integrals.

Witten conjecture, now a theorem due to Kontsevic and many others, thus can be simply stated as:

Theorem (Witten, Kontsevic)

$$\frac{\partial}{\partial t_1}U = U\frac{\partial U}{\partial t_0} + \frac{1}{12}\frac{\partial^3 U}{\partial t_0^3} \qquad \textit{(KdV equation).}$$

The first proof of this theorem was given By Kontsevic (1992) using a certain cellular decomposition of $M_{g,n}$ and matrix integrals.

There is a particularly beautiful proof of the theorem by Mirzakhani (2007). She uses symplectic and hyperbolic geometry.

Witten conjecture, now a theorem due to Kontsevic and many others, thus can be simply stated as:

Theorem (Witten, Kontsevic)

$$\frac{\partial}{\partial t_1} U = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3} \qquad \textit{(KdV equation).}$$

The first proof of this theorem was given By Kontsevic (1992) using a certain cellular decomposition of $M_{g,n}$ and matrix integrals.

There is a particularly beautiful proof of the theorem by Mirzakhani (2007). She uses symplectic and hyperbolic geometry.

Here we shall discuss a proof by Kazarian and Lando (2007), which uses Hurwitz numbers and the ELSV formula.

Hurwitz numbers

Let $(b_1, \ldots, b_n)$ be an ordered partition of d. Consider connected branched covers S of the Riemann Sphere $\mathbb{C}P^1$ with the following branching data:

- ∞ has n pre-images with ramification indices $b_1, \ldots, b_n$,
- for any other branch point the branching is simple, that is the ramification indices above it are $2, 1, 1, \ldots, 1$.

Hurwitz numbers

Let $(b_1, \ldots, b_n)$ be an ordered partition of d. Consider connected branched covers S of the Riemann Sphere $\mathbb{C}P^1$ with the following branching data:

- ∞ has n pre-images with ramification indices $b_1, \ldots, b_n$,
- for any other branch point the branching is simple, that is the ramification indices above it are $2, 1, 1, \ldots, 1$.

If genus of S is g then the number of branch points m other than ∞ is given by the Riemann-Hurwitz formula

$$m = 2g - 2 + n + d.$$

Hurwitz numbers

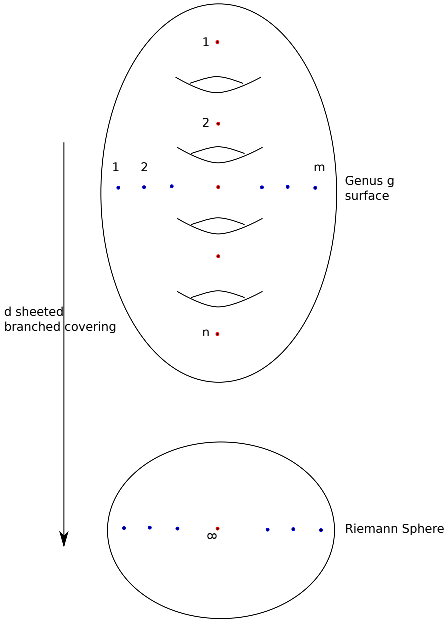
Let $(b_1, \ldots, b_n)$ be an ordered partition of d. Consider connected branched covers S of the Riemann Sphere $\mathbb{C}P^1$ with the following branching data:

- ∞ has n pre-images with ramification indices $b_1, \ldots, b_n$,
- for any other branch point the branching is simple, that is the ramification indices above it are $2, 1, 1, \ldots, 1$.

If genus of S is g then the number of branch points m other than ∞ is given by the Riemann-Hurwitz formula

$$m = 2g - 2 + n + d.$$

The complex structure on S is determined by the topological type of the covering.


1
2
1
2
m
Genus g
surface
d sheeted
branched covering
n
∞
Riemann Sphere

Hurwitz numbers

If we fix the images of the ramification points on $\mathbb{C}P^1$, then there are only finitely many distinct covers upto isomorphism. Denote that number by $h_{g,b_1,\ldots,b_n}$. These numbers are called Hurwitz numbers.

Hurwitz numbers

If we fix the images of the ramification points on $\mathbb{C}P^1$, then there are only finitely many distinct covers upto isomorphism. Denote that number by $h_{g,b_1,\ldots,b_n}$. These numbers are called Hurwitz numbers.

Up to a combinatorial factor $h_{g,b_1,\ldots,b_n}$ counts the number of ways of factoring a transitive d-permutation with conjugacy class $(b_1, \ldots, b_n)$ into m transpositions.

Hurwitz numbers

If we fix the images of the ramification points on $\mathbb{C}P^1$, then there are only finitely many distinct covers upto isomorphism. Denote that number by $h_{g,b_1,\ldots,b_n}$. These numbers are called Hurwitz numbers.

Up to a combinatorial factor $h_{g,b_1,\ldots,b_n}$ counts the number of ways of factoring a transitive d-permutation with conjugacy class $(b_1, \ldots, b_n)$ into m transpositions.

The Hurwitz numbers are interesting in their own right and can be thought of as certain Gromov-Witten invariants of $\mathbb{C}P^1$.

The ELSV formula

Ekedahl-Lando-Shapiro-Vainshtein (ELSV) formula gives an amazing relation between the Hurwitz numbers and some intersection numbers on $\overline{M}_{g,n}$

$$h_{g,b_1,\ldots,b_n} = m! \prod_{i=1}^{n} \frac{b_i^{b_i}}{b_i!} \int_{\overline{M}_{g,n}} \frac{1 - \lambda_1 + \cdots \pm \lambda_g}{(1 - b_1\psi_1)\cdots(1 - b_n\psi_n)}.$$

The ELSV formula

Ekedahl-Lando-Shapiro-Vainshtein (ELSV) formula gives an amazing relation between the Hurwitz numbers and some intersection numbers on $\overline{M}_{g,n}$

$$h_{g,b_1,\ldots,b_n} = m! \prod_{i=1}^{n} \frac{b_i^{b_i}}{b_i!} \int_{\overline{M}_{g,n}} \frac{1 - \lambda_1 + \cdots \pm \lambda_g}{(1 - b_1\psi_1)\cdots(1 - b_n\psi_n)}.$$

The numerator in the integral is just the total Chern class of the dual of the Hodge bundle that is $c(H^\vee)$.

The ELSV formula

Ekedahl-Lando-Shapiro-Vainshtein (ELSV) formula gives an amazing relation between the Hurwitz numbers and some intersection numbers on $\overline{M}_{g,n}$

$$h_{g,b_1,\ldots,b_n} = m! \prod_{i=1}^{n} \frac{b_i^{b_i}}{b_i!} \int_{\overline{M}_{g,n}} \frac{1 - \lambda_1 + \cdots \pm \lambda_g}{(1 - b_1\psi_1)\cdots(1 - b_n\psi_n)}.$$

The numerator in the integral is just the total Chern class of the dual of the Hodge bundle that is $c(H^\vee)$.

The integral is understood as expanding the denominator as a power series and only picking up monomials of the correct degree that is $3g - 3 + n$ in the entire product.

Consider the generating function for Hurwitz numbers

$$H(x, s_1, s_2, \ldots) = \sum_{g=0}^{\infty} \sum_{b_1,\ldots,b_n} h_{g,b_1,\ldots,b_n} \frac{x^m}{m!} \frac{s_{b_1} \cdots s_{b_n}}{n!}.$$

Consider the generating function for Hurwitz numbers

$$H(x, s_1, s_2, \ldots) = \sum_{g=0}^{\infty} \sum_{b_1,\ldots,b_n} h_{g,b_1,\ldots,b_n} \frac{x^m}{m!} \frac{s_{b_1} \cdots s_{b_n}}{n!}.$$

Okounkov (2000) showed that e^H satisfies the Kadomtsev-Petviashvili (KP) hierarchy, which in particular shows that H satisfies the KP equation.

$$\frac{\partial^2 H}{\partial s_2^2} = \frac{\partial^2 H}{\partial s_1 \partial s_3} - \frac{1}{2}\left(\frac{\partial^2 H}{\partial s_1^2}\right)^2 - \frac{1}{12}\frac{\partial^4 H}{\partial s_1^4}.$$

Consider the generating function for Hurwitz numbers

$$H(x, s_1, s_2, \ldots) = \sum_{g=0}^{\infty} \sum_{b_1,\ldots,b_n} h_{g,b_1,\ldots,b_n} \frac{x^m}{m!} \frac{s_{b_1} \cdots s_{b_n}}{n!}.$$

Okounkov (2000) showed that e^H satisfies the Kadomtsev-Petviashvili (KP) hierarchy, which in particular shows that H satisfies the KP equation.

$$\frac{\partial^2 H}{\partial s_2^2} = \frac{\partial^2 H}{\partial s_1 \partial s_3} - \frac{1}{2}\left(\frac{\partial^2 H}{\partial s_1^2}\right)^2 - \frac{1}{12}\frac{\partial^4 H}{\partial s_1^4}.$$

Kazarian and Lando use this equation to deduce the KdV equation for U.

The method of proof is the following:

The method of proof is the following:

- By a simple combinatorial technique K-L eliminate the λ classes from the ELSV formula and obtain the following explicit formula

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \sum_{b_1=1}^{d_1+1} \cdots \sum_{b_n=1}^{d_n+1} \left(\frac{1}{m!} \prod_{i=1}^{n} \frac{(-1)^{d_i - b_1 + 1}}{(d_i - b_1 + 1)! b_i^{b_i - 1}} \right) h_{g, b_1, \ldots b_n}.$$

The method of proof is the following:

- By a simple combinatorial technique K-L eliminate the λ classes from the ELSV formula and obtain the following explicit formula

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \sum_{b_1=1}^{d_1+1} \cdots \sum_{b_n=1}^{d_n+1} \left(\frac{1}{m!} \prod_{i=1}^{n} \frac{(-1)^{d_i - b_1 + 1}}{(d_i - b_1 + 1)! b_i^{b_i - 1}} \right) h_{g, b_1, \ldots b_n}.$$

- This gives a simple relation between the generating functions U and H after some clever change of variables.

The method of proof is the following:

- By a simple combinatorial technique K-L eliminate the λ classes from the ELSV formula and obtain the following explicit formula

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \sum_{b_1=1}^{d_1+1} \cdots \sum_{b_n=1}^{d_n+1} \left(\frac{1}{m!} \prod_{i=1}^{n} \frac{(-1)^{d_i - b_1 + 1}}{(d_i - b_1 + 1)! b_i^{b_i - 1}} \right) h_{g, b_1, \ldots b_n}.$$

- This gives a simple relation between the generating functions U and H after some clever change of variables.
- Finally KdV equation for U is deduced from the KP equation for H.

References

1. P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Etudes Sci. Publ. Math.(1969)
2. E. Witten, *Two-dimensional gravity and intersection theory on moduli spaces*, Surveys in Differential Geometry (1991)
3. M. Kontsevich, *Intersection theory on the moduli space of curves and the Airy function*, Comm. Math. Phys.(1992)
4. A. Okounkov, *Toda equations for Hurtwitz numbers*, Math. Res. Lett.(2000)
5. T. Ekedahl, S. K. Lando, M. Shapiro, A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. math.(2001)
6. M. Mirzakhani, *WeilPetersson volumes and intersection theory on the moduli space of curves*, J. Amer. Math. Soc.(2007)
7. M.E. Kazarian, S.K. Lando, *An algebro-geometric proof of Witten's conjecture*, J. Amer. Math. Soc.(2007)