An inverse problem for the relativistic Schrödinger equation with partial boundary data

Venky P. Krishnan[1]
vkrishnan@math.tifrbng.res.in
Based on joint work with Manmohan Vashisth

TIFR Centre for Applicable Mathematics, Bangalore, India
http://math.tifrbng.res.in[1]

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The Calderón inverse problem

The Calderón inverse problem

Motivated by the potential application of using electrical prospection in oil exploration, Alberto Calderón asked the following inverse problem: By making steady state voltage and current measurements at the boundary of a medium, can one recover the conductivity of that medium?

In mathematical terms:

$$\nabla \cdot \gamma \nabla u = 0 \text{ on } \Omega, \quad u|_{\partial\Omega} = f$$

where $0 < c < \gamma \in L^\infty(\Omega)$. γ is the conductivity of the medium.

Given boundary measurement operator

$$N_\gamma : f \to \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

the question is can one uniquely recover the conductivity γ?

The map N_γ is the Dirichlet to Neumann map. Also known as the voltage to current map.

Interested in the inversion of the non-linear map: $\gamma \to N_\gamma$.

The Calderón inverse problem

In his fundamental paper, Calderón was not able to tackle the inversion of this non-linear problem.
He linearized the problem and studied the corresponding linear map near constant conductivities.
Consider the linearized map (linearized near constant conductivities). That is, consider the Frechét derivative of the following map:

$$\gamma \to Q_\gamma,$$

where Q_γ is defined as:

$$Q_\gamma(f) = \int_\Omega \gamma |\nabla u|^2 \mathrm{d}x,$$

with

$$\nabla \cdot (\gamma \nabla u) = 0, \quad u|_{\partial\Omega} = f.$$

By integration by parts: N_γ and Q_γ are equivalent information.
That is,

$$\int_\Omega \nabla \cdot \gamma \nabla u v \mathrm{d}x + \int_\Omega \gamma \nabla u \cdot \nabla v \mathrm{d}x = \int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu} v \mathrm{d}s.$$

The Calderón inverse problem

We get the linearized map:

$$\mathrm{d}\, Q_\gamma(f)|_{\gamma=1}(\delta) = \int\limits_\Omega \delta |\nabla u|^2 \mathrm{d}x,$$

where $\Delta u = 0, \quad u|_{\partial\Omega} = f$.

The above integral vanishes for all u satisfying $\Delta u = 0$, $u|_{\partial\Omega} = f$ if and only if

$$\int\limits_\Omega \delta \nabla u_1 \cdot \nabla u_2 = 0,$$

with $\Delta u_1 = \Delta u_2 = 0$.

The question is: If

$$\int\limits_\Omega \delta \nabla u_1 \cdot \nabla u_2 = 0,$$

with $\Delta u_1 = \Delta u_2 = 0$, does it imply $\delta \equiv 0$?

The Calderón inverse problem

Calderón showed the answer is yes, through an extremely simple but very powerful technique:

He looked for special solutions to $\Delta u = 0$ of the form

$$u(x) = e^{\mathrm{i}x\cdot\zeta}$$

where ζ is a complex number.

Any harmonic function has to necessarily satisfy $\zeta \cdot \zeta = 0$ (not dot product).

Fix any $\xi \in \mathbb{R}^n$ and let $\zeta_1 = \mathrm{i}\eta + \frac{\xi}{2}$ and $\zeta_2 = -\mathrm{i}\eta + \frac{\xi}{2}$, where η is such that $|\eta| = \frac{|\xi|}{2}$ and $\eta \cdot \xi = 0$. Then let

$$u_1(x) = e^{\mathrm{i}x\cdot\zeta_1}, \quad u_2(x) = e^{\mathrm{i}x\cdot\zeta_2}$$

We then obtain that the Fourier transform: $\frac{|\xi|^2}{2}\widehat{\delta}(\xi) = 0$ a.e. in $\mathbb{R}^n$ which then implies that $\delta = 0$ a.e. in $\mathbb{R}^n$.

Study of the non-linear problem

The invertibility of the non-linear problem was shown by Sylvester and Uhlmann.

They introduced a very powerful and very versatile technique for looking for special solutions to lower order perturbations of Laplacians or its powers.

These type of solutions are what are called Complex Geometric Optics (CGO) solutions.

Study of the non-linear problem

Sylvester and Uhlmann solved the nonlinear problem by converting the conductivity equation to the Schrödinger equation:

$$(-\Delta + q)v = 0.$$

This can be achieved by letting u in the conductivity equation to be of the form $u = \gamma^{-1/2}v$. Then straightforward computation gives:

$$(-\Delta + q)\, v = 0, \quad v|_{\partial\Omega} = g,$$

where $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$ and $g = \gamma^{1/2}f$.

The inverse problem: Given (the Dirichlet to Neumann map) $N_q : g \to \frac{\partial v}{\partial \nu}$ where v is the solution of

$$(-\Delta + q)v = 0, \quad v|_{\partial\Omega} = g$$

determine q.

Study of the nonlinear problem

Sylvester-Uhlmann look for solutions to Schrödinger equation of the form by modifying the solutions Calderón used in his original work:

$$v(x) = e^{ix\cdot\zeta}\left(a + r(x,\zeta)\right),$$

where $\zeta \in \mathbb{C}^n$ satisfies $\zeta \cdot \zeta = 0$, a is such that $\nabla a \cdot \zeta = 0$ and r is a correction term satisfying the equation:

$$e^{-ix\cdot\zeta}\Delta\left(e^{ix\cdot\zeta} r(x)\right) = -q.$$

Note r is a solution to the above equation depending on the parameter ζ and it was shown by Sylvester-Uhlmann that as $|\zeta| \to \infty$, $r \to 0$.

More precisely: r satisfies the following estimate:

$$\|r\|_{L^2(\Omega)} \leq \frac{C}{|\zeta|}, \quad \|\nabla r\|_{L^2(\Omega)} \leq C.$$

The solutions $v(x)$ of the form above are what are called CGO solutions.

Uniqueness of the non-linear problem

To show uniqueness: Consider two q_1 and q_2 with corresponding solutions v_1 and v_2. By integration by parts one has the following identity:

$$\int q v_1 v_2 = 0$$

for all v_i solutions to $(-\Delta + q_i)v_i = 0$, $v_i|_{\partial\Omega} = g$. Here $q = q_1 - q_2$.

That is, this identity follows from

$$\int_{\Omega} \Delta u v \mathrm{d}x + \int_{\Omega} \nabla u \cdot \nabla v \mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \mathrm{d}s.$$

We need to show that the product of such solutions is dense in a suitable space.

The idea then is to find solutions v_1 and v_2 that are close to $e^{\mathrm{i}x\cdot\xi}$.

Uniqueness of the non-linear problem

In dimensions $n \geq 3$, the proof is easier.

Fix any $\xi \in \mathbb{R}^n$ and let ξ, ω_1, ω_2 be an orthogonal vectors with $|\omega_1| = |\omega_2| = 1$.

Let $\zeta = s\,(\omega_1 + \mathrm{i}\omega_2)$.

Now let $v_1(x) = e^{\mathrm{i}x\cdot\zeta_1}(1 + r_1(x, \zeta_1))$ and $v_2(x) = e^{-\mathrm{i}x\cdot\zeta}\left(e^{\mathrm{i}x\cdot\xi} + r_2(x, \zeta_2)\right)$.

Now due to the decay of r_1 and r_2, we observe that we have that the Fourier transform of $q = q_1 - q_2$ is 0.

Hence $q = 0$ a.e. in $\mathbb{R}^n$.

The theorem is valid in 2-dimensions as well.

Other related problems

Instead of Schrödinger equation, one can consider Magnetic Schrödinger equation

$$((D+W)\cdot(D+W)+q)\,u=0, \quad u|_{\partial\Omega}=f$$

The Dirichlet-to-Neumann map is

$$N_{W,q}: f \to \frac{\partial u}{\partial \nu}|_{\partial\Omega} + \mathrm{i}(W\cdot\nu)f.$$

Question: Can one determine the potential q and magnetic field W from Dirichlet-to-Neumann map $N_{W,q}$?

W cannot be determined due to gauge invariance associated to the problem. That is consider the magnetic Schrödinger equation with W replaced by $W' = W + \nabla\varphi$ where $\varphi \in C_c^\infty(\Omega)$. Then the solution to the equation with W' is $v = ue^{-\mathrm{i}\varphi}$. Since φ (as well as all its derivatives) are 0 at the boundary, $N_{W,q} = N_{W',q}$. Therefore one can only determine a component of W from boundary measurements.

From boundary measurements, the potential q can be uniquely determined.

Other related problems

Partial data problems: From a physical point of view, the entire boundary may not be accessible.

In such cases, measurements can be made only on part of the boundary.

Such cases have been studied earlier by Bukhgeim-Uhlmann and Kenig-Sjöstrand-Uhlmann.

In case measurements can be made on slightly more than half of the boundary, the CGO solutions similar to what was introduced earlier will work.

In general, a more refined version of CGO solutions is required. This was introduced by Kenig-Sjöstrand-Uhlmann in their fundamental paper. The weights in the CGO solutions introduced by these authors are called limiting Carleman weights.

A hyperbolic inverse problem

Some notation

- $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$.
- For $T > \operatorname{diam}(\Omega)$, let $Q_T := (0, T) \times \Omega$ and denote its lateral boundary by $\partial Q_T := (0, T) \times \partial\Omega$.
- Denote boundary of Q_T by $[\partial Q_T] := \partial Q_T \cup (\{0\} \times \overline{\Omega}) \cup (\{T\} \times \overline{\Omega})$.
- $\nu(x)$ is outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$.

A hyperbolic PDE

We consider the following operator:

$$\mathcal{L}_{\mathcal{A},q}u := [(\partial_t + A_0(t,x))^2 - \sum_{j=1}^{n}(\partial_j + A_j(t,x))^2 + q(t,x)]u;$$

where

- $A_j \in C_c^\infty(Q_T)$ and Scalar potential: $q \in L^\infty(Q_T)$
- Vector potential: $\mathcal{A} := (A_0, A_1, \cdots, A_n)$
- $A := (A_1, A_2, \cdots, A_n)$.

This is the so-called relativistic Schrödinger equation that is of importance in general relativity and quantum mechanics.

A hyperbolic PDE

Consider the following initial boundary value problem (IBVP):

$$\begin{cases} \mathcal{L}_{\mathcal{A},q} u(t,x) = 0; & (t,x) \in Q_T \\ u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x); & x \in \Omega \\ u(t,x) = f(t,x); \ (t,x) \in \partial Q_T. \end{cases}$$

Let

- $\phi \in H^1(Q_T)$, $\psi \in L^2(Q_T)$
- $f \in H^1(\partial Q_T)$ such that $f(0,x) = \phi(x)$ for $x \in \partial\Omega$.

Then

- there exists a unique solution $u \in C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$
- $\partial_\nu u \in L^2(\partial Q_T)$.

Reference: Lasiecka-Lions-Triggiani (1986) or the book "Inverse Boundary Spectral Problems" by Katchalov-Kurylev-Lassas.

Input-output operator

We define the *input-output* operator $\Lambda_{\mathcal{A},q}$ by

$$\Lambda_{\mathcal{A},q}(\phi, \psi, f) := (\partial_\nu u|_{\partial Q_T}, u|_{t=T}, \partial_t u|_{t=T})$$

where u is the solution to the hyperbolic PDE above. The operator

$$\Lambda_{\mathcal{A},q} : H^1(\Omega) \times L^2(\Omega) \times H^1(\partial Q_T) \to L^2(\partial Q_T) \times H^1(\Omega) \times L^2(\Omega)$$

is well-defined. This follows from the well-posedness result stated above.

Problem of interest

We are interested in the unique determination of the coefficients $(\mathcal{A}, q)$ in the hyperbolic PDE from the (partial) knowledge of input-output operator $\Lambda_{\mathcal{A},q}$.

Let $(\mathcal{A}^{(1)}, q_1)$ and $(\mathcal{A}^{(2)}, q_2)$ be two sets of vector and scalar potentials, and denote the corresponding input-output operator by $\Lambda_{\mathcal{A}^{(i)},q_i}$, i=1,2.

The uniqueness question is: If $\Lambda_{\mathcal{A}^{(1)},q_1} = \Lambda_{\mathcal{A}^{(2)},q_2}$, then $(\mathcal{A}^{(1)}, q_1) = (\mathcal{A}^{(2)}, q_2)$?

Even with knowledge on the input-output operator on the full boundary, the answer is no.

Gauge Invariance

Definition (Salazar 2013)

The vector and scalar potentials $(\mathcal{A}^{(1)}, q_1)$ and $(\mathcal{A}^{(2)}, q_2)$ are said to be gauge equivalent if there exists $g \in C^\infty(\bar{Q})$ such that $g(t,x) = e^{\Phi(t,x)}$ with $\Phi \in C_c^\infty(Q)$ and

$$\begin{cases} (\mathcal{A}^{(2)} - \mathcal{A}^{(1)})(t,x) = -\frac{\nabla_{t,x} g(t,x)}{g(t,x)} = -\nabla_{t,x}\Phi(t,x) \\ q_1(t,x) = q_2(t,x) \end{cases}$$

where $\nabla_{t,x} := (\partial_t, \nabla_x) = (\partial_t, \partial_1, \partial_2, \cdots, \partial_n)$.

Gauge Invariance

Proposition (Salazar 2013)

Let $(\mathcal{A}^{(1)}, q_1)$ *and* $(\mathcal{A}^{(2)}, q_2)$ *be gauge-equivalent and* $u(t,x)$ *is solution to*

$$\begin{cases} \mathcal{L}_{\mathcal{A}^{(1)},q_1} u(t,x) = 0 \quad in \;\; Q_T \\ u(0,x) = \phi(x);\;\; \partial_t u(0,x) = \psi(x) \quad in \;\; \Omega \\ u|_{\partial Q_T} = f \end{cases} \tag{1}$$

then $v(t,x) = e^{\Phi(t,x)} u(t,x)$ *solves*

$$\begin{cases} \mathcal{L}_{\mathcal{A}^{(2)},q_2} v(t,x) = 0 \quad in \;\; Q_T \\ v(0,x) = \phi(x); \partial_t v(0,x) = \psi(x) \quad in \;\; \Omega \\ v|_{\partial Q_T} = f. \end{cases} \tag{2}$$

Furthermore, $\Lambda_{\mathcal{A}^{(1)},q_1} = \Lambda_{\mathcal{A}^{(2)},q_2}$.

Our partial data set-up

Fix an $\omega_0 \in \mathbb{S}^{n-1}$. Define

$$\partial\Omega_{\pm,\omega_0} := \{x \in \partial\Omega : \ \pm\omega_0 \cdot \nu(x) \geq 0\}$$
$$\text{and} \ \ \text{denote} \ (\partial Q_T)_{\pm,\omega_0} := (0, T) \times \partial\Omega_{\pm,\omega_0}.$$

- Let F' and G' be open neighborhoods of $\partial\Omega_{+,\omega_0}$ and $\partial\Omega_{-,\omega_0}$ respectively in $\partial\Omega$.
- Denote $F := (0, T) \times F'$ and $G := (0, T) \times G'$.

The partial boundary data, we consider is

$$\Lambda^*_{\mathcal{A}^{(i)},q_i}(\phi, \psi, f) := (\partial_\nu u_i|_G, u_i|_{t=T}) \text{ for } i = 1, 2$$

where u_i is the solution to the hyperbolic PDE when $(\mathcal{A}, q) = (\mathcal{A}^{(i)}, q_i)$.

Statement of the main result

Theorem (K. and Vashsith, preprint 2018, arXiv:1801.04866)

Let $\left(\mathcal{A}^{(1)}, q_1\right)$ and $\left(\mathcal{A}^{(2)}, q_2\right)$ be two sets of vector and scalar potentials, respectively. If

$$\Lambda^*_{\mathcal{A}^{(1)},q_1}(\phi,\psi,f) = \Lambda^*_{\mathcal{A}^{(2)},q_2}(\phi,\psi,f)$$

for $(\phi,\psi,f) \in H^1(\Omega) \times L^2(\Omega) \times H^1(\partial\Omega_T)$, then there exists a function $\Phi \in C_c^\infty(Q_T)$, such that

$$(\mathcal{A}^{(1)} - \mathcal{A}^{(2)})(t,x) = \nabla_{t,x}\Phi(t,x) \quad \text{and } q_1(t,x) = q_2(t,x), \ (t,x) \in Q_T.$$

The above result can be proved for $\mathcal{A}^{(i)} \in W^{1,\infty}(Q_T)$ provided they are identical on $[\partial Q_T]$, (See Kian (2016) for the case $A = 0$).

Existing results in this direction

For the case $\mathcal{A} = 0$, this problem has been studied by

- Bukhgeím and Klibanov (1981) time-independent case
- Rakesh and Symes (1988) time-independent case
- Ramm and Rakesh(1991) time-dependent case
- Yamamoto and his collaborators (time-independent case)
- Kian (2016) time-dependent case with partial data
- Kian and Hu (2017), time-dependent singular coefficient

For the case where $A_j = 0$ for $1 \leq j \leq n$ (that is for the case of time-derivative perturbation only), the problem has been studied by

- Isakov (1991) time-independent
- Kian (2016) time-dependent partial data

For general coefficient $\mathcal{A}$

- Eskin(2006) time-independent coefficients (Boundary Control method)
- Stefanov and Yang (2017) (Local uniqueness)
- Salazar (2013) time-dependent with full data (Using Geometric Optics solutions)

Sketch of the proof

To prove the above theorem, we proceed as follows:

1. Derive an integral identity using the solution to the adjoint problem.
2. Construction of Geometric optics (GO) solutions using an interior Carleman estimate.
3. Control of boundary data on the unknown part using a boundary Carleman estimate.
4. Substituting the GO solutions in the integral identity, we end up with a light ray transform.
5. Inverting the light ray transform, gives the required uniqueness result.

Integral identity

For $\phi \in H^1(\Omega)$, $\psi \in L^2(\Omega)$ and $f \in H^1(\partial Q_T)$, let $u_i \in H^1(Q_T)$ for $i = 1, 2$ be the solution to the following IBVP:

$$\begin{cases} \mathcal{L}_{\mathcal{A}^{(i)}, q_i} u_i(t,x) = 0; & (t,x) \in Q_T \\ u_i(0,x) = \phi(x), \quad \partial_t u_i(0,x) = \psi(x); & x \in \Omega \\ u_i(t,x) = f(t,x); \ (t,x) \in \partial Q_T. \end{cases}$$

Denote:

- $u(t,x) := (u_1 - u_2)(t,x)$
- $\tilde{q}_i := \partial_t A_0^{(i)} - \nabla_x A^{(i)} + |A_0^{(i)}|^2 - |A^{(i)}|^2 + q_i$
- $\tilde{q} := \tilde{q}_2 - \tilde{q}_1$
- $\mathcal{A} := \mathcal{A}^{(2)} - \mathcal{A}^{(1)}$, i.e. $A_0 := A_0^{(2)} - A_0^{(1)}$, $A := A^{(2)} - A^{(1)}$

Integral identity

The difference of solutions $u \in H^1(Q_T)$ satisfies:

$$\mathcal{L}_{\mathcal{A}^{(1)},q_1} u(t,x) = -2A \cdot \nabla_x u_2 + 2A_0 \partial_t u_2 + \tilde{q} u_2 \tag{3}$$
$$u(0,x) = \partial_t u(0,x) = u|_{\partial Q_T} = 0.$$

Let $v \in H^1(Q_T)$ be the solution to the following adjoint problem

$$\mathcal{L}^*_{\mathcal{A}^{(1)},q_1} v(t,x) = 0. \tag{4}$$

Multiplying (3) by v and integrating over Q_T, we get

$$\int_{Q_T} \big(-2A \cdot \nabla_x u_2 + 2A_0 \partial_t u_2 + \tilde{q} u_2 \big) v(t,x) \mathrm{d}x \mathrm{d}t =$$
$$\int_{\Omega} \partial_t u(T,x) v(T,x) \mathrm{d}x - \int_{\partial Q_T \setminus G} \partial_\nu u(t,x) v(t,x) \mathrm{d}S_x \mathrm{d}t.$$

In deriving the identity above, we have used

$$u|_{\partial Q_T} = 0,\ u(T,x) = 0,\ \partial_\nu u|_G = 0 \text{ and } L^*_{\mathcal{A}^{(1)},q_1} v(t,x) = 0.$$

Interior Carleman Estimate

Our goal now is to produce solutions u_2 and v of a special form (the so-called GO solutions). This is achieved using the following Carleman estimate:

Lemma (Kian, and K. and Vashisth)

Let $\varphi(t,x) = t + x\cdot\omega$ *and* $\mathcal{L}_\varphi := h^2 e^{-\varphi/h}\mathcal{L}_{\mathcal{A},q}e^{\varphi/h}$. *There exists an* $h_0 > 0$ *such that*

$$||v||_{L^2(\mathbb{R}^{1+n})} \leq \frac{C}{h}||\mathcal{L}_\varphi v||_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})}, \tag{5}$$

and

$$||v||_{L^2(\mathbb{R}^{1+n})} \leq \frac{C}{h}||\mathcal{L}^*_\varphi v||_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \tag{6}$$

for all $v \in C_c^\infty(Q),\ 0 < h \leq h_0$.

where we denote by $H^s_{\mathrm{scl}}(\mathbb{R}^{1+n})$, the Sobolev space of order s with the norm given by

$$||u||^2_{H^s_{\mathrm{scl}}(\mathbb{R}^{1+n})} = \left|\left|\langle hD\rangle^s u\right|\right|^2_{L^2(\mathbb{R}^{1+n})} = \int_{\mathbb{R}^{1+n}} (1+h^2\tau^2+h^2|\xi|^2)^s\,|\widehat{u}(\tau,\xi)|^2\,\mathrm{d}\tau\mathrm{d}\xi.$$

Interior Carleman estimate

The above estimate follows from the Carleman estimate:

Lemma

Let $\varphi(t,x) = t + x\cdot\omega$ *and* $\mathcal{L}_\varphi := h^2 e^{-\varphi/h}\mathcal{L}_{\mathcal{A},q}e^{\varphi/h}$. *There exists an* $h_0 > 0$ *such that*

$$\|u\|_{H^1_{\mathrm{scl}}(Q)} \leq \frac{C}{h}\|\mathcal{L}_\varphi u\|_{L^2(Q)},$$

for all $u \in C_c^\infty(Q)$ *and* $0 < h \leq h_0$.

Here we denote by $H^1_{\mathrm{scl}}(Q)$, the semiclassical Sobolev space of order 1 on Q with the following norm

$$\left|\left|u\right|\right|_{H^1_{\mathrm{scl}}(Q)} = \left|\left|u\right|\right|_{L^2(Q)} + \left|\left|h\nabla_{t,x}u\right|\right|_{L^2(Q)}.$$

The Carleman estimate in negative index Sobolev space follows from shifting the index to the left using suitable pseudodifferential operators.

Construction of GO solutions

Using the above Carleman estimate together with a standard Functional Analysis argument, we can prove existence of solutions to the hyperbolic equation:

Proposition

Let $\mathcal{L}_\varphi$ be as above. For $h > 0$ small enough and $v \in L^2(Q)$, there exists a solution $u \in H^1(Q)$ of

$$\mathcal{L}_\varphi u = v,$$

satisfying the estimate

$$\|u\|_{H^1_{\mathrm{scl}}(Q)} \leq \frac{C}{h} \|v\|_{L^2(\mathbb{R}^{1+n})},$$

where $C > 0$ is a constant independent of h.

Proposition

Let $\mathcal{L}_{\mathcal{A},q}$ *be our hyperbolic operator.*

- *(Exponentially decaying solutions) There exists an* $h_0 > 0$ *such that for all* $0 < h \leq h_0$*, we can find* $v \in H^1(Q)$ *satisfying* $\mathcal{L}_{-\mathcal{A},\overline{q}} v = 0$ *of the form*

$$v_d(t,x) = e^{-\frac{\varphi}{h}} \left(B_d(t,x) + h R_d(t,x;h) \right), \tag{7}$$

where $\varphi(t,x) = t + x \cdot \omega$*,*

$$B_d(t,x) = \exp\left(-\int_0^\infty (1,-\omega) \cdot \mathcal{A}(t+s, x - s\omega) \mathrm{d}s \right) \tag{8}$$

with $\zeta \in (1,-\omega)^\perp$ *and* $R_d \in H^1(Q)$ *satisfies*

$$\|R_d\|_{H^1_{\mathrm{scl}}(Q)} \leq C. \tag{9}$$

Construction of GO solutions

Proposition

Let $\mathcal{L}_{\mathcal{A},q}$ be our hyperbolic operator.

- *(Exponentially growing solutions) There exists an $h_0 > 0$ such that for all $0 < h \leq h_0$, we can find $v \in H^1(Q)$ satisfying $\mathcal{L}_{\mathcal{A},q} v = 0$ of the form*

$$v_g(t,x) = e^{\frac{\varphi}{h}} \left(B_g(t,x) + hR_g(t,x;h) \right), \tag{10}$$

where $\varphi(t,x) = t + x \cdot \omega$,

$$B_g(t,x) = e^{-i\zeta\cdot(t,x)} \exp\left(\int_0^\infty (1,-\omega) \cdot \mathcal{A}(t+s, x - s\omega) \mathrm{d}s \right) \tag{11}$$

with $\zeta \in (1,-\omega)^\perp$ and $R_g \in H^1(Q)$ satisfies

$$\|R_g\|_{H^1_{\mathrm{scl}}(Q)} \leq C. \tag{12}$$

Boundary Carleman estimate

Recall the integral identity derived above:

$$\int_{Q_T} \big(-2A \cdot \nabla_x u_2 + 2A_0 \partial_t u_2 + \tilde{q} u_2 \big) v(t,x) \mathrm{d}x \mathrm{d}t =$$

$$\int_{\Omega} \partial_t u(T,x) v(T,x) \mathrm{d}x - \int_{\partial Q_T \setminus G} \partial_\nu u(t,x) v(t,x) \mathrm{d}S_x \mathrm{d}t.$$

We still need to estimate the two terms on the right hand side of the above integral identity. This is achieved using the following boundary Carleman estimate.

Boundary Carleman estimate

Lemma

Let $\varphi(t,x) := t + x\cdot\omega$, where $\omega \in \mathbb{S}^{n-1}$ is fixed. Assume $\mathcal{L}_{\mathcal{A},q}$ be as above. Then the Carleman estimate

$$
\begin{aligned}
& h\left(e^{-\varphi/h}\partial_\nu\varphi\partial_\nu u, e^{-\varphi/h}\partial_\nu u\right)_{L^2(\Sigma_{+,\omega})} \\
& + h\left(e^{-\varphi(T,\cdot)/h}\partial_t u(T,\cdot), e^{-\varphi(T,\cdot)/h}\partial_t u(T,\cdot)\right)_{L^2(\Omega)} \\
& + \|e^{-\varphi/h}u\|^2_{L^2(Q)} + \|he^{-\varphi/h}\partial_t u\|^2_{L^2(Q)} + \|he^{-\varphi/h}\nabla_x u\|^2_{L^2(Q)} \\
& \leq C\Big(\|he^{-\varphi/h}\mathcal{L}_{\mathcal{A},q}u\|^2_{L^2(Q)} + \left(e^{-\varphi(T,\cdot)/h}u(T,\cdot), e^{-\varphi(T,\cdot)/h}u(T,\cdot)\right)_{L^2(\Omega)} \\
& + h\left(e^{-\varphi(T,\cdot)/h}\nabla_x u(T,\cdot), e^{-\varphi(T,\cdot)/h}\nabla_x u(T,\cdot)\right)_{L^2(\Omega)} \\
& + h\left(e^{-\varphi/h}\left(-\partial_\nu\varphi\right)\partial_\nu u, e^{-\varphi/h}\partial_\nu u\right)_{L^2(\Sigma_{-,\omega})}\Big)
\end{aligned}
$$

holds for all $u \in C^2(Q)$ with

$$u|_\Sigma = 0,\ \ u|_{t=0} = \partial_t u|_{t=0} = 0, \text{ and } h \text{ small enough.}$$

Integral identity

Using the boundary Carleman estimate, the GO solution expressions for u_2 and v, and letting $h \to 0$, we have the following equality:

$$\int\limits_{\mathbb{R}^{n+1}} e^{-\zeta\cdot(t,x)}\tilde{\omega}\cdot\mathcal{A}(t,x)\exp\Big(\int\limits_0^\infty \tilde{\omega}\cdot\mathcal{A}(t+s,x-s\omega)ds\Big)\mathrm{d}x\mathrm{d}t = 0, \tag{13}$$

for all $\zeta \in (1,-\omega)^\perp$ and ω near ω_0:

Light ray transform

Use the decomposition $\mathbb{R}^{1+n} = (1, -\omega)^{\perp} \oplus \mathbb{R}(1, -\omega)$ and Fubini's theorem in (13), we deduce that

$$\int_{\mathbb{R}} \tilde{\omega} \cdot \mathcal{A}(t' + s, x' - s\omega)\, ds = 0, \quad \forall\ (t', x') \in \mathbb{R}^{1+n}\ \&\ \omega \text{ near } \omega_0.$$

This is the so-called light ray transform.

Our uniqueness question is reduced to showing invertibility of this light ray transform.

Note that the light ray transform (acting on vector fields) has a natural kernel.

Uniqueness

- Use Fourier transform techniques, (see also (Stefanov(2017), RabieniaHaratbar(2017)), we show that if the light ray transform vanishes for all $(t', x') \in \mathbb{R}^{1+n}$ & ω near ω_0, we have that there exists $\Phi(t,x) \in C_c^\infty(Q_T)$ such that

$$\mathcal{A}(t,x) = \nabla_{t,x}\Phi(t,x).$$

- Using the uniqueness of the vector potential term (modulo potential fields), uniqueness for q can be proved given the vanishing of the light ray transform acting on functions.

Thank you very much for your attention