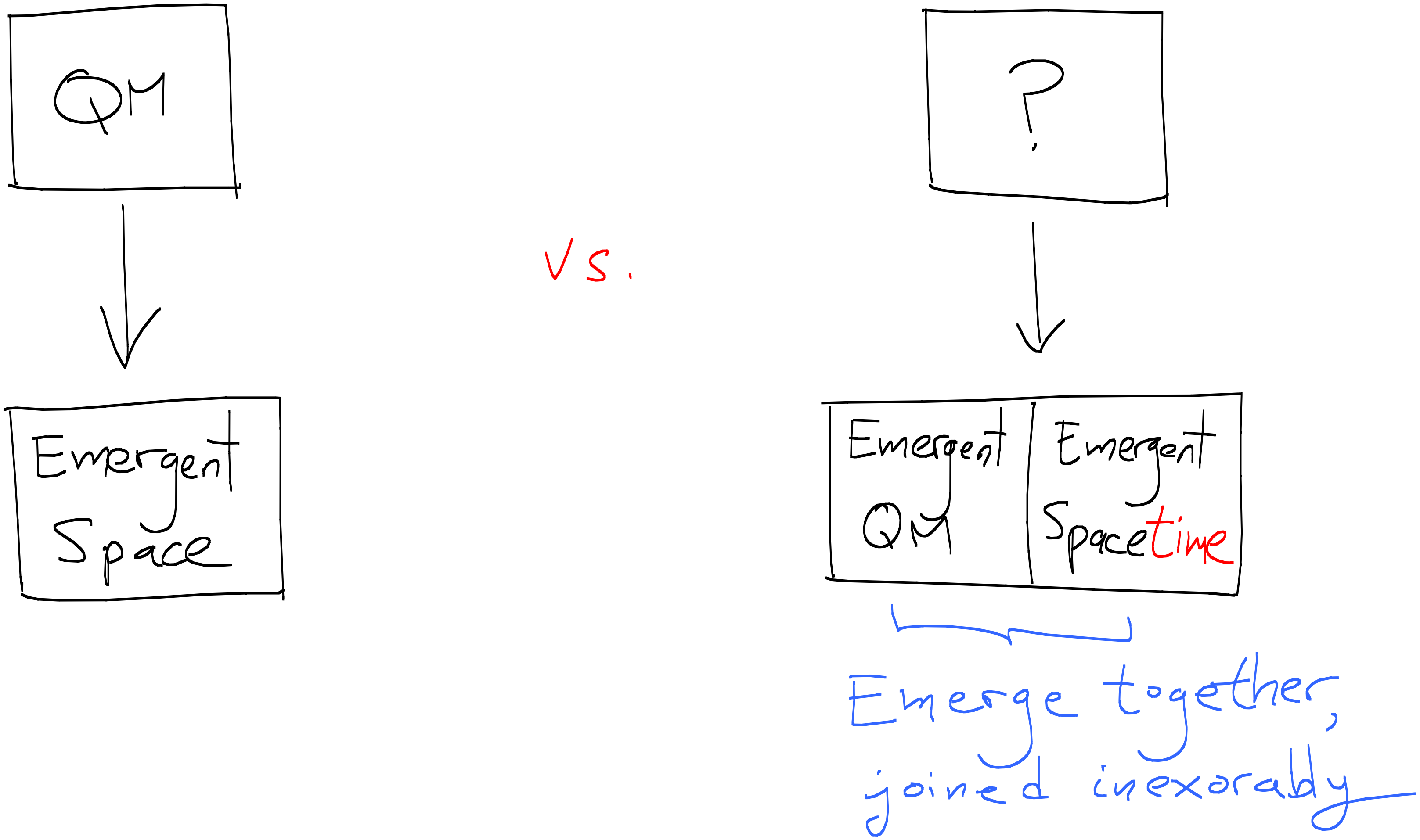
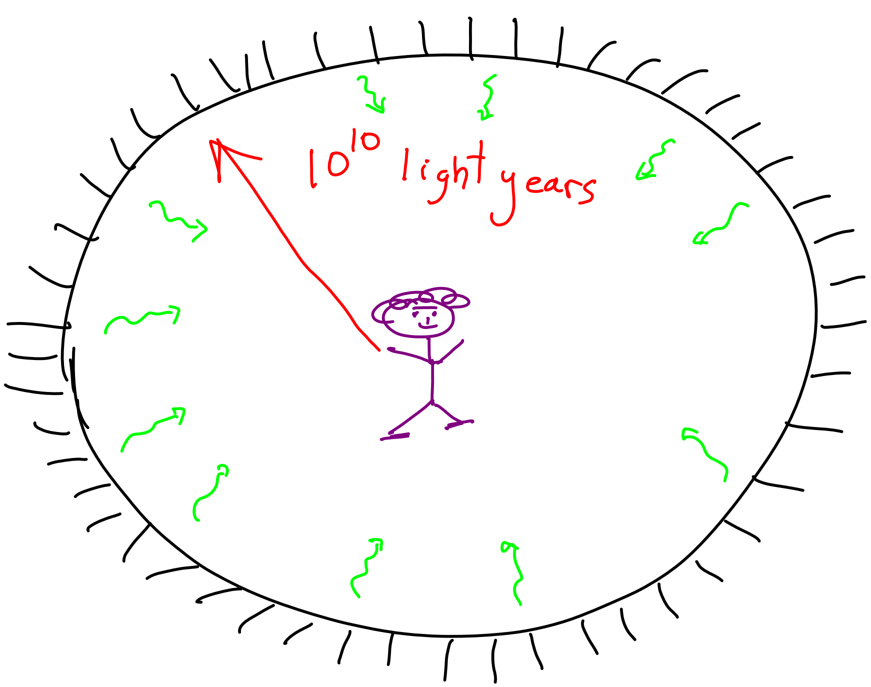
Space-Time, Quantum Mechanics

and

Positive Geometry


QM
Emergent Space
vs.
?
Emergent QM
Emergent Spacetime
Emerge together, joined inexorably

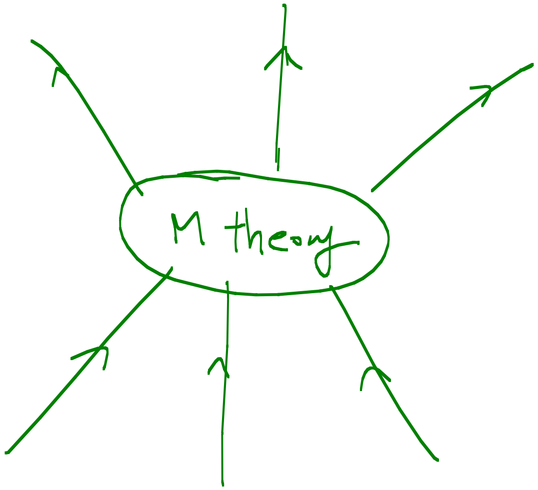


What are the correct observables??

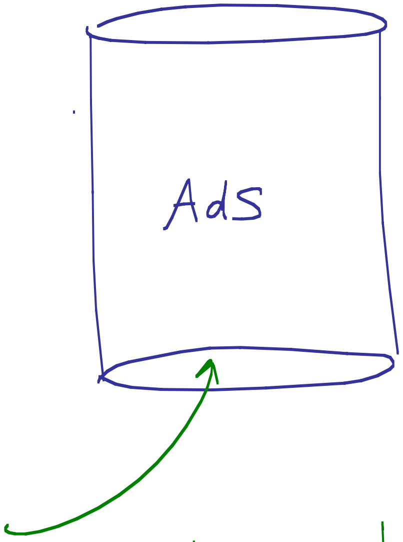
Emergent
Extension of

Space-Time
Quantum Mechanics?

A Huge Tension



vs.



Any Old Quantum Theory

One Unified Theory!

Landscape of connected solutions. UNIFIED in FLAT SPACE

Is a different theory in AdS!

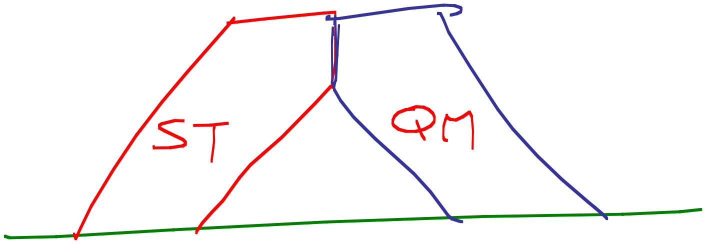
Other Circumstantial Clues

(4+0) more natural

QFT in (3+1) dim

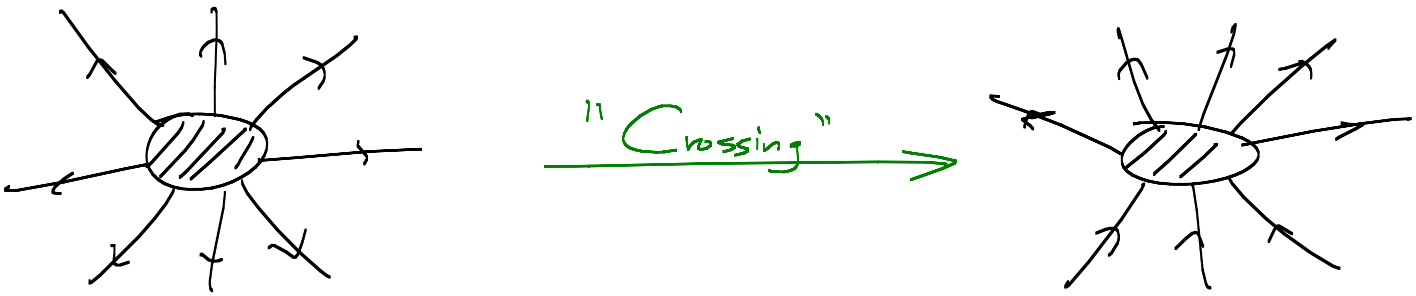
(2+2) more natural

"Causal" + "Probabilistic" intimately tied together



They Buttress Each Other, making each other more rigid + robust

Clues in Scattering Amplitudes



Most natural object:
no "in", "out";
complex momenta

"in" → "out"
+ Unitarity hand-in-hand

Locality + Unitarity
totally intertwined

Mystery of Time

How is Causality reflected in S-Matrix?

How is Cosmological Time reflected in Ψ_{Univ}?

WE STILL DON'T KNOW,
NOT EVEN IN PERT TH

New Strategy: Look For

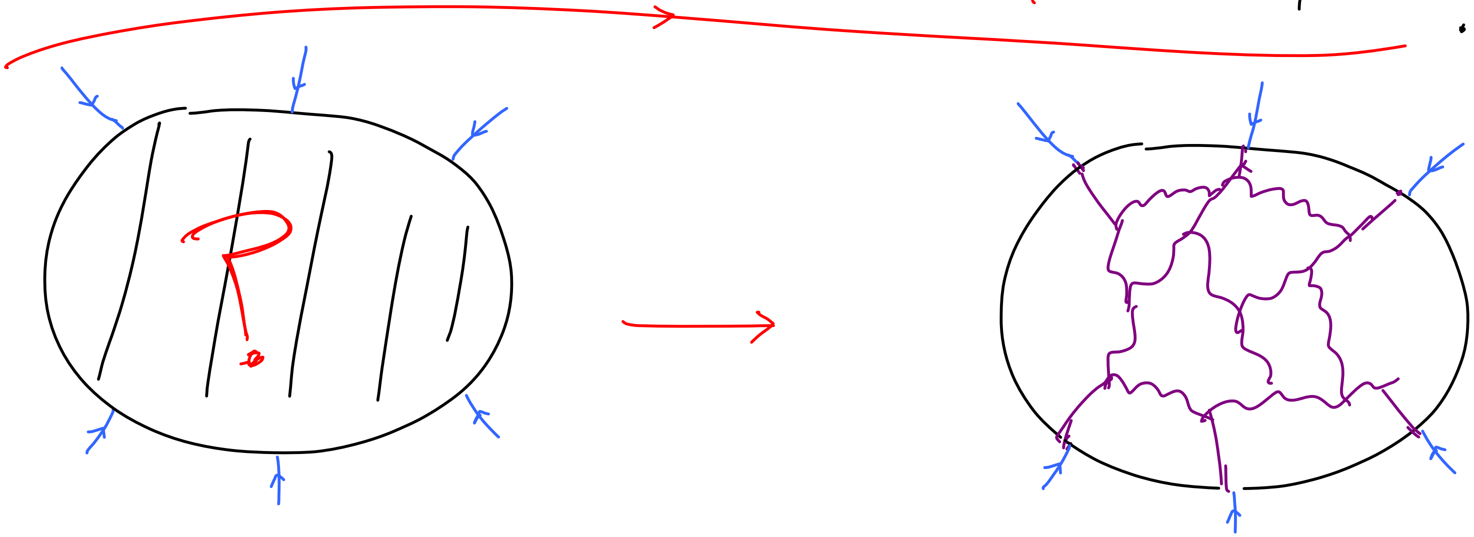
NEW PRINCIPLES, LAWS

from which CAUSAL, UNITARY

evolution — local Spacetime Physics + QM,

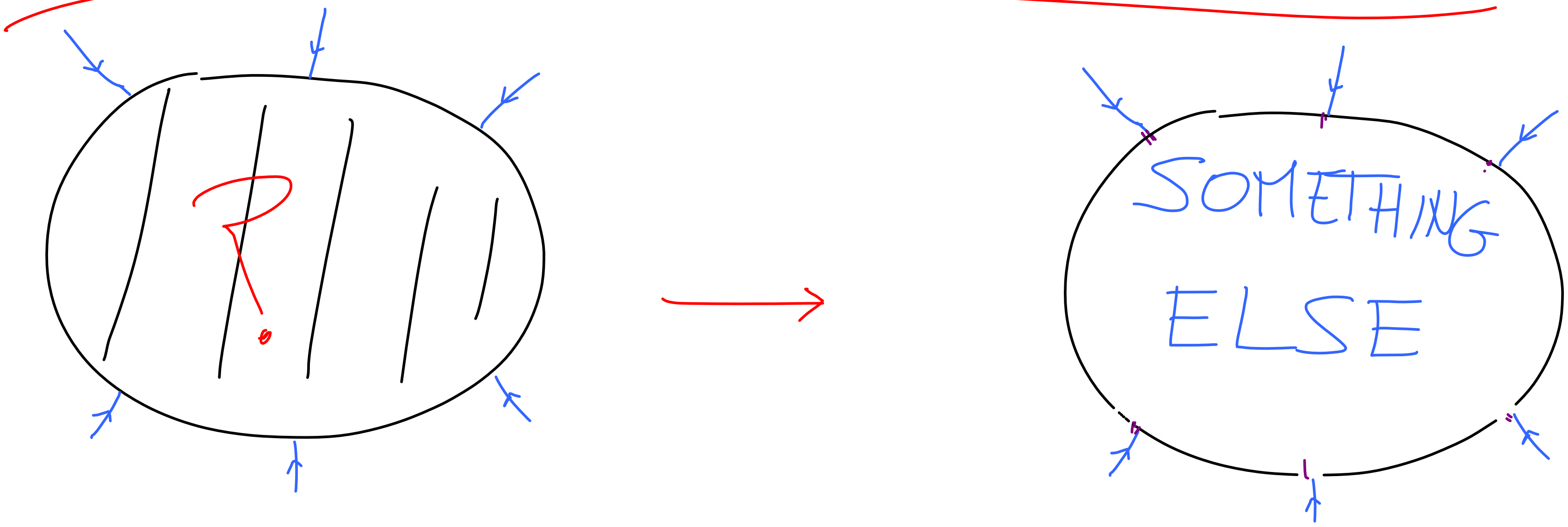
emerge together.

What is the Q to which A is the Answer?



Local, Unitary Evolution in Spacetime

What is the Q to which A is the Answer?



The Canvas

Kin. Space

* Physical momenta
* "Twistor" variables
* "Celestial Sphere"

⋮

Note Unlike e.g ∂AdS:

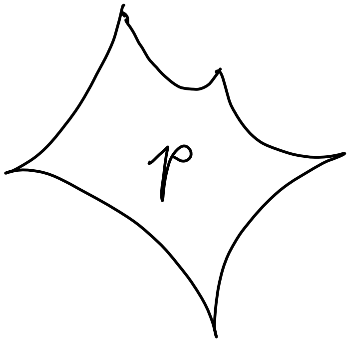
NO TIME

NO LOCALITY

WHAT IDEAS BREATHE PHYSICS-LIFE INTO THIS SPACE?

Positive Geometries

Canonical Forms



↔

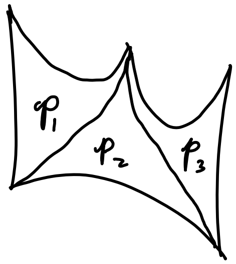
Ω_P : unique form with logarithmic singularities on (+ only on) boundaries of P $\left\{ \text{locally } \Omega \to \frac{dx_1}{x_1} \cdots \frac{dx_D}{x_D} \right\}$

Region P w/ boundaries of all Codimension

Real Geometry

Complex Form

“Triangulation”

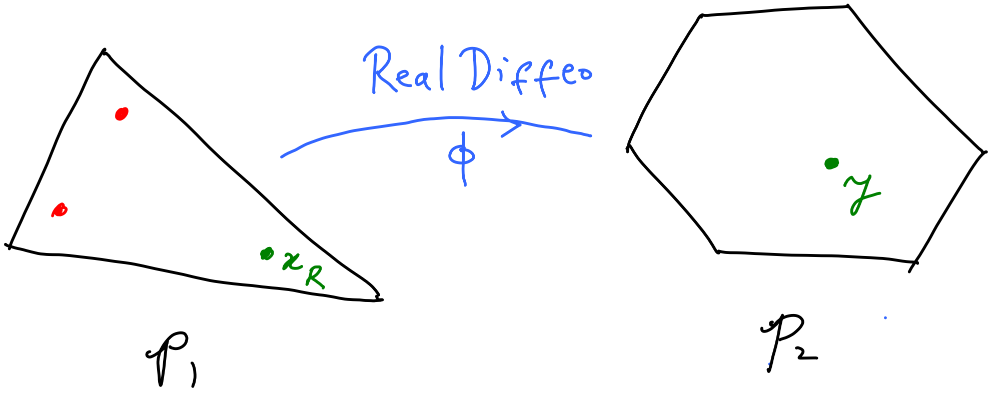


$\mathcal{P}$ tiled by $\mathcal{P}_i$

$$\Downarrow$$

$$\Omega_{\mathcal{P}} = \sum_i \Omega_{\mathcal{P}_i}$$

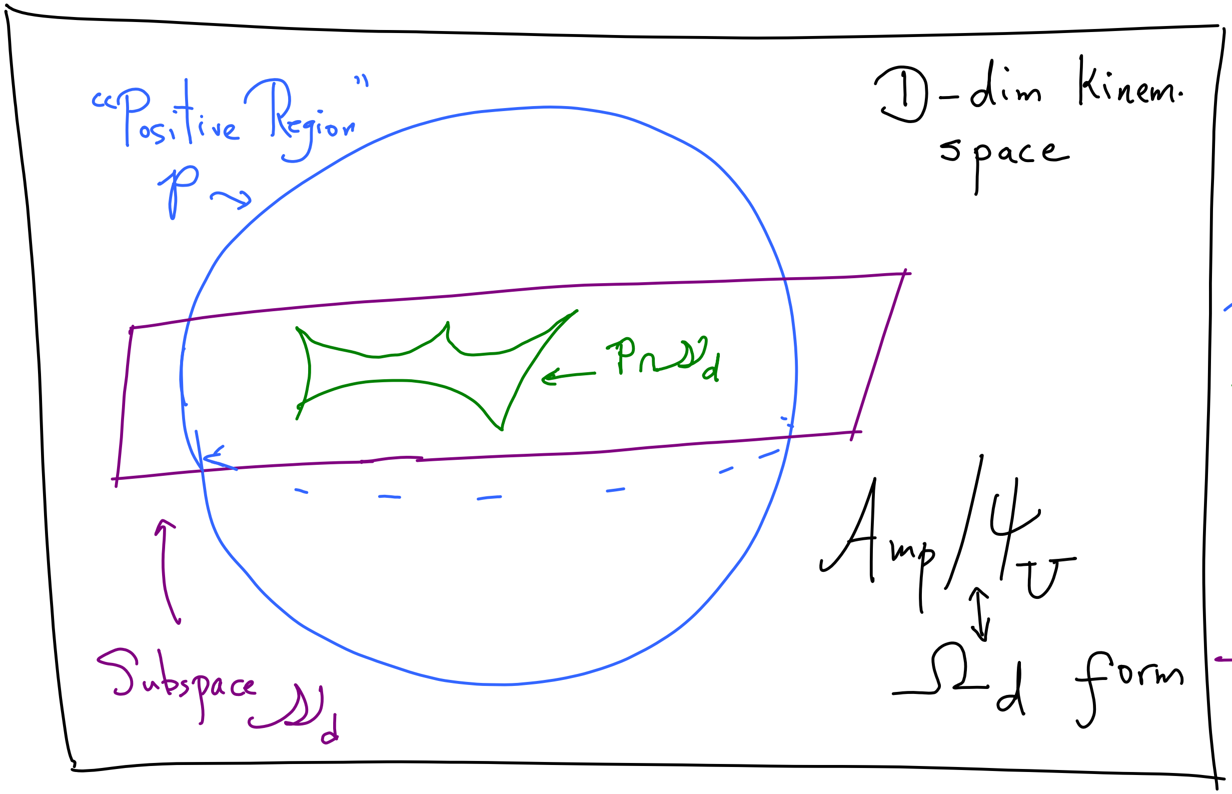
“Push-Forward”



$$\Omega_{\mathcal{P}_2}[y] = \sum_{\phi^{-1}(y)} \Omega_{\mathcal{P}_1}[\phi^{-1}(y)]$$

Magic: single $\underline{\text{real}}$ solution $x_R \in \mathcal{P}_1$ iff $y_R \in \mathcal{P}_2$

General Picture



Ω_d fixed thusly:

$\mathcal{S}_d$ intersects $\mathcal{P}$ in a POSITIVE GEOMETRY

Ω_d Pulls Back to CANONICAL FORM

Many (Likely Related) Examples

- * Amplituhedra: Positive Geometry of Planar $\mathcal{N}=4$ SYM
- * Associahedra + beyond: Positive Geometry of Factorization + Color
- * Cosmological Polytopes: Positive Geometry of Univ.
- * Positive Geometry of EFT: Univ predictions from consistent UV
  e.g. Quant. pred. from Quant Gr.
  in the real world
- * Positive Geometry of CFT: Geometry underlying conformal bootstrap

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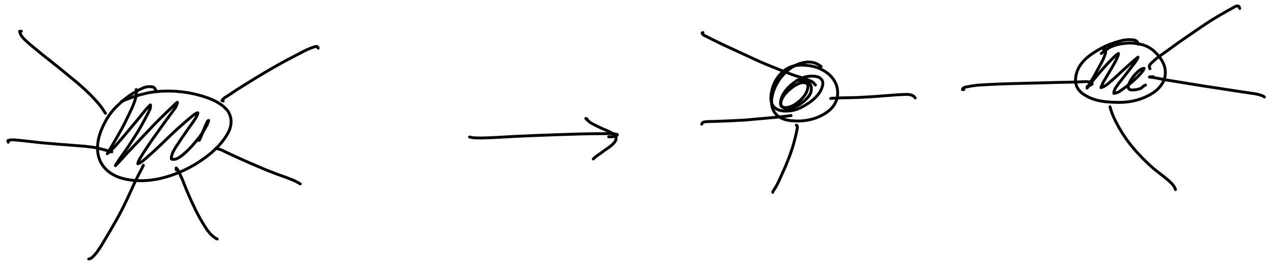
Universal Feature of Tree Amplitudes:

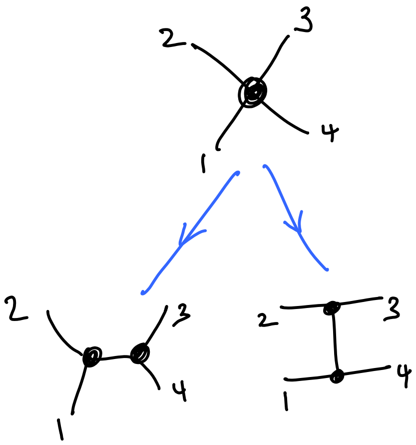
Factorization

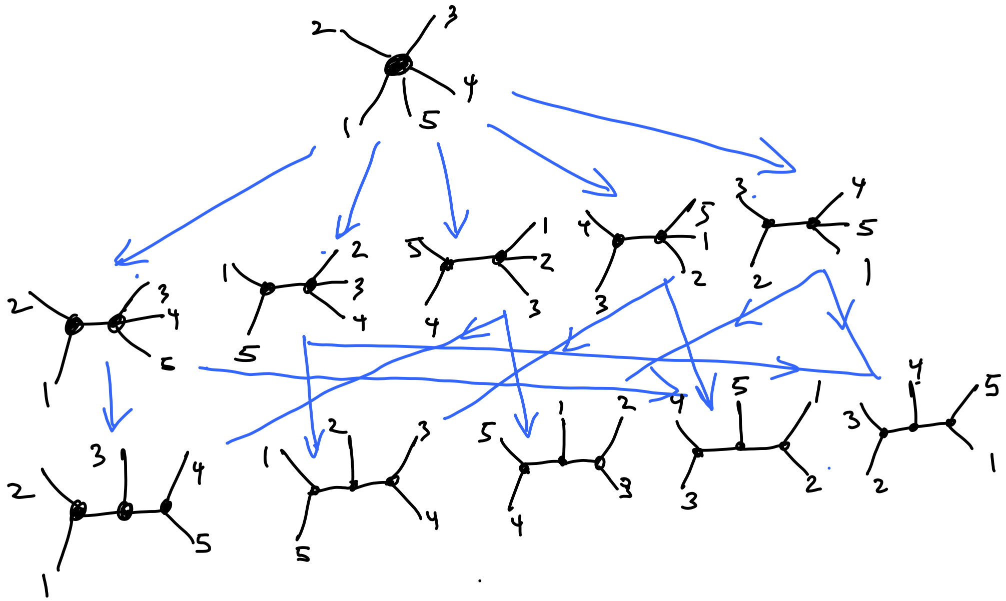


Locality

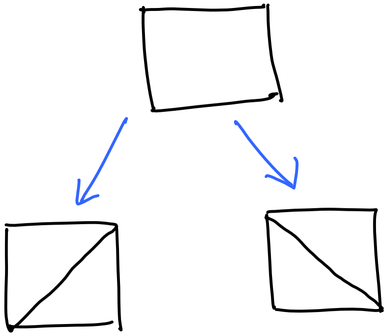
Unitarity

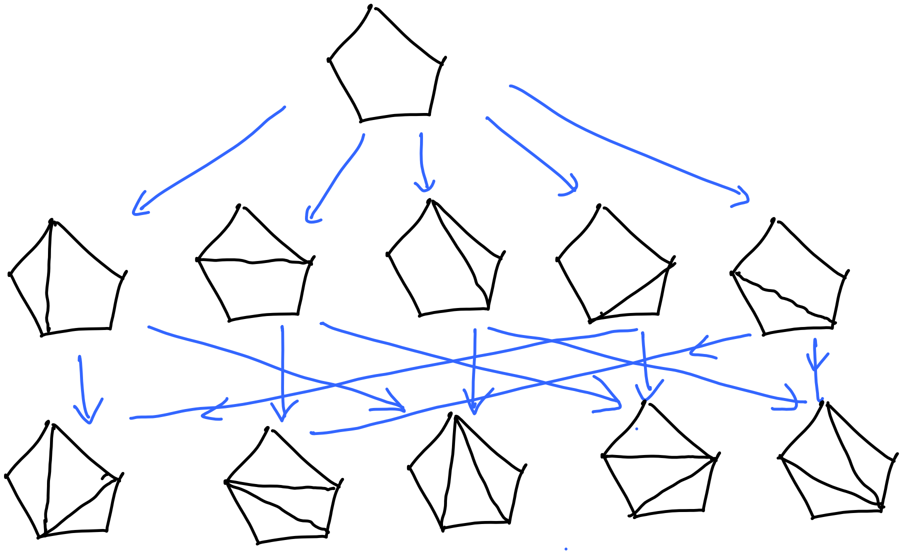
The Associahedron



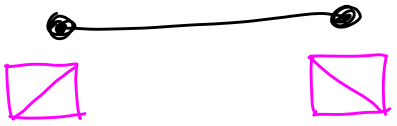


The Associahedron

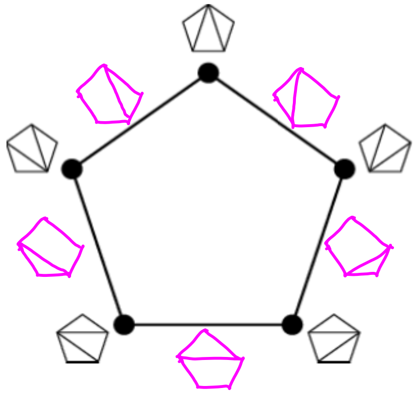




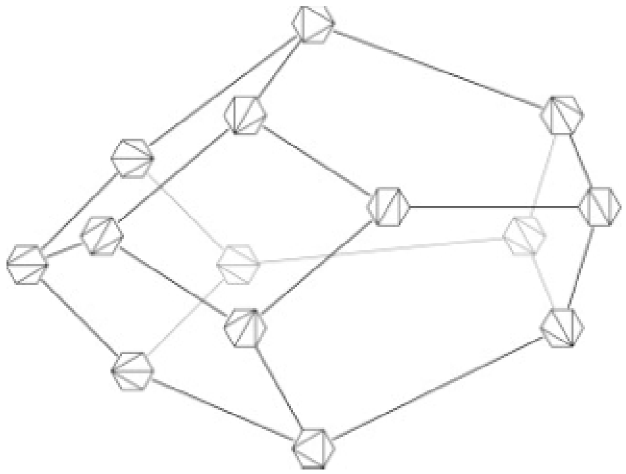
Quite beautifully, this network of inclusion relationships is that of a convex polytope in (n-3) dimensions:



$n=4$



$n=5$

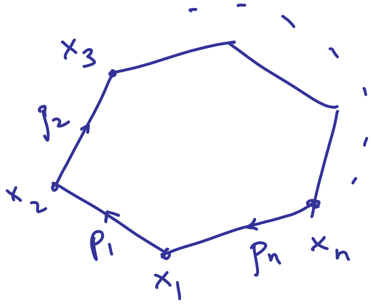


$n=6$

Mandelstam Kinematics

* n particles ; $s_{ab} = p_a \cdot p_b$; $\sum_{b \neq a} s_{ab} = 0$; $\binom{n}{2} - n = \frac{n(n-3)}{2}$ dim space.

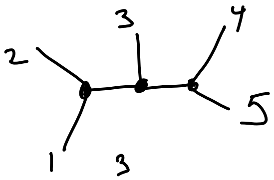
* Order



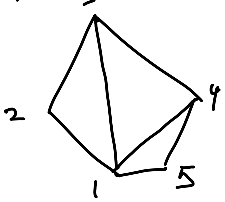
$$p_a^\mu = x_{a+1}^\mu - x_a^\mu$$

$$X_{ij} \equiv (x_i - x_j)^2 = (p_i + \cdots + p_{j-1})^2$$

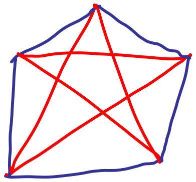
*



$$\frac{1}{s_{12}\; s_{123}}$$



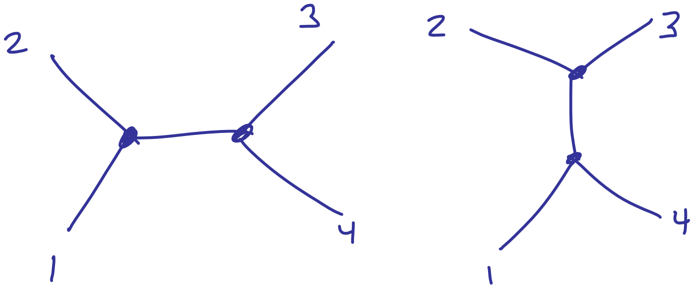
$$\frac{1}{X_{13}\, X_{14}}$$



diagonals $= \frac{n(n-3)}{2}$

X_{ij} a basis for kinematic space

Scattering Forms



$$\frac{1}{s} + \frac{1}{t}$$

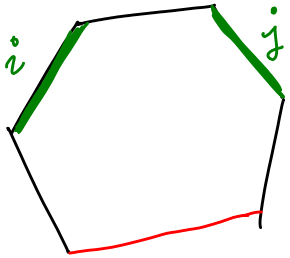
$$\frac{ds}{s} - \frac{dt}{t} \quad \left(= d\log\frac{s}{t} \right)$$

Projectivity

Associahedron In Mandelstam Space!

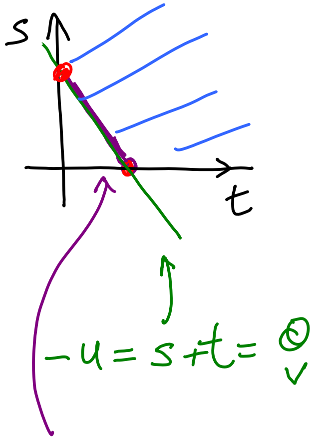
"Positive Region" $X_{ij} \geq 0$ $\{\text{All poles} \geq 0\}$

Linear Subspace:



$-2p_i\cdot p_j = c_{ij}$ fixed,

with $c_{ij} > 0$.



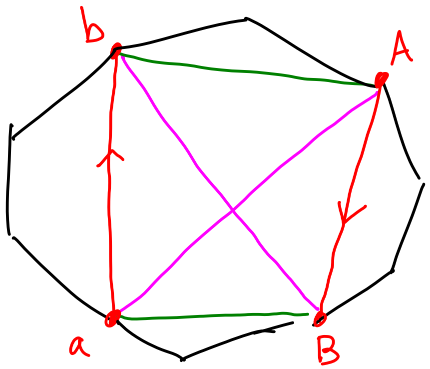
$-u = s+t = c$

$n=4$

Assoc.

The intersection of this Subspace with Positive Region is (n−3) dimensional — and is an associahedron!

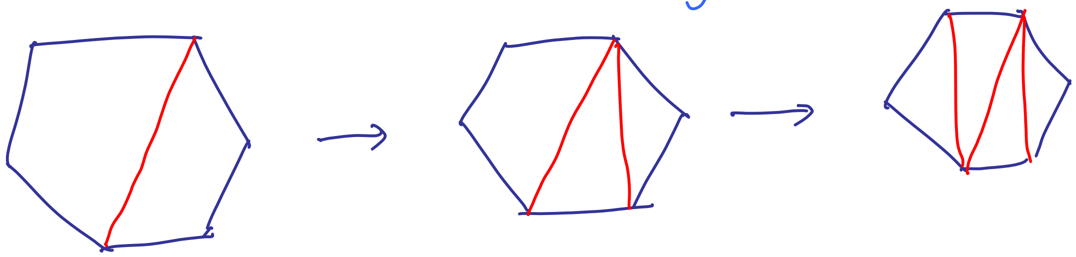
Why?



$$-2p_{a\to b}\cdot p_{A\to B} = \sum -2p_i\cdot p_J > 0$$

$$= X_{bB} + X_{aA} - X_{bA} - X_{aB}.$$

So, we can't set crossing diagonals X_{bB}, X_{aA} both to 0! Thus boundaries correspond to triangulations,



precisely defn of associahedron

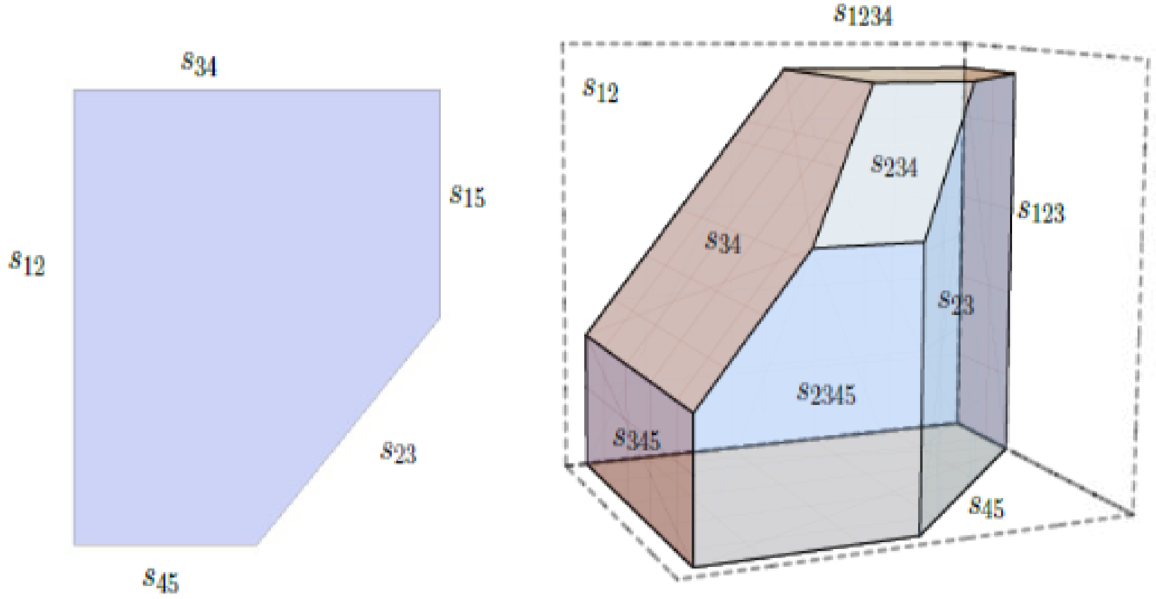
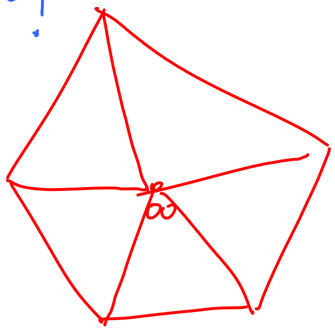


Figure 2: Pictures for n=5 (left) and n=6 (right) associahedra, where we have labeled every facet by the corresponding vanishing planar variable.

- * There is a unique $(n-3)$ form on the big space that pulls back to canonical form of associahedron on subspace, + gives amplitude!
- * Projective invariance = Hidden symmetry of ϕ^3_{BA}
- * Feynman Diagrams: one triangulation, breaks this symmetry term-by-term

Spurious Poles @ ∞!

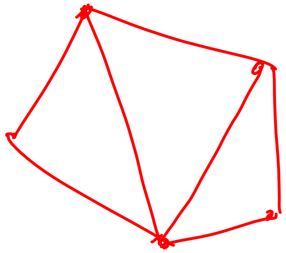
$$\frac{1}{s_{12}s_{123}}+\frac{1}{s_{23}s_{234}}+\frac{1}{s_{34}s_{345}}+\frac{1}{s_{45}s_{451}}+\frac{1}{s_{51}s_{512}}$$



Dual
Assoc.
Feynman Diag.

$$\frac{s_{12}+s_{234}}{s_{12}s_{34}s_{234}}+\frac{s_{12}+s_{234}}{s_{12}s_{234}s_{23}}+\frac{s_{12}-s_{123}+s_{23}}{s_{12}s_{23}s_{123}}$$

No sp poles @ ∞

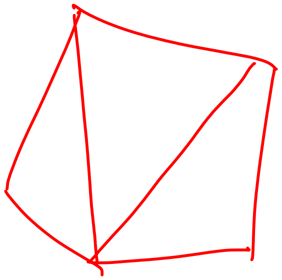


Dual
Assoc.
proj. manifest

$$\frac{(X_{1,3}+X_{2,5})(X_{1,4}+X_{3,5})d^2X}{X_{1,3}X_{3,5}(X_{1,4}X_{2,5}-X_{1,3}X_{3,5})}$$

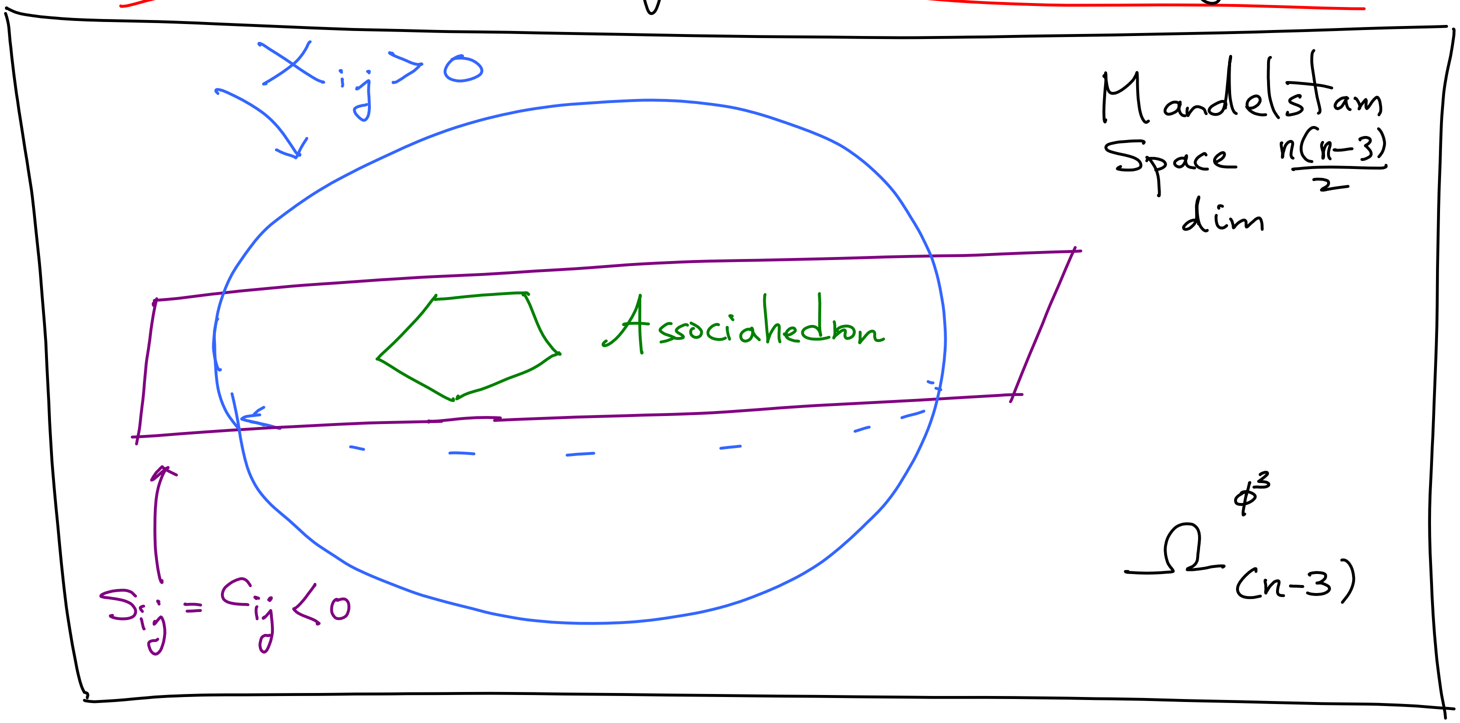
$$\frac{(X_{1,3}+X_{2,5})^2(X_{2,4}-X_{2,5}+X_{3,5})d^2X}{X_{2,5}(-X_{1,4}X_{2,5}-X_{1,3}X_{2,4}+X_{1,3}X_{2,5})(X_{1,4}X_{2,5}-X_{1,3}X_{3,5})}$$

$$\frac{(X_{1,3}-X_{1,4}+X_{2,4})(-X_{2,4}+X_{1,4}+X_{2,5})d^2X}{X_{1,4}X_{2,4}(-X_{1,4}X_{2,5}-X_{1,3}X_{2,4}+X_{1,3}X_{2,5})}$$



Assoc.
proj + "soft"
limit manifest

Bi-Adjoint $(\Phi_{aA})^3$ theory



Associahedron is "Amplituhedron" of Φ^3_{aA} theory

$\mathcal{N}=4$ SYM

$\mathcal{P}$: Config. of $\{Z_1, \ldots, Z_n\}$ has fixed "binary code"

$\Rightarrow$ physical poles > 0

+ maximal "winding #"

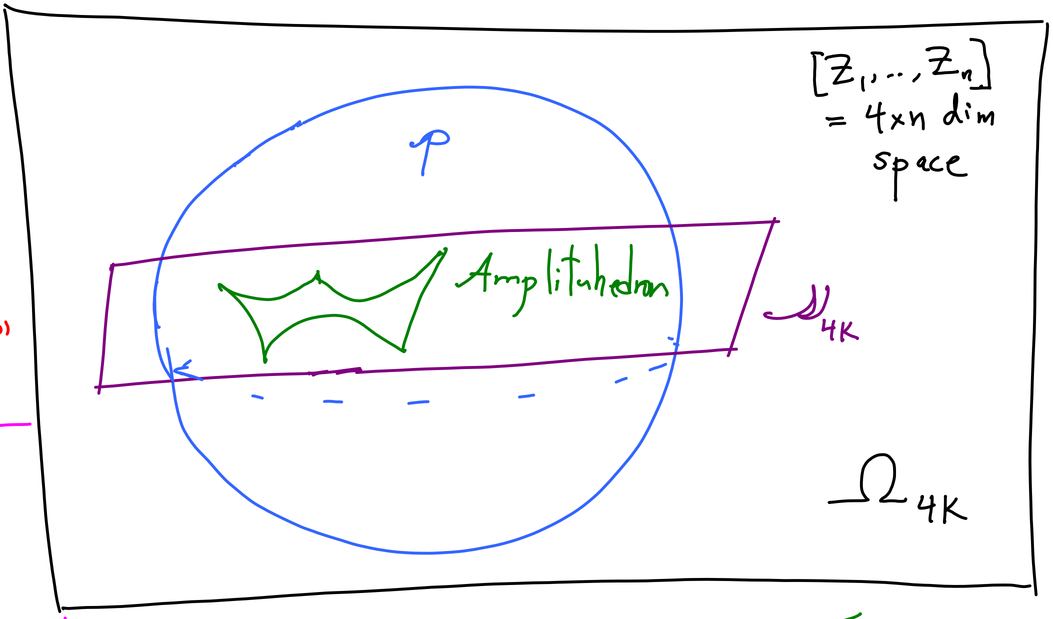
In full:

$\langle i\, i{+}1\, j\, j{+}1 \rangle > 0$

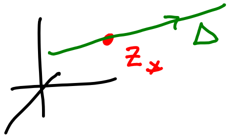
$\{\langle 1234 \rangle, \ldots, \langle 123n \rangle\}$ has k sgn flips

$\langle AB_\alpha\, i\, i{+}1 \rangle > 0$, $\langle AB_\alpha AB_\beta \rangle > 0$

$\{\langle AB_\alpha 12 \rangle, \ldots, \langle AB_\alpha 1n \rangle\}$ has k+2 sgn flips



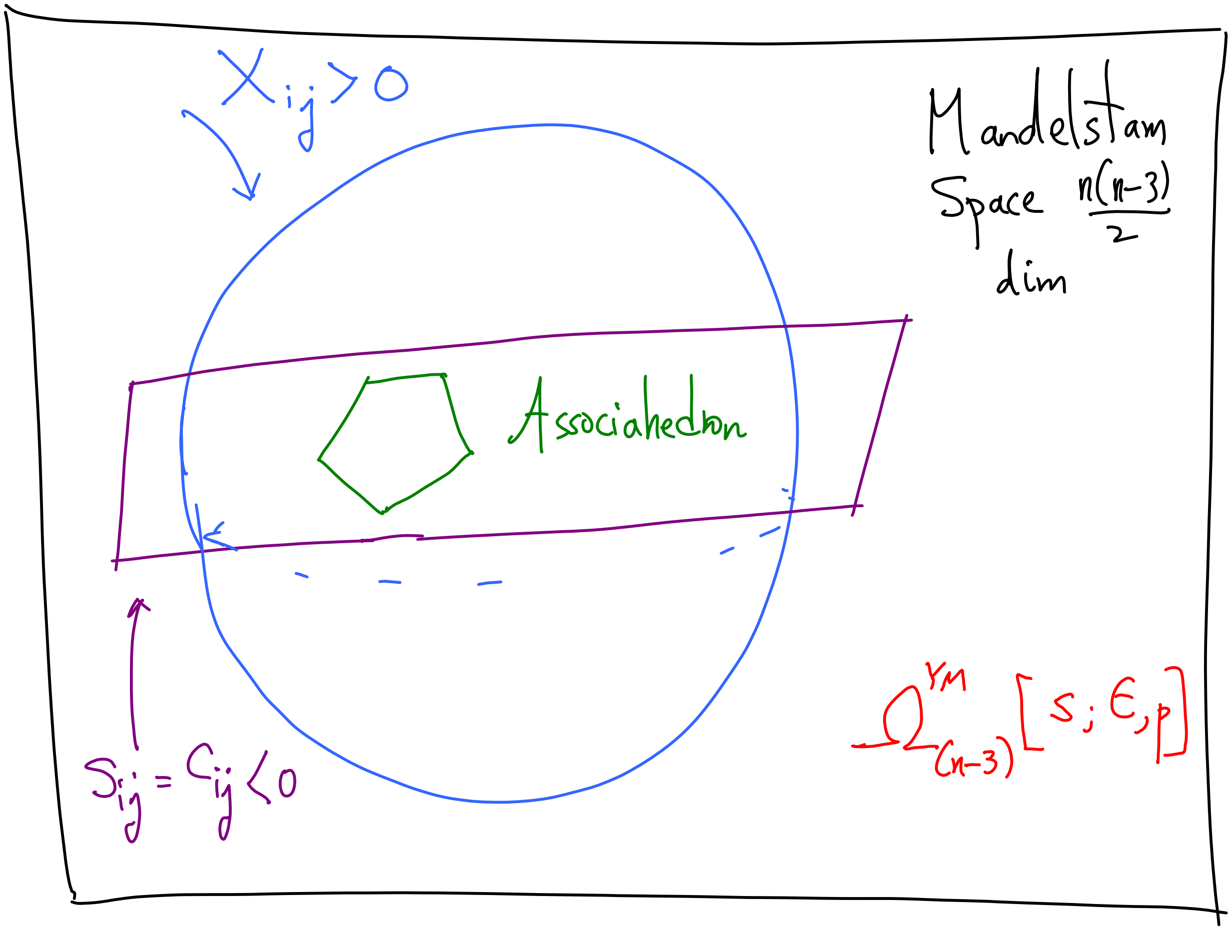
$\mathcal{S}_{4K}$: Affine subspace



$$Z^I_a = Z^I_{*a} + y^I_\alpha \Delta^\alpha_a$$

$$\begin{pmatrix} Z_* \\ \hline \Delta \end{pmatrix} \cap G_+(4+K, n)$$

Gluons



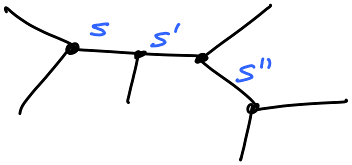
- $(-1)^P$ under Perm.
- On-shell GI
- Minimal pole @ ∞

$\downarrow$

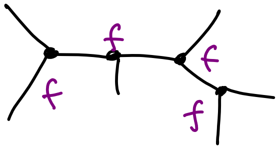
Ω^{YM} is Unique!

{non-trivial!}

Color IS Kinematics (Form)!



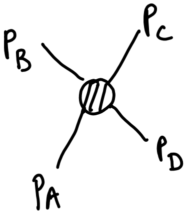
$\rho \equiv ds \wedge ds' \wedge ds''$



$f\,f\,f\,f$

Exactly the same Algebraic Relations

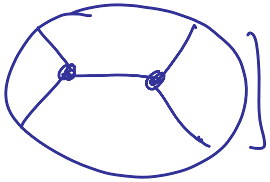
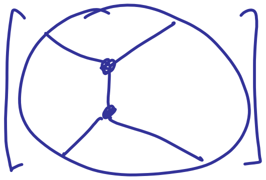
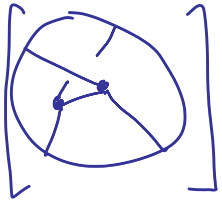
Why?



$$(p_A+p_B)^2+(p_A+p_C)^2+(p_A+p_D)^2 = p_A^2+p_B^2+p_C^2+p_D^2$$

So $\left[d(p_A+p_B)^2 + d(p_A+p_C)^2 + d(p_A+p_D)^2\right]$

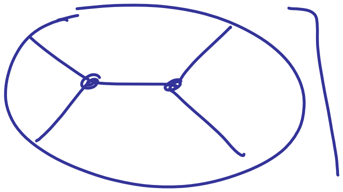
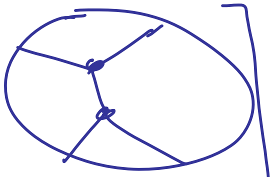
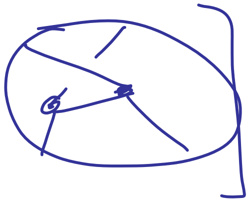
$\wedge\ dp_A^2\, dp_B^2\, dp_C^2\, dp_D^2$

Hence $\rho\Big[$  $\Big] + \rho\Big[$  $\Big] + \rho\Big[$  $\Big] = 0$!

Projective Avatar of Color-Kin

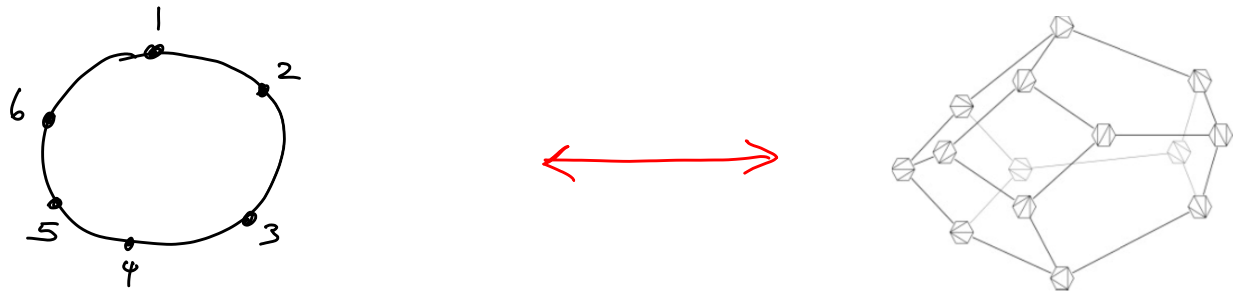
$$\Omega = \sum_{\substack{\text{all cubic}\\ \text{graphs}}} c_\Gamma \prod_{\text{prop in }\Gamma} \left(\frac{dS}{S}\right)$$

Ω well-defined projectively $\Longrightarrow$

$c\,[$  $] + c\,[$  $] + c\,[$  $] = 0\,!$

Also: $\Omega^{\text{proj}} = \sum_{\text{orderings}} c(\pi)\,\Omega[\pi_1, \ldots, \pi_n]$

Worldsheet Associahedron

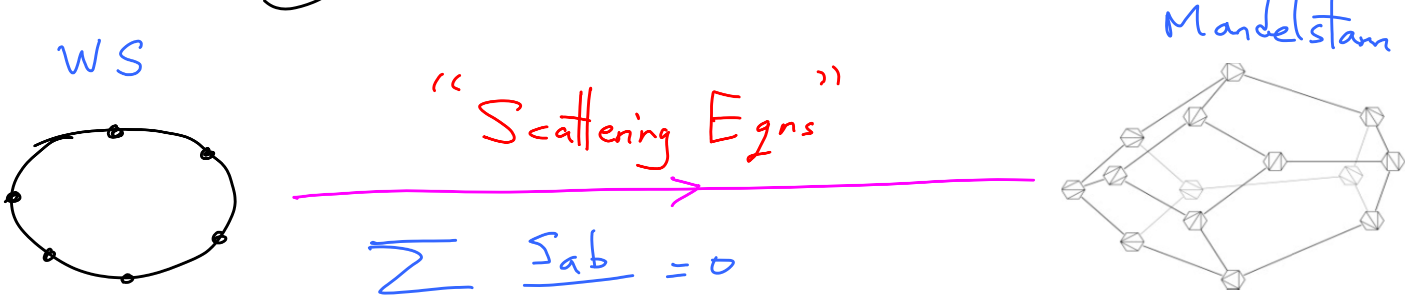


$$\left.{\scriptstyle 2}\updownarrow\underbrace{\begin{pmatrix}\sigma_1 & \cdots & \sigma_n\end{pmatrix}}_{\longleftarrow\, n\, \longrightarrow}\right/ \begin{matrix}GL(1)^n \\ \times SL(2)\end{matrix}$$

$$\Omega^{WS}_{(n-3)} = \left.\frac{d^2\sigma_1 \cdots d^2\sigma_n}{(12)(23)\cdots(n1)}\right/ \begin{matrix}GL(1)^n \\ \times SL(2)\end{matrix}$$

$\langle \sigma_a \sigma_b \rangle > 0 \quad a < b$

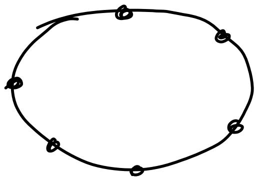
Pushing Forward From Worldsheet



$$\sum_{b\neq a}\frac{s_{ab}}{\sigma_a-\sigma_b}=0$$

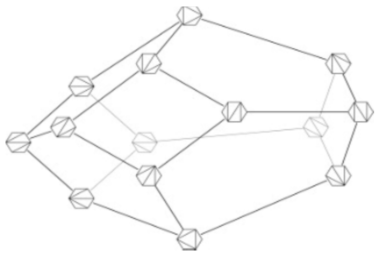
On $(n-3)$ mandelstam subspace ⇢ solve for s's in terms of σ's → Mandelstam associahedron is literally image of WS! Conversely: single real solution for σ's in WS assoc, iff s_{ab} live in some Mandelstam assoc.

Pushing Forward From Worldsheet



Scattering Eqns

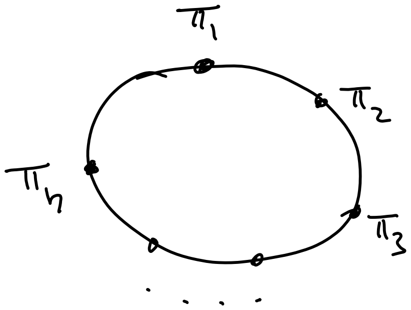
$$\sum_{b\neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0$$



$$\begin{tikzcd}\Omega^{WS}_{(n-3)} \arrow[r, "\text{Push forward}"] & \Omega^{(\phi_{aA})^3}_{(n-3)}\end{tikzcd}$$

= CHY formula

Gluons from Worldsheet



$$\tilde{\Omega}^{WS}_{(n-3)} = \sum_{\pi} \Omega^{WS}_{(n-3)}[\pi_1, \ldots, \pi_n]\, \mathcal{N}(\epsilon, p)$$

Push forward on SE $\Big\downarrow$ Demand minimal # p + On-shell Gauge Inv.

$\tilde{\Omega}^{WS}_{(n-3)}$ is Unique! + Pushes forward to $\Omega^{YM}_{(n-3)}$

Gluons ⟶ Open Strings

Q: Why not integrate form on WS, instead of pushing forward? A: it would be log divergent!

Canonical way of dealing w/ this (c.f. Gelfand)

$$\Omega^{WS}_{\alpha'} = \Omega^{WS} \times \underbrace{\prod_{(ab)} (\sigma_a \sigma_b)^{\alpha' s_{ab}}}_{\text{Koba-Nielsen}} \qquad \left\{ \sum_{b \neq a} s_{ab} = 0 \text{ for } SL(2) \text{ weight} \right\}$$

$$(\alpha')^{n-3} \int_{WS} \Omega^{WS}_{\alpha'} = \text{Superstring Gluon Amps!}$$

The Canvas

D-dim kinem.
space

- Null momenta..
- Twistors ....

} Amplitudes

- Spatial Momenta for Ψ_{Univ}

WHAT
BREATHES
PHYSICS
LIFE
INTO
THIS SPACE?

In Our Examples

Combinatorics ↔ Positive Geometries ↔ Canonial Forms

Emergent

Space-Time + QM

(Hand in Hand)