Geometry of Surface group representations

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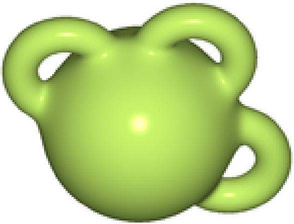
Topological Classification of Surfaces



Theorem (Riemann 1851, Moebius 1863, Jordan 1866, Poincaré 1882, Klein 1882): Any closed orientable surface is homeomorphic to a sphere with g handles for some non-negative integer g.
$g =$genus.
Proof: Dehn and Heegaard (1907).

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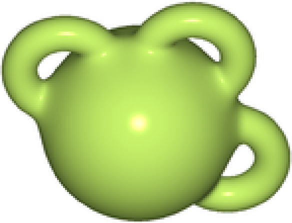

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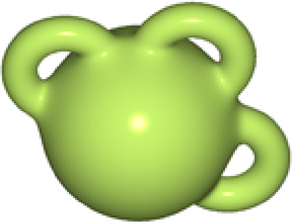

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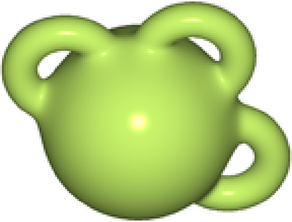

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Geometric Structures on Surfaces

1. Differential Geometry: Constant curvature metrics: +1 ($g = 0$), 0 ($g = 1$), -1 ($g \geq 2$).
2. Complex Geometry: Riemann surfaces : transition functions complex analytic.
3. Algebraic Geometry: Solution sets to algebraic equations: Varieties in $\mathbb{C}P^n$.

Uniformization theorem establishes a dictionary between these structures.

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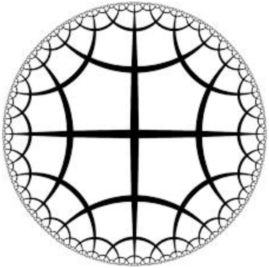
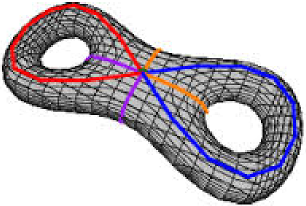
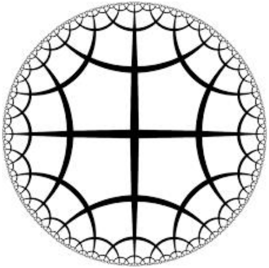
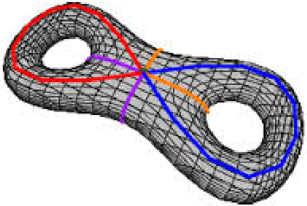




Constant curvature -1 metric on closed surface S of genus at least 2 $\Leftrightarrow$ **discrete faithful** representation $\rho : \pi_1(S) \to PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\} = Isom^+(\mathbf{H}^2)$





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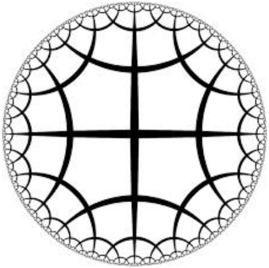

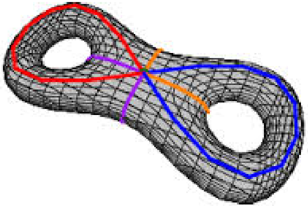

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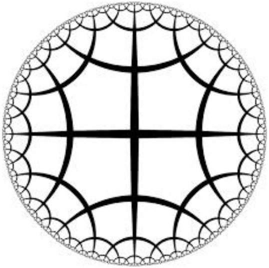

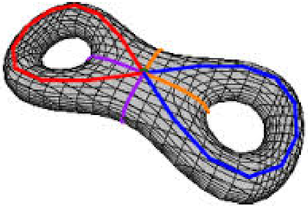

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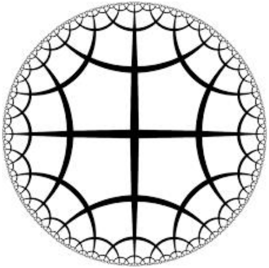

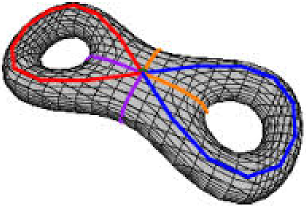

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Space of such structures: **Teichmüller space/Moduli space**

1. Differential Geometry: Isometry classes of constant curvature -1 metrics with topological marking.
2. Complex Geometry: Conformal/complex analytic isomorphism classes of Riemann surfaces with topological marking.
3. Algebraic Geometry: Moduli space: Smooth projective algebraic curves up to (algebraic) isomorphism.
4. Representations: Discrete faithful representations $\rho : \pi_1(S) \to PSL(2, \mathbb{R}) / \sim$, where $\sim$ means "up to (global) conjugation by an element of $PSL(2, \mathbb{R})$".

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$PSL(2,\mathbb{C}) = Isom^+(\mathbf{H}^3)$

Look at space of discrete faithful $\rho : \pi_1(S) \to PSL(2,\mathbb{C})$ equipped with the usual (algebraic) topology of (pointwise) convergence. Denote as $AH(S)$ – analog of Teichmüller space. Let $\Gamma = \rho(\pi_1(S))$ – Kleinian surface group.

Theorem

(Thurston-Bonahon): *$M = \mathbf{H}^3/\Gamma$ is homeomorphic to a product $S \times \mathbb{R}$.*

But geometrically, a lot of variety. So 3-dimensional analog of Teichmüller theory becomes the study of hyperbolic structures on $S \times \mathbb{R}$ up to isometry.

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Broadly, 3d hyperbolic geometric structures are of two kinds.

1. Quasi-Fuchsian/ Convex cocompact/ undistorted/ quasi-isometrically (qi) embedded:
   Distances in $\mathbf{H}^2$ (denote d_2) and $\mathbf{H}^3$ (denote d_3) are linearly comparable: There exist (k, ϵ) such that
   $\frac{1}{k} d_2(g.o, h.o) - \epsilon \leq d_3(\rho(g).O, \rho(h).O) \leq k\, d_2(g.o, h.o) + \epsilon$.
   $QF(S) = Teich(S) \times Teich(S)$ **(Bers' simultaneous uniformization theorem, 1960)**
2. Limits of these in the algebraic topology. **(Bers' density conjecture proved by Brock-Canary-Minsky, 2012)**

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1) Representation $\Gamma = \rho(\pi_1(S))$ *is Anosov,*
2) qi-embedded orbits $\Gamma.O \subset G/K$ *(the associated symmetric space) in a strong sense,*
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Other non-compact semisimple Lie groups G: Higher Teichmüller theory

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1. *Other (non-discrete, non-faithful) representations: Geometric understanding?* Goldman's work
2. *Analog of elements of limiting representations in $AH(S) \setminus QF(S)$?*
3. *What if asymptotic embedding $\partial i : S^1 \to G/B$ is replaced by requiring only a continuous (higher Cannon-Thurston?) map $\partial i : S^1 \to G/B$?*

Unifying framework–Higgs bundles (Hitchin-Simpson-Corlette-Donaldson):

1. *Algebraic geometry: vector bundles with Higgs field;*
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