Black Holes II

G. Srinivasan

Schwarzschild metric

- Schwarzschild's solution of Einstein's equations for the gravitational field describes the curvature of space and time near a spherically symmetric massive body.

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)} dr^2 - r^2 \left(\sin^2\theta \, d\phi^2 + d\theta^2\right)$$

Inside the horizon

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)}dr^2 - r^2\left(\sin^2\theta d\phi^2 + d\theta^2\right)$$

- The Schw. Metric cannot be extended inside the horizon.

$$\left(1 - \frac{2GM}{rc^2}\right) < 0$$

- The dt^2 term becomes NEGATIVE, and dr^2 term becomes POSITIVE.
- t-coordinate becomes SPACELIKE. r-cord becomes TIMELIKE.
- r = constant line cannot serve as the ‘radial coordinate’.
- **The only TIMELIKE trajectories are those with decreasing r.**

The Lemaitre coordinates (1933)

Choose co-ords { R, τ, θ, φ } $\quad r = \left(\frac{3}{2}(R - c\tau)\right)^{\frac{2}{3}} r_g^{\frac{1}{3}}$

$$ds^2 = c^2 d\tau^2 - \frac{dR^2}{\left[\frac{3}{2r_g}(R - c\tau)\right]^{\frac{2}{3}}} - \left[\frac{3}{2}(R - c\tau)\right]^{\frac{4}{3}} r_g^{\frac{2}{3}} (d\theta^2 + \sin^2\theta d\phi^2)$$

In these coords, there is no singularity at r_g : $\quad r_g = \frac{3}{2}(R - c\tau)$

τ is timelike and R is spacelike everywhere.

The metric is not stationary; τ enters explicitly.

The Lemaitre coordinates (1933)

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In these coordinates, $r_g = \frac{3}{2}(R - c\tau)$

All metric coefficients are finite.

The reference frame can be extended to r = 0, that is R = cτ.

The Lemaitre coordinates (1933)

$$ds^2 = c^2 d\tau^2 - \frac{dR^2}{\left[\frac{3}{2r_g}(R - c\tau)\right]^{\frac{2}{3}}} - \left[\frac{3}{2}(R - c\tau)\right]^{\frac{4}{3}} r_g^{\frac{2}{3}} (d\theta^2 + \sin^2\theta\, d\phi^2)$$

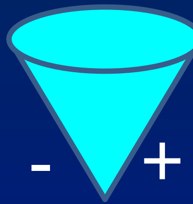
At $R = c\tau$ ($r=0$), there is a true spacetime singularity – infinite curvature.

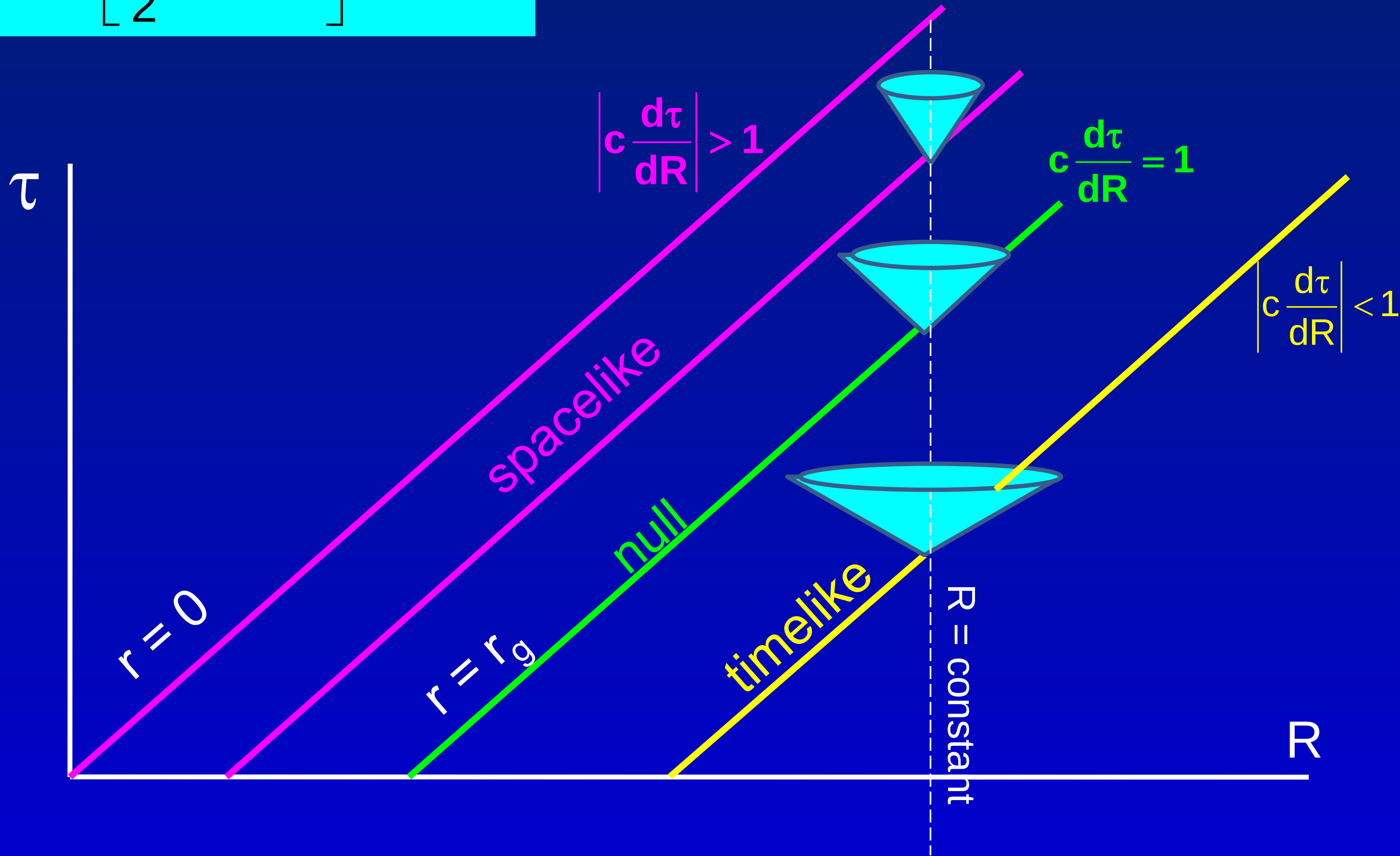
Each falling body with R=constant moves in time τ to smaller r. The particle reaches r_g over time τ, keeps falling, and reaches the singularity at r=0.

World lines of radial light rays can be found from ds=0, $d\theta=0$, $d\phi=0$.

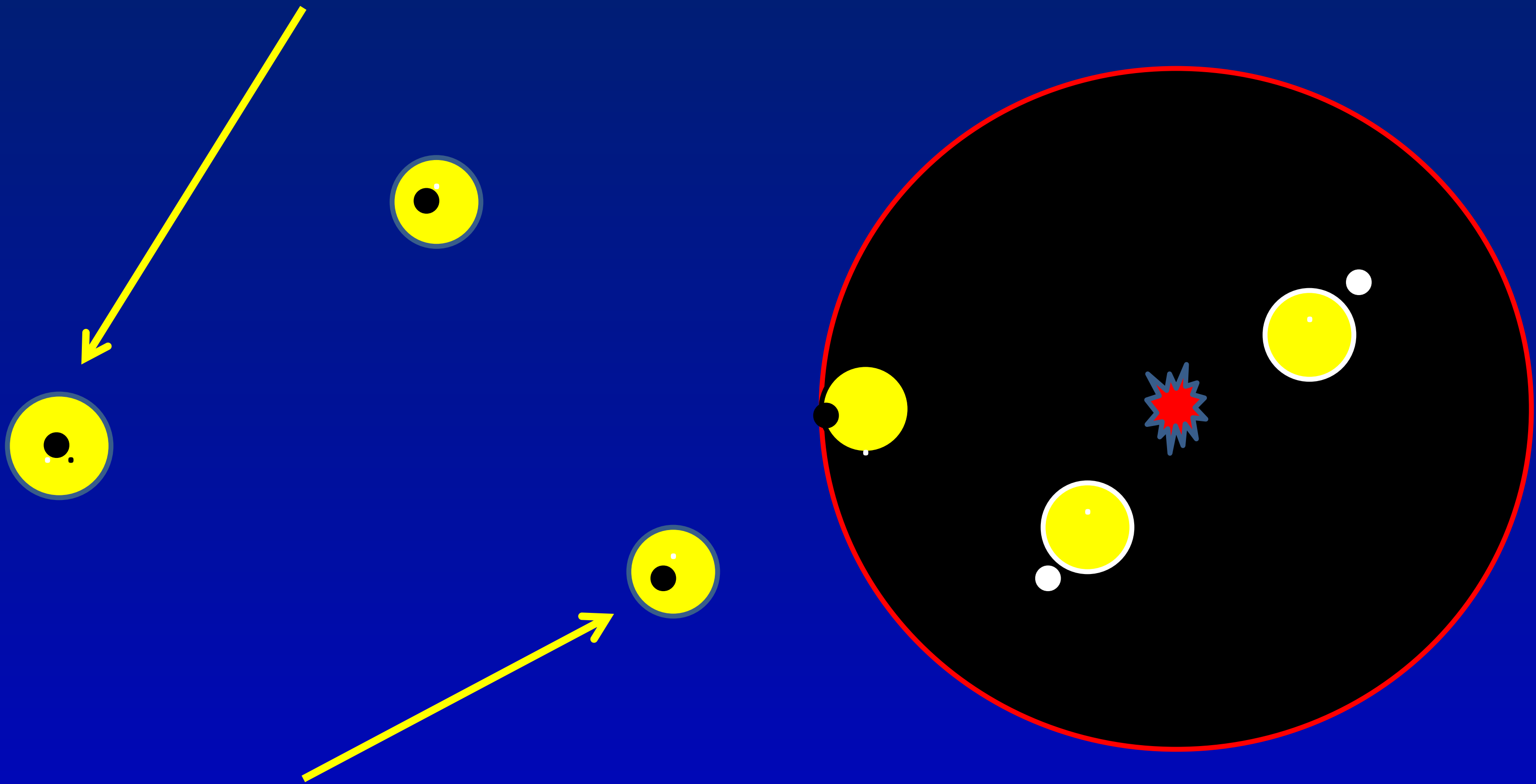
$$c\frac{d\tau}{dR} = \pm\left[\frac{r_g}{\frac{3}{2}(R-c\tau)}\right]^{\frac{1}{3}} = \pm\sqrt{\frac{r_g}{r}}$$

Light signals:

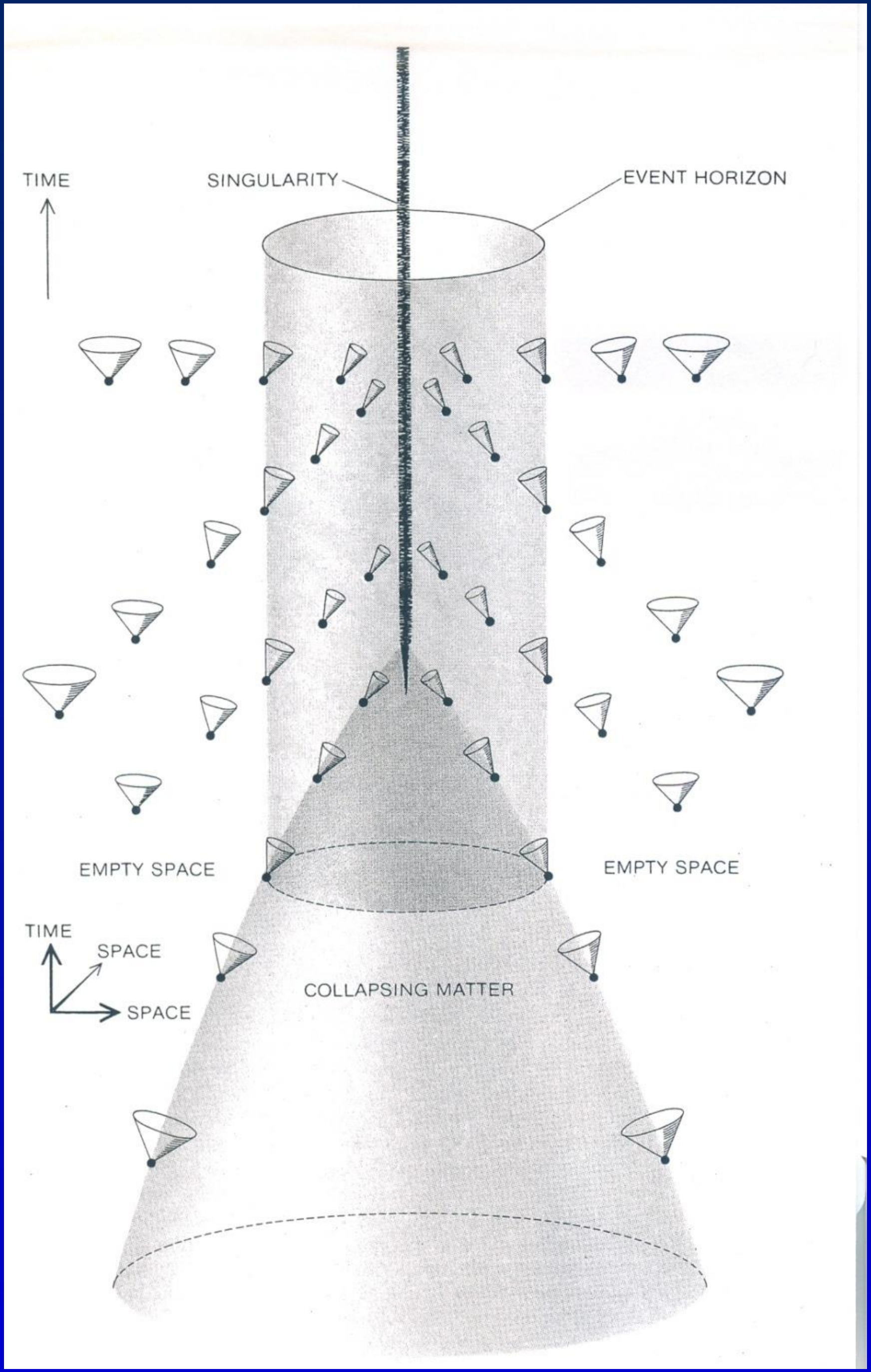




At large distances, the point of emission lies at the centre of the expanding spherical wavefront surface.

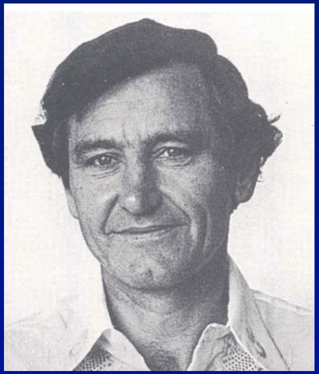


At short distances from the event horizon, the wavefront surface is displaced by the strong gravity. At precisely the horizon, the spherical wavefront surface touches the horizon INTERNALLY. Inside, the expanding wavefront detaches itself from the point of emission. Light emitted in any direction is pulled towards the central singularity.



Roger Penrose

Rotating Black Holes



In 1964, a New Zealand mathematician, Roy Kerr, discovered an exact solution for the geometry of spacetime around a rotating star. This was the second exact solution after Karl Schwarzschild's discovery in 1916.

The Kerr metric describes every conceivable rotating BH!

Let us introduce the "angular momentum parameter" 'a',

$$a = \frac{J}{M}$$

where J is the angular momentum. **The maximum value of 'a' is 'M'**. Such a BH is referred to as the "extreme Kerr BH".

The Kerr Metric

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \frac{1}{\left(1 - \frac{2GM}{rc^2} + \frac{a^2}{r^2}\right)}dr^2 - \{\ldots\}r^2d\phi^2 + \{\ldots\}dtd\phi$$

Equatorial plane

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)}dr^2 - r^2\left(\sin^2\theta d\phi^2 + d\theta^2\right)$$

Schwarzschild metric

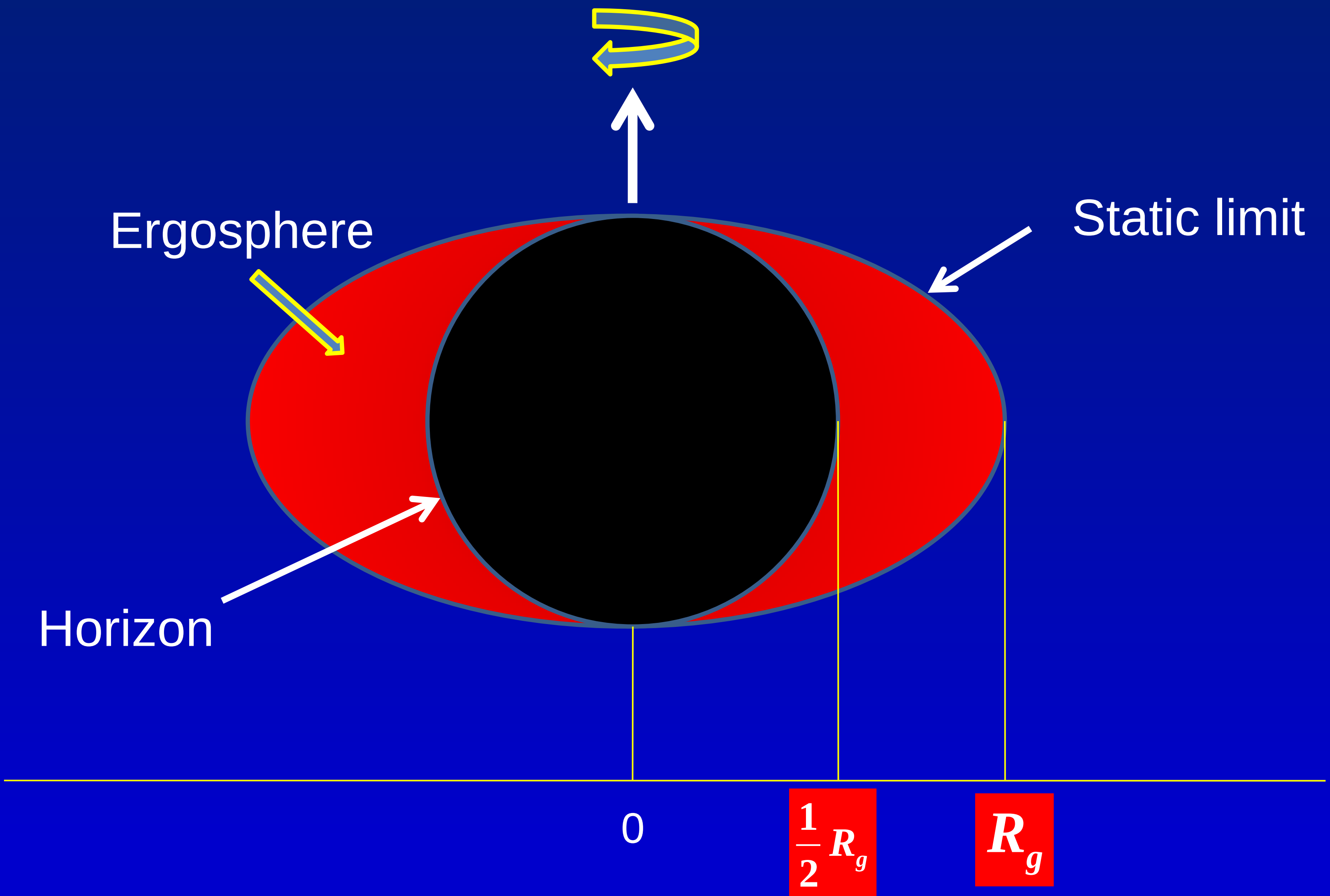
Schwarzschild metric

- At the HORIZON, the coefficient of the dr^2 term $\to \infty$.

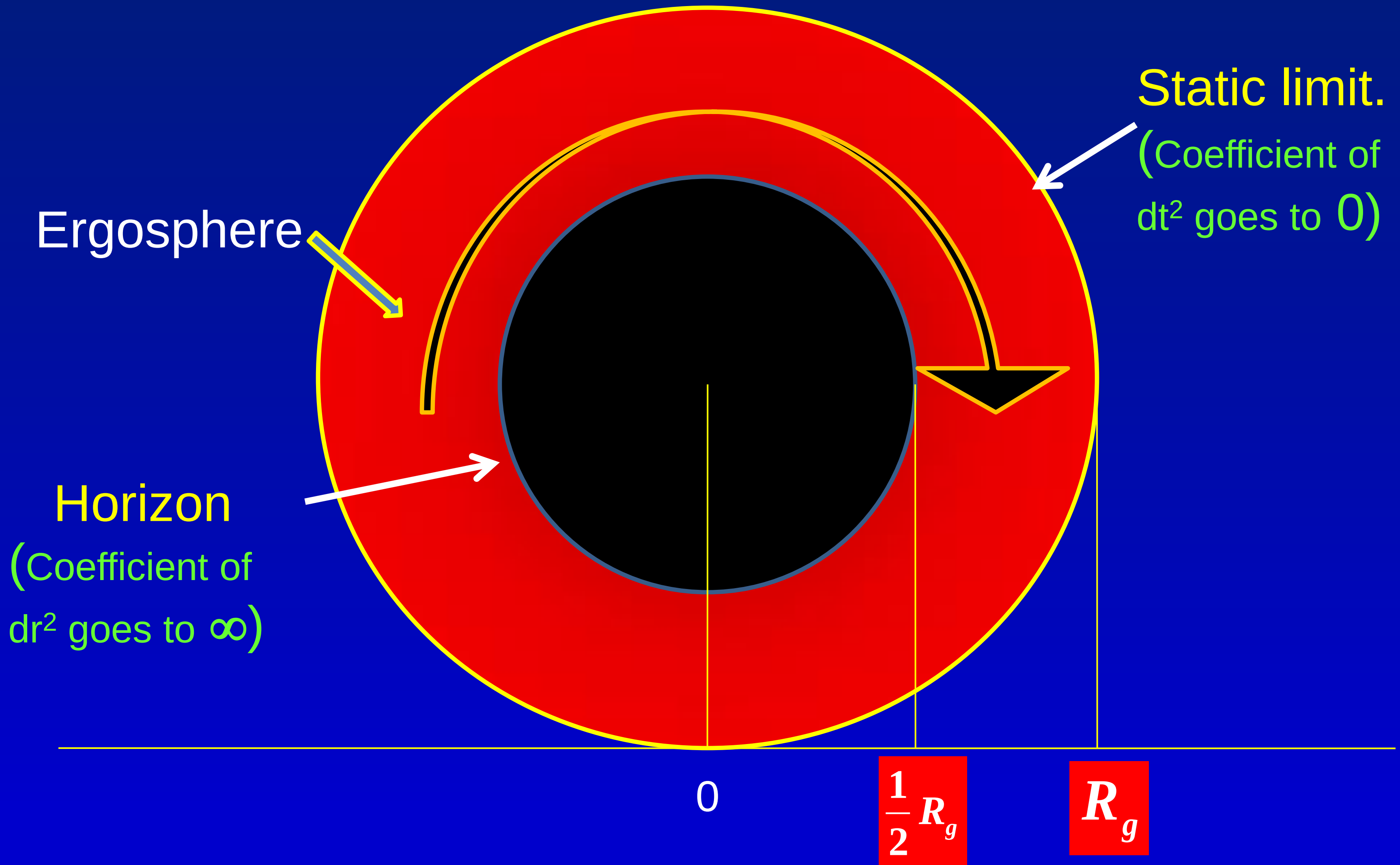
$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)} dr^2 - r^2 \left(\sin^2 \theta d\phi^2 + d\theta^2\right)$$

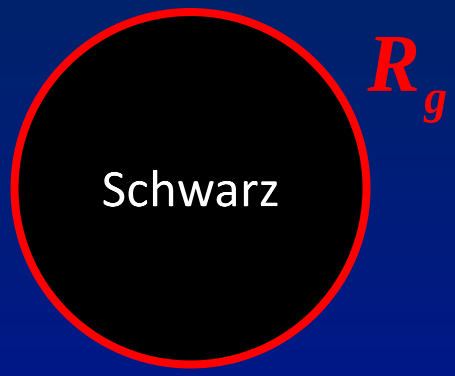
- The coefficient of the dt^2 term goes to ZERO at the same radius. This is NOT so for the Kerr metric.

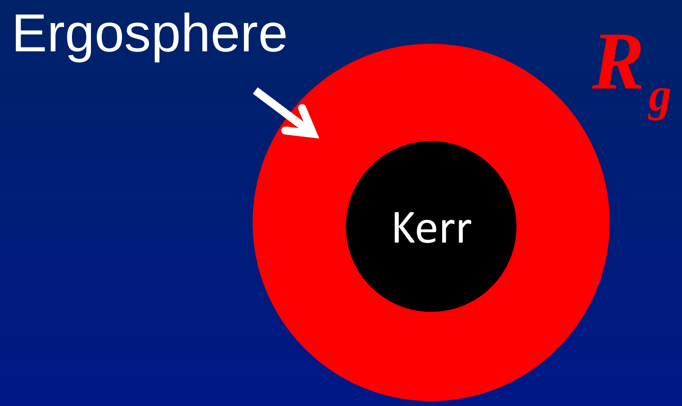
Maximally Rotating Black Hole



Extreme Kerr Black Hole: a=M

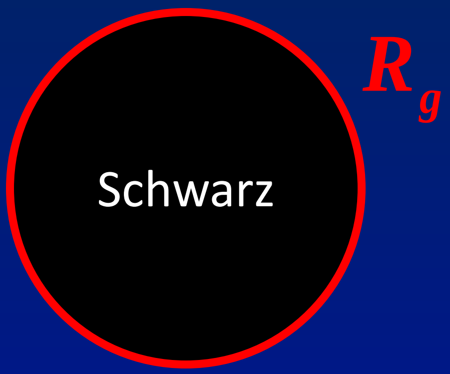






- Inside R_g particles (and light) cannot be at rest w.r.t. infinity. r = constant not allowed. Particles (and light) have to move radially towards the centre.

- As seen by a distant observer, it will take an infinite time for the particle to reach R_g , which is the EVENT HORIZON.



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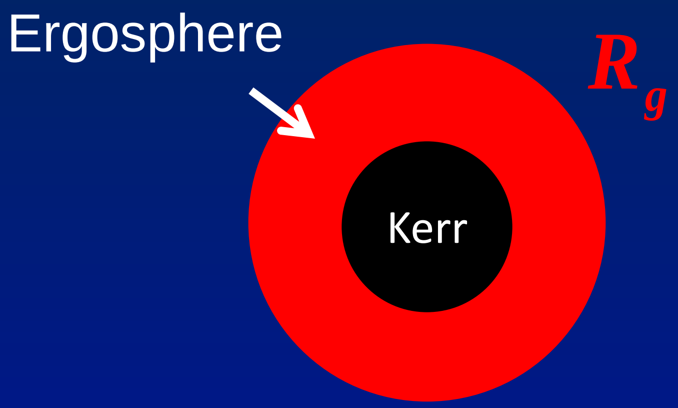
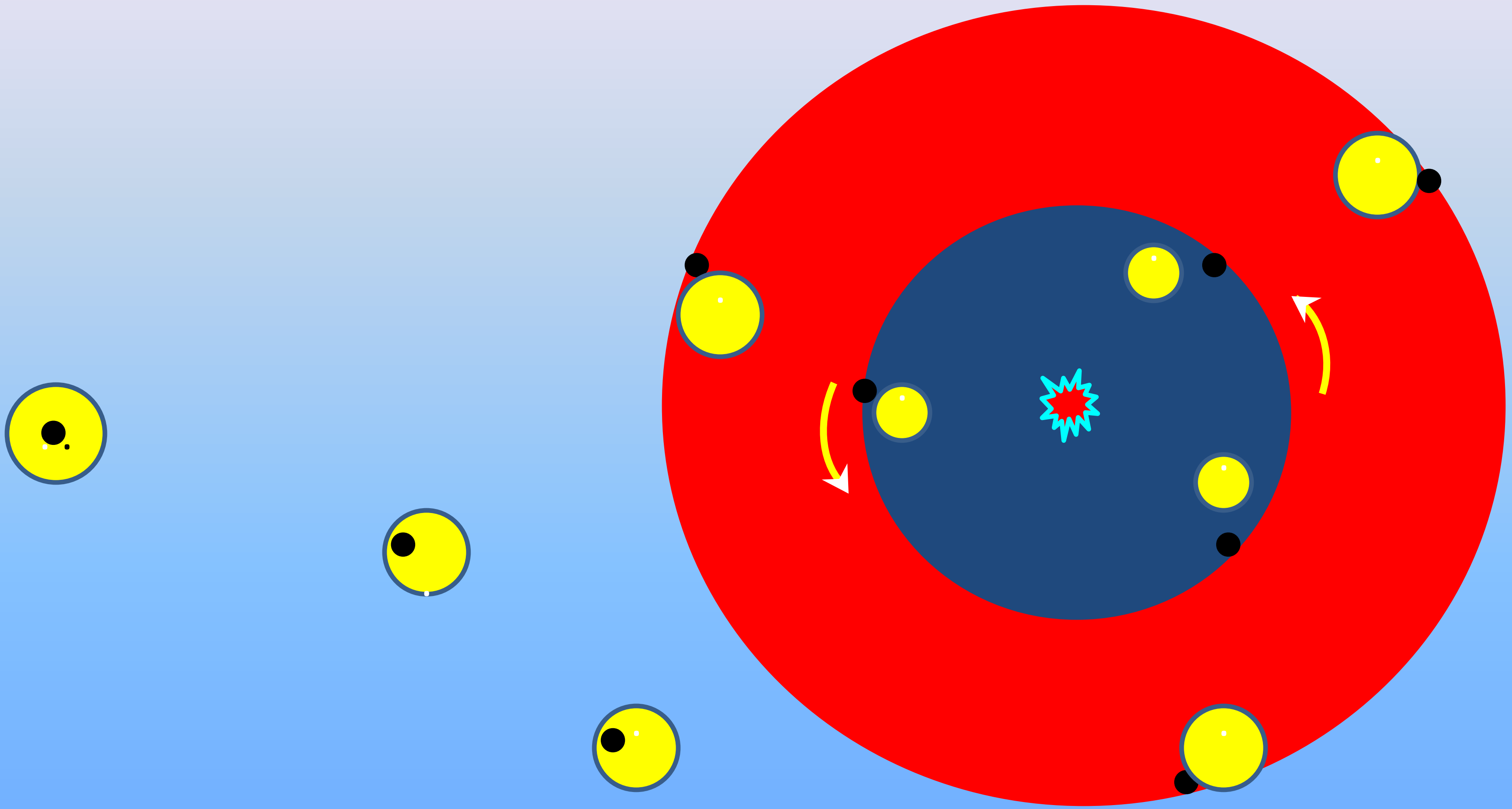
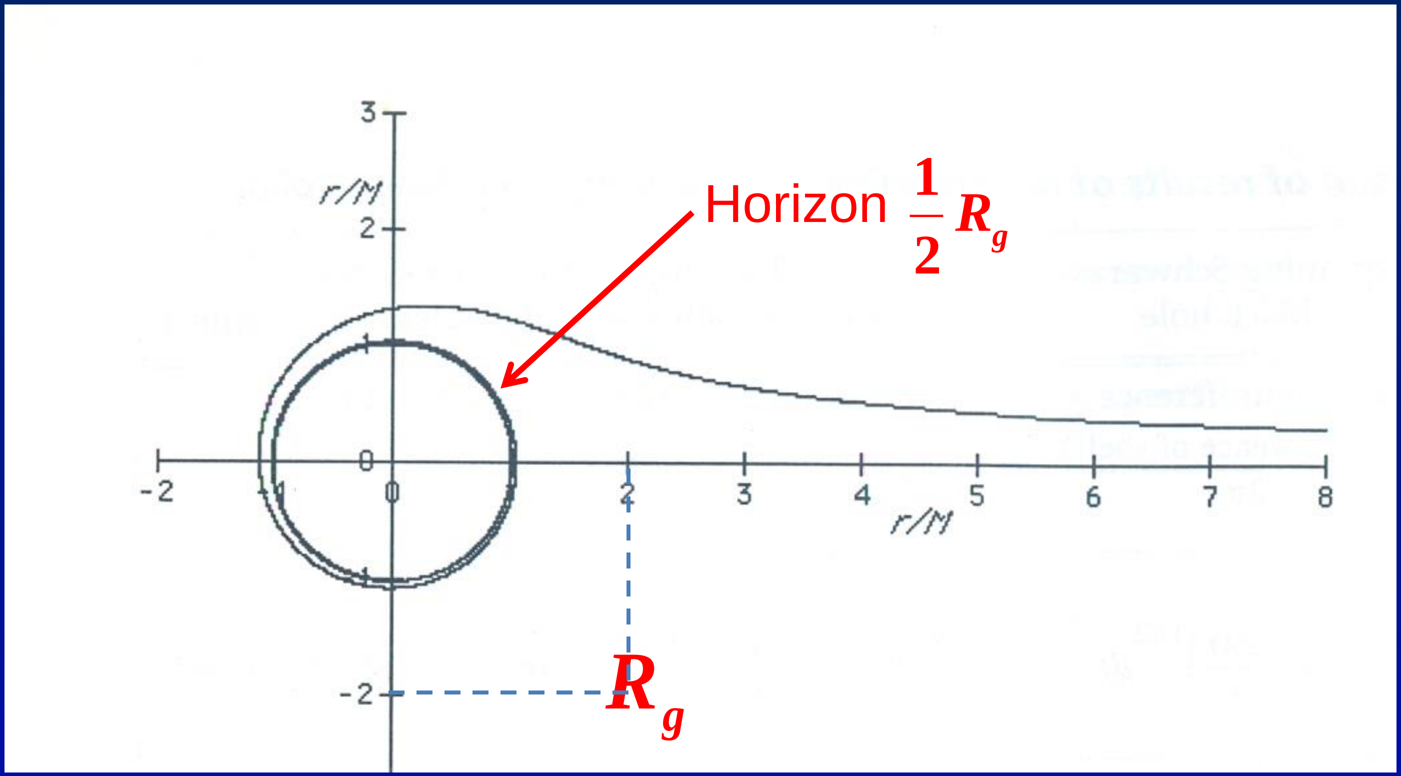


- Inside the Ergosphere, **ϕ = constant** is NOT possible.
- Particles (and light) must rotate around the axis of symmetry.
- r = const. is possible for particles (and light). They can also move with decreasing or increasing r.
- They can emerge from Ergosphere into external space!
- Particles can reach the Ergosphere in a finite time – EXCEPT THE POLES.
- Time to reach the HORIZON is, again, infinite.

The shearing of the expanding wavefront due to dragging of inertial frames around a rotating black hole.

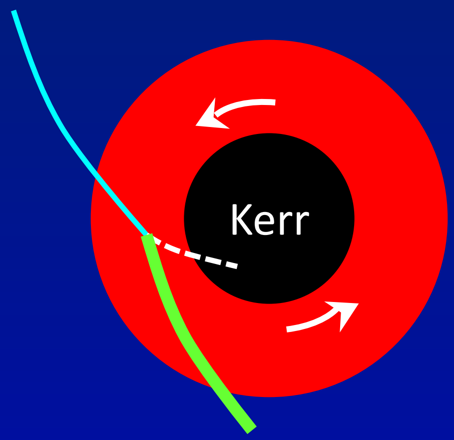




Computer plot of the trajectory in space of a stone dropped from rest far from a Kerr black hole. Initially, the stone has zero angular momentum. But a far away observer will see the stone SPIRAL IN to the horizon at r=M and circulate there for ever.

Here is an example of angular motion without angular momentum !

Extraction of energy – The Penrose process



- **Negative energy states can exist in the Ergosphere.**
- A particle with positive energy enters the Ergosphere, and SPLITS into two particles.
- One has negative energy, and the other positive energy.
- Negative energy particle falls into Horizon.
- **Positive energy particle comes out with larger energy.**

What is inside a black hole?

There is nothing
inside a Black Hole!

What is inside a black hole?

"The star thus tends to close itself off from any communication with a distant observer; only its gravitational field persists."

Oppenheimer and Snyder (1939)

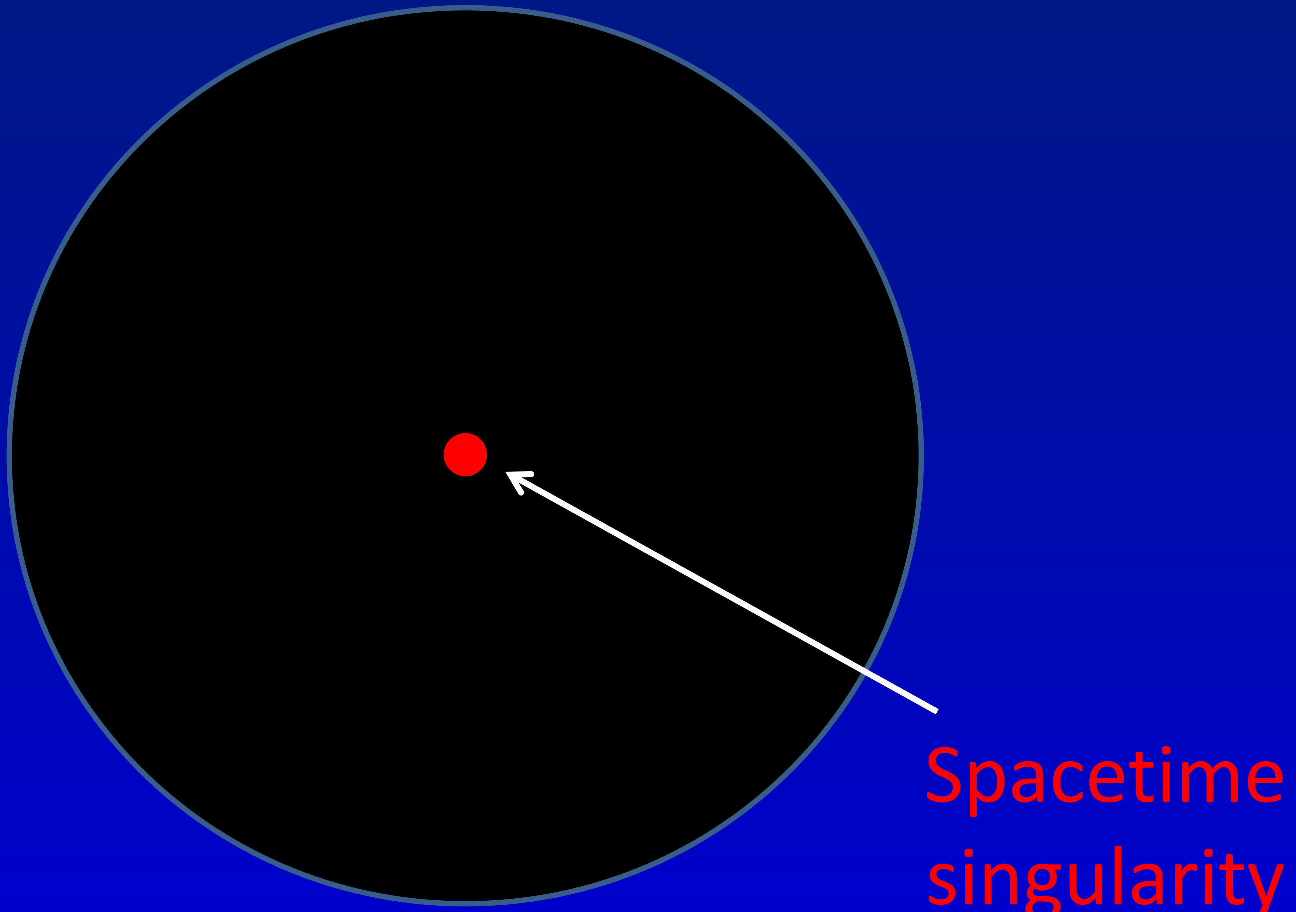
- Once a star collapses to the critical radius, it cannot arrest its collapse.
- According to General relativity, it will continue to collapse till it becomes of zero radius!

- Therefore, there is NOTHING inside a black hole!

What is inside a black hole?

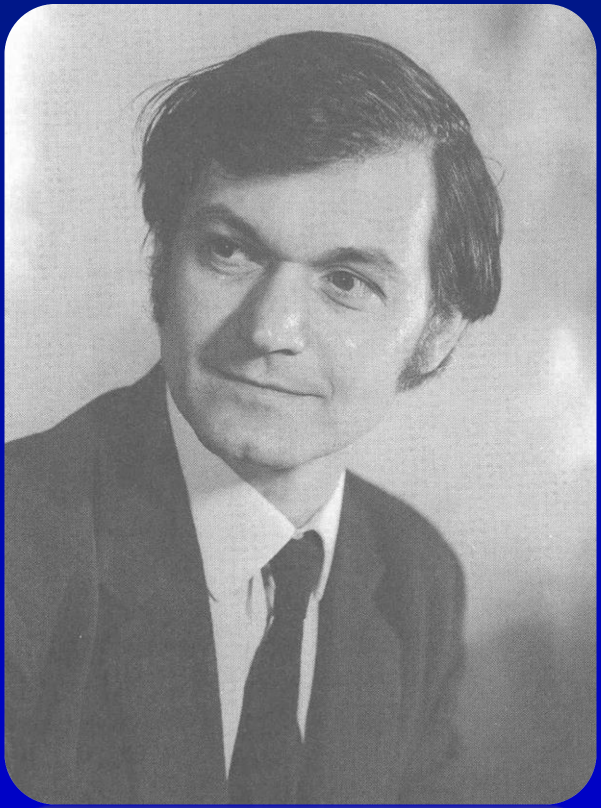
After creating the black hole, the spherical star continues imploding to infinite density and zero volume, whereupon it creates and merges into a space-time singularity.

What is inside a black hole?



The Golden Age of Black Holes

Roger Penrose



Singularity Theorem

Every black hole must have a singularity inside it.

Roger Penrose,
1964

Singularity Theorem

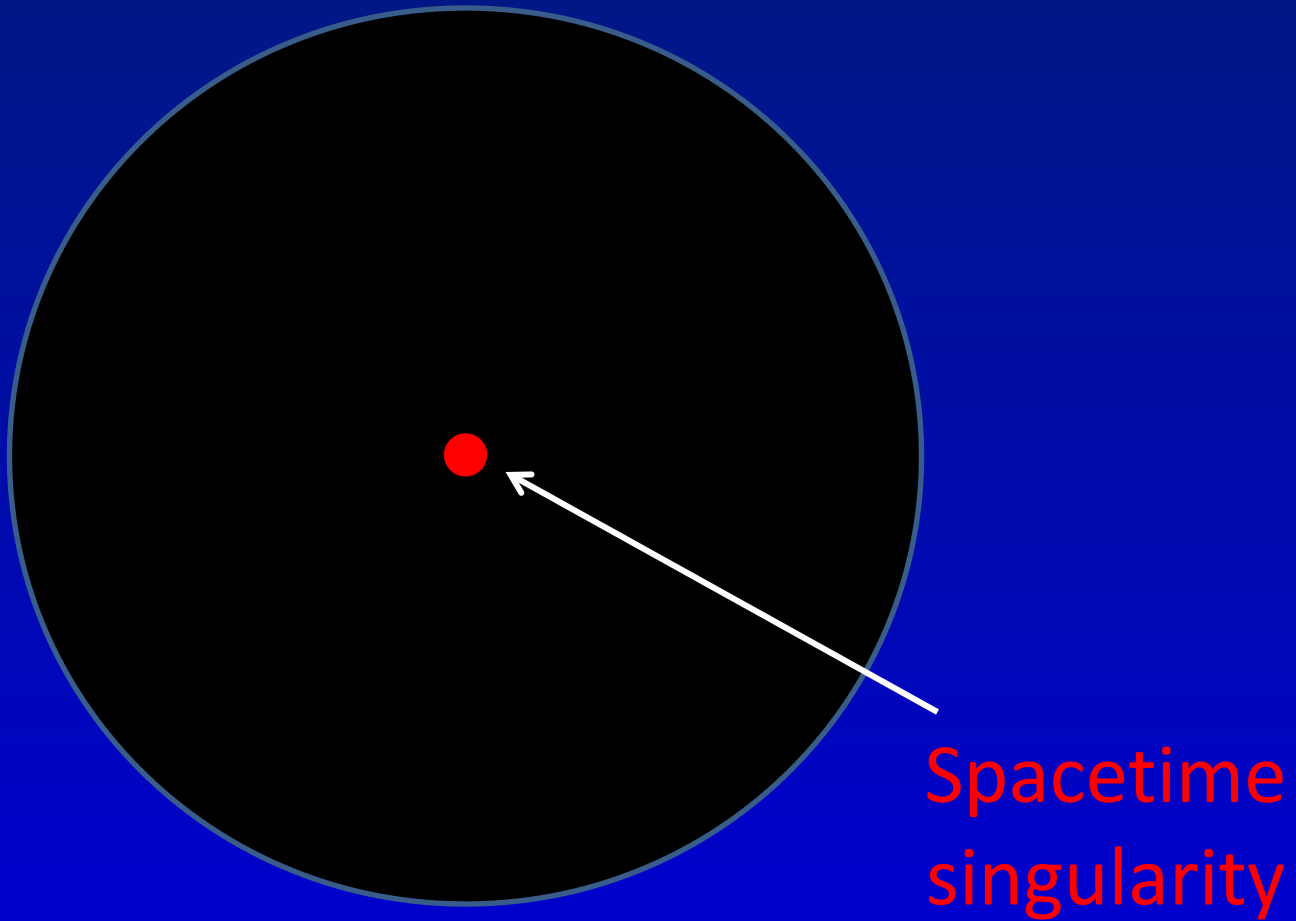
Roger Penrose, 1964

Every black hole must have a singularity inside it.

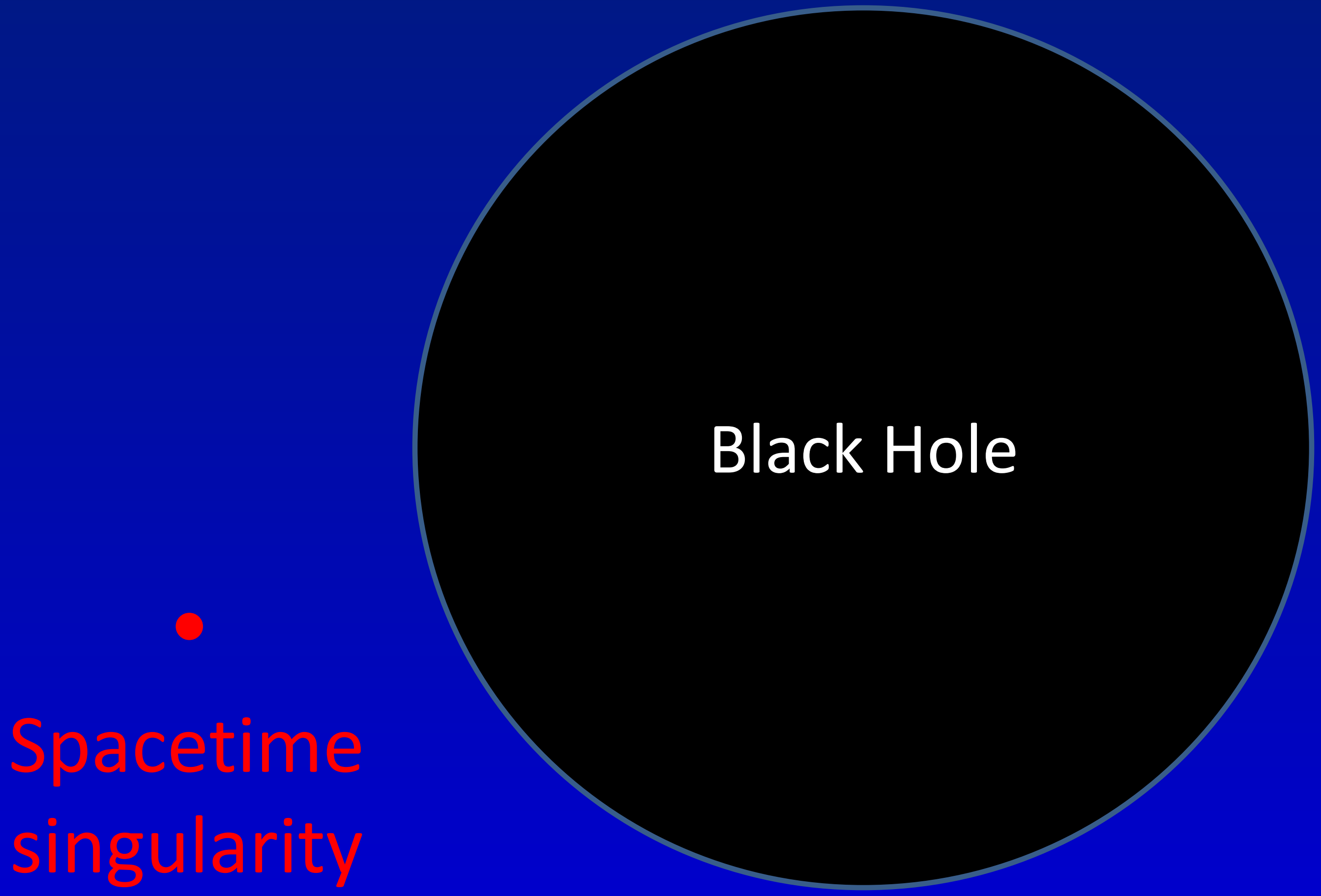
This was a statement with sweeping power.
It did not deal with idealized imploding stars.
It did not assume that the random deformations were small.

It dealt with every imploding star imaginable.
It dealt with stars that inhabit our Universe! (Kip Thorne)

What is inside a black hole?



Naked Singularity?

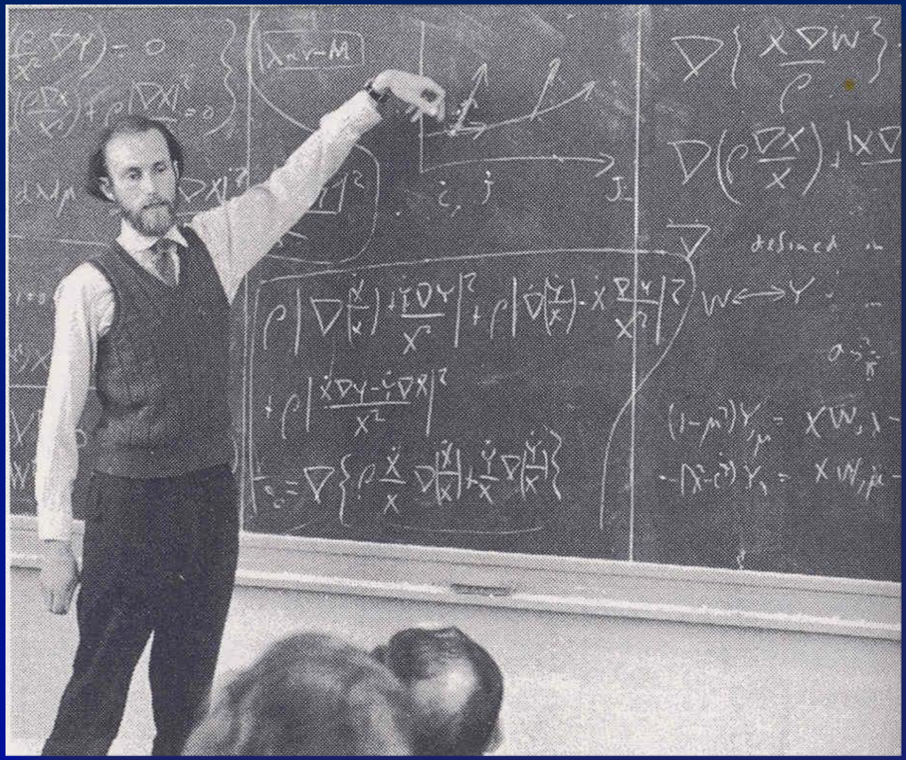


Cosmic Censorship Principle

No imploding object can ever form a naked singularity.

If a singularity is formed it must be clothed by a black hole horizon so that no one in the external Universe can see it.

Roger Penrose,
1969



The Kerr solution represents not just one special type of spinning black hole, but every spinning black hole that can possibly exist!

Brandon Carter

The four laws of BH Mechanics

- <u>Zeroth Law</u>

The surface gravity κ of a stationary BH is constant over the horizon.

- <u>First Law</u>

Any two neighbouring solutions containing BHs are related by

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega \delta J$$

$$dU = Tds + pdV$$

$$\kappa \Leftrightarrow T$$

$$A \Leftrightarrow S$$

The four laws of BH Mechanics

- Second Law

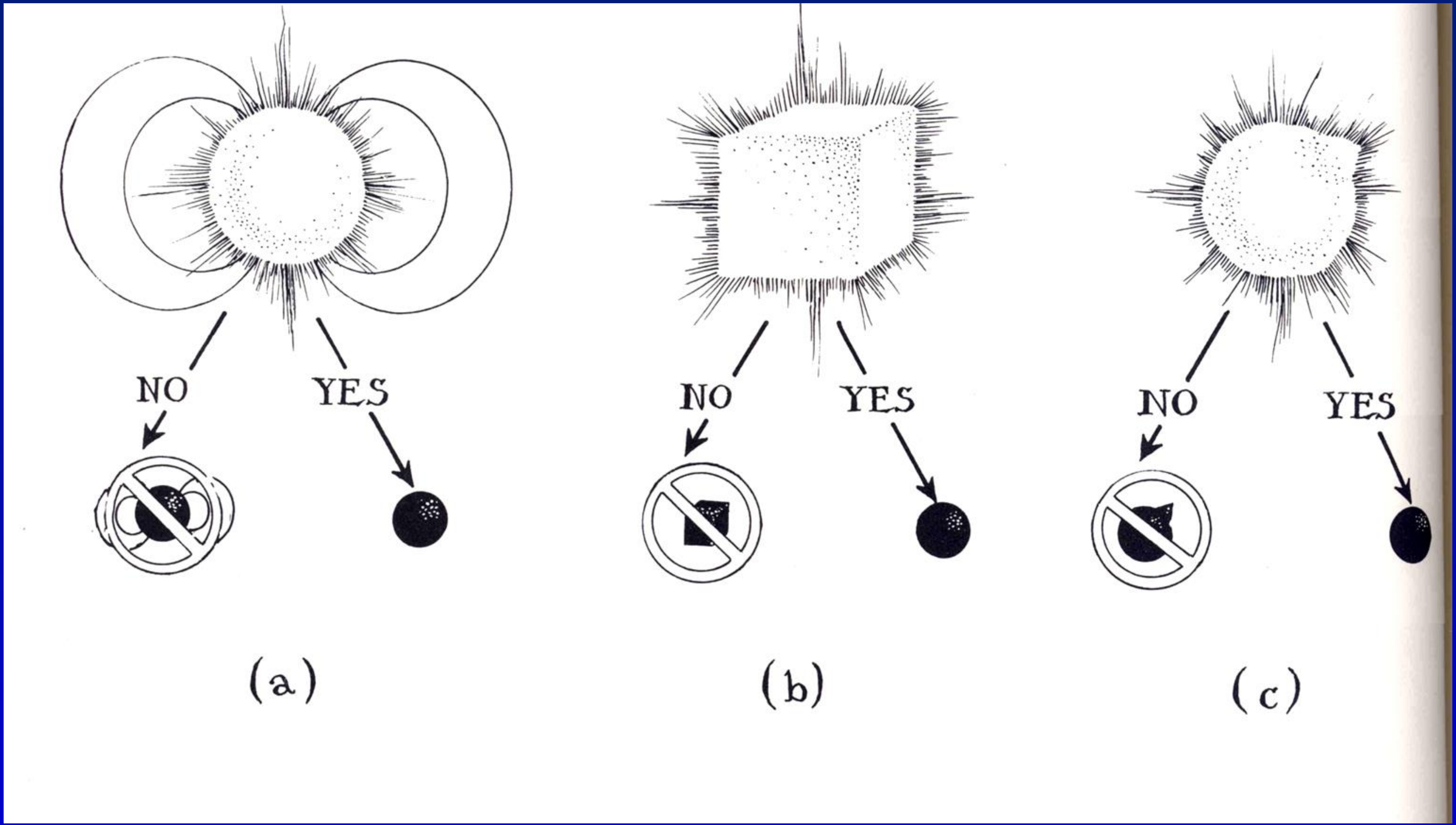
$$dA \geq 0 \qquad A_3 > A_1 + A_2$$

The area of a BH can never decrease.

- Third Law

It is impossible by any process, no matter how idealized, to reduce the surface gravity κ to zero by a finite sequence of operations.

Black holes have no hair!



“No Hair” Theorems

- A neutral and nonrotating BH is uniquely characterized by its mass M. W. Israel, 1968.
- All stationary BHs are AXIALLY SYMMETRIC. Hawking, 1970.
- All charged, rotating BHs are uniquely characterized by their Mass, Charge and Angular momentum. Carter, Hawking, Israel, 1972.

Black holes are uniquely characterized by their mass, charge & angular momentum.

Carter, Hawking and Israel, 1972

- All the deformations are radiated away as gravitational waves.
- All other quantum attributes of the matter are lost! (Baryon number, Lepton number, Strangeness etc.).
- Almost all the conservation laws discovered in the last century are transcended by BHs!

In my entire scientific life. . . . . the most shattering experience has been the realization that an exact solution of Einstein’s equations, the KERR METRIC, provides the absolutely exact representation of untold numbers of massive black holes that populate the Universe.

S. Chandrasekhar

The Quantum Twist

The Entropy of Black Holes

Beckenstein, 1972

- The entropy of a BH cannot be zero.
- The entropy of a Black Hole must be proportional to its surface area.

$$\text{Entropy} = \left(\frac{k_B c}{G h}\right) \text{Area of BH}$$

Entropy ~ 10^{79} for a 10 $M_{\odot}$ Black Hole.

$$\text{Thermodynamics} \Leftrightarrow \text{Classical Radiation} \Rightarrow (\boldsymbol{h})$$

Max Planck, 1900

$$\text{Thermodynamics} \Leftrightarrow \text{Classical Gravity} \Rightarrow (\boldsymbol{h})$$

Beckenstein, 1972

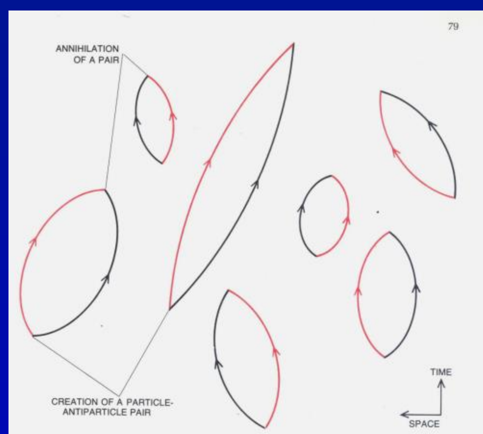
The Entropy of a Black Hole

- But Hawking was skeptical. A Black Hole cannot have entropy unless it is at a finite temperature.
- A classical Black Hole absorbs everything. It does not emit anything. It is a PERFECT SINK.
- A ‘perfect sink’ must be at the absolute zero of temperature.
- Therefore, the temperature of a Black Hole must be 0 K!

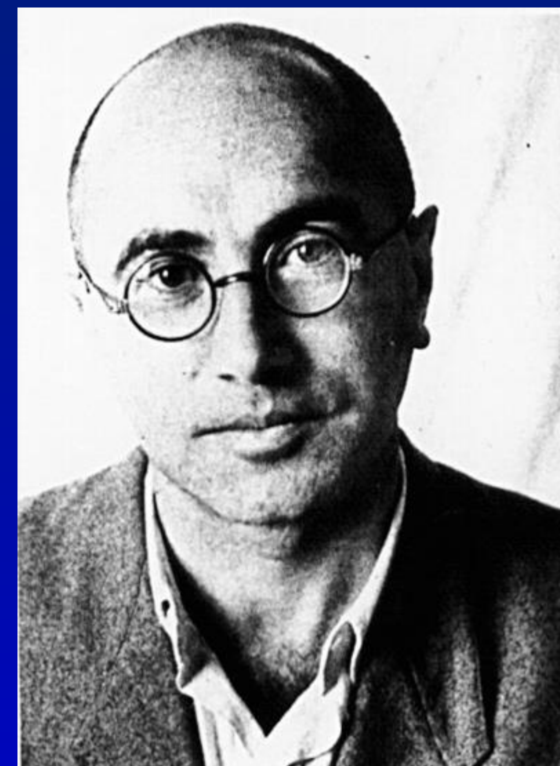
A deep insight

“A spinning BH must radiate!”

“A spinning metal sphere emits electromagnetic radiation”.



“Virtual Pairs” materialize from the vacuum by extracting energy from the rotating sphere.



Ya. B. Zeldovich

Super radiance from spinning BHs

- In papers published in 1970, 1972, Zeldovich had made the following extraordinary prediction.

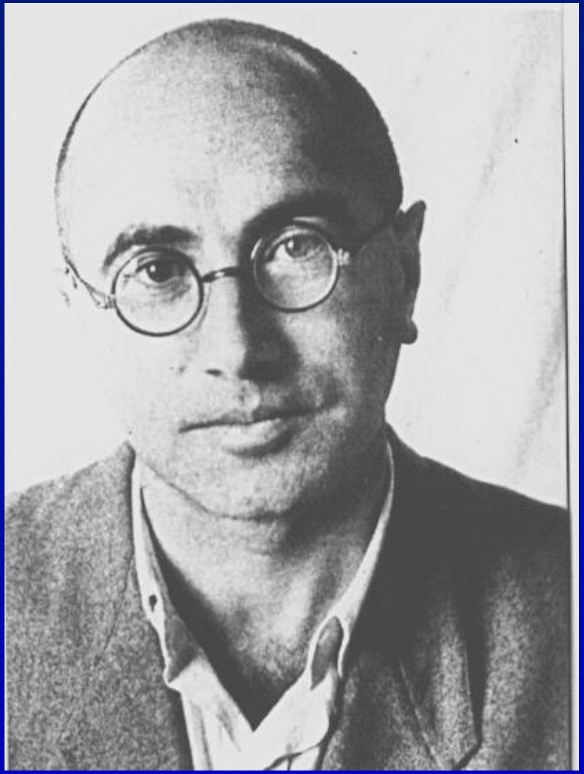
- Super radiance is a phenomenon in which waves incident on a rotating BH in "certain modes" will be scattered back with increased intensity. Rotational energy is extracted.
- In a particle picture this corresponds to an increased number of particles, and therefore to stimulated emission.
- Zeldovich had argued that, following Einstein, one should expect a steady rate of spontaneous emission also in the super radiant modes.

Will a non-rotating Black Hole radiate?

During the summer of 1973, Hawking visited Zeldovich in Moscow. During that conversation, Zeldovich told him of his ideas about spontaneous emission from rotating black holes in the "super radiant modes".

He asked Hawking the following question:

"Will a non-rotating BH radiate?"

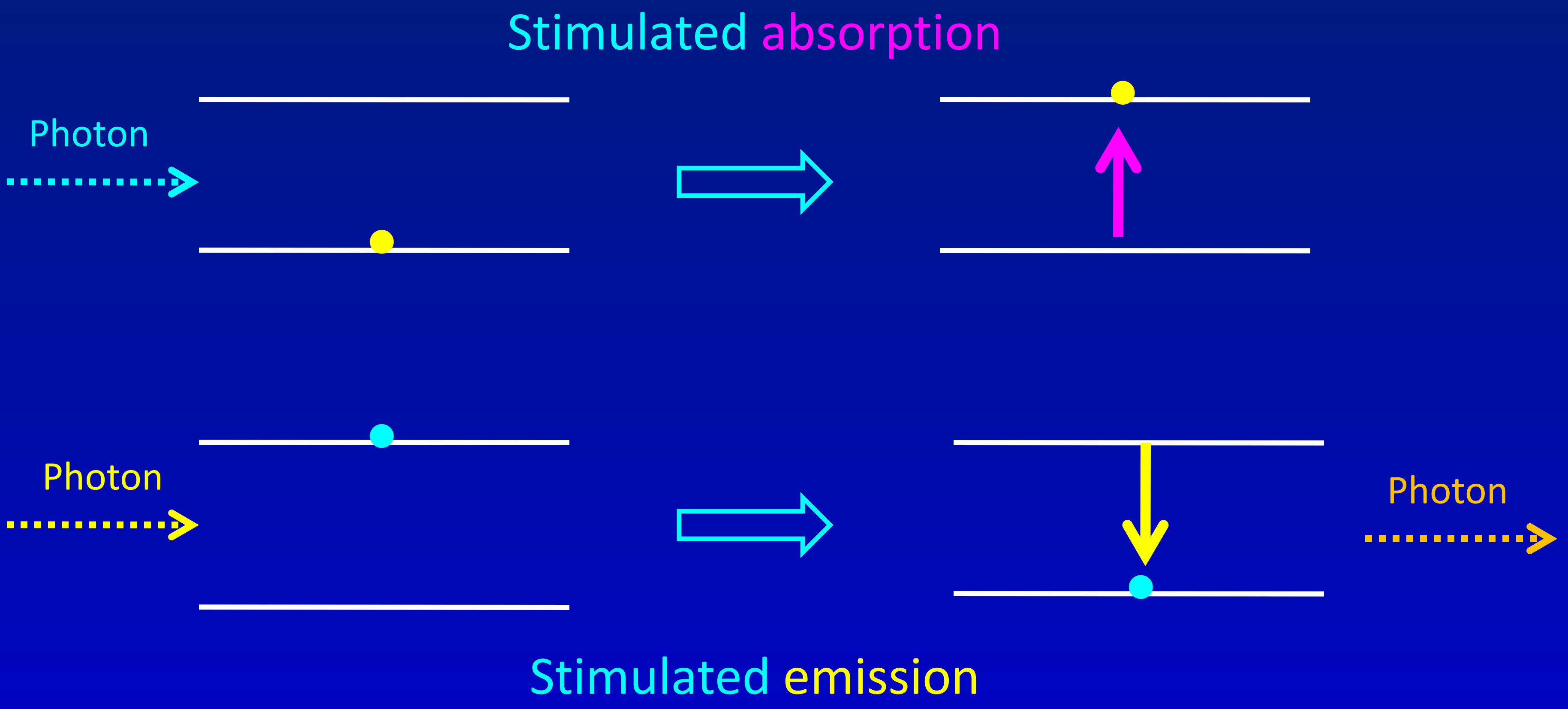


Ya. B. Zeldovich

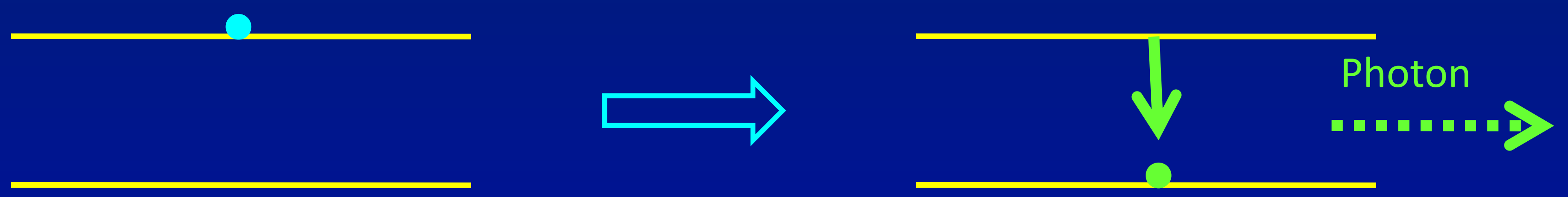
Spontaneous emission of radiation

- In 1917, Einstein introduced the novel idea of spontaneous emission of radiation.

Stimulated emission & absorption



Spontaneous emission of radiation



EINSTEIN: Whenever there is stimulated emission and absorption, there MUST be SPONTANEOUS emission also.

We now know that the electron 'jumps' because the vacuum tickles it!

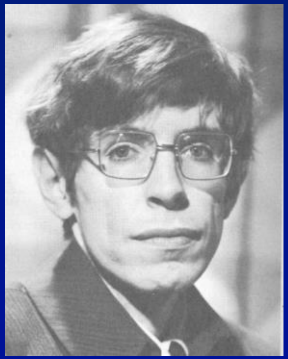
Spontaneous emission of radiation

- In 1917, Einstein showed that there is a fundamental relation between Absorption, Stimulated Emission and Spontaneous Emission.

- It is this relation that leads to Planck's distribution for the spectrum of Black Body radiation.

- Zeldovich argued that since a black hole can be "stimulated" to emit, it must also spontaneously emit.

A Great Discovery!



Stephen Hawking (1974)

- A nonrotating BH <u>will</u> radiate.
- It will emit particles and radiation in ALL modes, as though it were a blackbody with a temperature T (which is proportional to the surface gravity κ).

Thermal Radiation from Black Holes

Stephen Hawking, 1974

Black Body Radiation

Number of photons with frequency ν is $\propto \dfrac{1}{e^{h\nu/kT} - 1}$

Black Hole Radiation

Number of particles created in a mode is $\propto \dfrac{1}{e^{2\pi\omega/\kappa} - 1}$

Thermal Radiation from Black Holes

Stephen Hawking, 1974

Black Body Radiation

$$(\text{Prob. of Emission}) = e^{-\frac{h\nu}{kT}} (\text{Prob. of Absorption})$$

Black Hole Radiation

$$(\text{Prob. of Emission}) = e^{-\frac{2\pi\omega}{\kappa}} (\text{Prob. of Absorption})$$

Hawking Radiation

- The important thing to appreciate is that even a stationary, non-rotating black hole will radiate.
- The energy needed for the creation of these particles comes at the expense of the rest mass energy of the black hole.
- Therefore, as the black hole radiates, it will become less massive.

Hawking’s Discovery

- Hawking tried his best to make this radiation go away. But it wouldn’t!
- What convinced him finally was the “thermal nature” of the radiation.

Hawking’s Discovery

- Hawking’s discovery is not mathematically rigorous. But it makes sense.

- Recall, when matter and radiation come into thermodynamic equilibrium, radiation loses all its memory: how it was produced, frequency at birth, direction, polarization etc.
- It remembers only the temperature of the matter.

- Similarly, Hawking radiation makes sense because the radiation has a thermal spectrum, and remembers only the mass of the black hole.

Hawking's Discovery

- Recall that when matter falls into a black hole, there is a tremendous loss of information.
- Black Holes are uniquely characterized by their **mass, charge** and **angular momentum**.
- The nature of the radiation emitted by a black hole reflects this loss of information.
- The Hawking radiation makes sense because the radiation has a thermal spectrum, and remembers only the mass of the black hole.

Hawking Radiation

- As the black hole radiates, it will become less massive, and therefore “hotter”!

$$T_{BH} \sim 10^{-6} \frac{M_{Sun}}{M_{BH}} \text{Kelvin}$$

- Eventually, the black hole will evaporate away.
- What will remain is a “naked singularity”.

- Has quantum mechanics transcended the Cosmic Censorship Principle?
- Will an evaporating black hole leave behind a NAKED SINGULARITY?
- A naked singularity would introduce an additional element of uncertainty in physics, over and above Heisenberg's Uncertainty Principle.

Since the particles emitted by a black hole come from a region of which the observer has very limited knowledge, he cannot definitely predict the position or the velocity of the particle or any combination of the two: all he can predict is the probability that certain particles will be emitted.

So, as Hawking put it, Einstein was doubly wrong!

God does not play dice.

Albert Einstein

God not only plays dice,

He throws them in places where we cannot see!

Stephen Hawking

Black holes are macroscopic objects with masses varying from a few solar masses to millions of solar masses. To the extent they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described exactly by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object. Macroscopic objects, as we see them all around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a single unique two-parameter family of solutions for their description, they are the simplest objects as well.

S. Chandrasekhar